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Branching Rules for n-fold Covering Groups of SL_2 over a Non-Archimedean Local Field

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Abstract. Let G be the n-fold covering group of the special linear group of degree two over a non-Archimedean local field. We determine the decomposition into irreducibles of the restriction of the principal series representations of G to a maximal compact subgroup. Moreover, we analyse those features that distinguish this decomposition from the linear case.

1 Introduction

In this paper, covering groups, also known in the literature as metaplectic groups, are central extensions of a simply connected simple and split algebraic group over a non-Archimedean local field F by the group of the n-th roots of unity, μ_n. The two-fold covering group of the symplectic group was first constructed by André Weil in 1964 [32]. The problem of determining the class of covering groups was then studied by Steinberg [31], Moore [21], and further examined by Matsumoto [18] for simply connected Chevalley groups. Around the same time, Kubota independently constructed n-fold covering groups of SL_2 [15] and GL_2 [16], by means of presenting an explicit 2-cocycle. Kubota's cocycle is expressed in terms of the n-th Hilbert symbol.

Representation theory of covering groups is an active area of research. There have been a number of studies in this area from different perspectives, and there are still many open questions. Among these studies are the work of H. Aritürk [1], D. A. Kazhdan, and S. J. Patterson [14], C. Moen [20], D. Joyner [11,12], G. Savin [28], M. Weissman and T. Howard [10], P. J. McNamara [19], and D. Szpruch [9]. Principal series representations of the n-fold covering group G of $\mathrm{SL}(2)$ over F are explicitly constructed in [19]. One of the open questions we address in this paper is to analyse the decomposition of the restriction to a maximal compact subgroup of these principal series representations. We refer to this decomposition as the K-type decomposition.

The study of the decomposition of the restriction of representations to a particular subgroup (branching rules) is a common technique in representation theory. In the theory of real Lie groups, restriction to maximal compact subgroups retains a lot of information from the representation; in fact, such a restriction is a key step towards classifying irreducible unitary representations. In the case of reductive groups over p-adic fields, investigating the K-type decomposition reveals a finer structure of

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the representation; for instance, in this decomposition, in many cases one can identify types, in the sense of [3], and typical representations, in the sense of [2, 22]. The K-type problem for reductive p-adic groups is visited and solved in certain cases, including the principal series representations of $\mathrm{GL}(3)$ [4, 5, 26], and $\mathrm{SL}(2)$ [23, 24], representations of $\mathrm{GL}(2)$ [6], and supercuspidal representations of $\mathrm{SL}(2)$ [25].

The main idea of our method, which is aligned with the one in [23], is to use Mackey theory to reduce the K-type problem to calculating the dimensions of certain finite-dimensional Hecke algebras (Propositions 4.4, 4.5, and 4.10). Despite the similarity with the linear group $\mathrm{SL}(2)$ in [23], the K-type decomposition (Theorem 6.1) is fairly different; there are several interesting features in this K-type decomposition that were not present in the linear case.

Let m denote the conductor of the central character of the metaplectic torus, and set $\underline{n} = n$ if n is odd, and $\underline{n} = \frac{n}{2}$ if n is even. The K-type decomposition (Theorem 6.1) consists of the level-m representations, and their complement, the tail. The tail does not detect the non-triviality of the cover; it consists of $\underline{n}$ copies of a lift to covering groups of similar terms in the K-type decomposition for the linear group (Corollary 4.11(i)). The K-type decomposition for the n-fold covering group of GL_2 is a side-product of understanding the tail (Corollary 5.3).

On the other hand, the level-m representations demonstrate interesting different traits. For instance, the level-m piece almost always consists of $\underline{n}$ multiplicity-free irreducible representations, with the exception of certain level-one (also called depth-zero) representations. These level-one representations, which arise from twists of metaplectic quadratic characters by the characters of the group $\mathcal{O}^\times/\mathcal{O}^{\times\underline{n}}$, are either reducible or appear with multiplicity two. In this case, the number of reducible level-one components interestingly depends on whether n is divisible by four (Proposition 4.8 and Corollary 4.11(ii)).

Apart from its intrinsic value, the K-type decomposition can be used to answer other questions about the metaplectic principal series representations. For instance, the question of when these representations are reducible has not yet been fully answered. The author found the reducibility points for unramified principal series representations of $\mathtt{G}$ in [13]. Note that, unlike the case for the linear group $\mathrm{SL}(2)$ [8], a concrete description of the irreducible pieces is not known. In an ongoing project, we have some early results that make it feasible to describe these irreducible pieces in terms of the K-type components we found in Theorems 6.1 and 6.2.

Another interesting, and yet open, problem in the representation theory of covering groups is about their theory of types. There are several approaches to this question, among which is to describe the typical representations, defined by Henniart in an Appendix to [2], of a maximal compact subgroup. This approach is also visited by [17, 22, 27] for certain linear groups. We conjecture, based on some preliminary work, that our level-m pieces in the K-type decomposition in Theorem 6.1 are typical. If one can extend the local Langlands conjecture to the covering groups, such results on typical representations have implications for the so-called "inertial Langlands correspondence".

This paper is organized as follows. In Section 2, we present Kubota's construction of the covering group $\mathtt{G}$ of $\mathrm{SL}(2)$. In Section 3, we overview the structure of this covering group and compute some subgroups of our interest. We compute the K-type decomposition for the principal series representations of $\mathtt{G}$ in Section 4. This decomposition is completed by considering a similar problem for the n-fold covering group of $\mathrm{GL}(2)$ in Section 5. Our main results, Theorem 6.1 and Corollary 6.2, are stated in Section 6.

2 Notation and Background

Let F be a non-Archimedean local field with the ring of integers $\mathcal{O}$ and let the maximal ideal $\mathfrak{p}$ of $\mathcal{O}$. Let $\kappa := \mathcal{O}/\mathfrak{p}$ be the residue field and $q = |\kappa|$ be its cardinality. Let $\mathcal{O}^\times$ denote the group of units in $\mathcal{O}$. We fix a uniformizing element ϖ of $\mathfrak{p}$. For every $x \in F^\times$, the valuation of x is denoted by $\mathrm{val}(x)$, and $|x| = q^{-\mathrm{val}(x)}$. Let $\mathbf{1}$ denote the trivial character on $\mathcal{O}^\times$, and let sgn denote the non-trivial character of $\mathcal{O}^\times/\mathcal{O}^{\times 2}$. We refer to $\mathbf{1}$ and sgn as quadratic characters of $\mathcal{O}^\times$. Let $n > 2$ be an integer such that $n | q-1$. Set $\underline{n} = n$ if n is odd, and $\underline{n} = \frac{n}{2}$ if n is even. We assume that F contains the group μ_n of n-th roots of unity.

Set $\underline{G} = \mathrm{GL}_2(F)$, and $G = \mathrm{SL}_2(F)$. Let B (resp. $\underline{B}$) be the standard Borel subgroup of G (resp. $\underline{G}$), let N (resp. $\underline{N}$) be its unipotent radical, and let T (resp. $\underline{T}$) be the standard torus in G (resp. $\underline{G}$). Set $K = \mathrm{SL}_2(\mathcal{O})$ (resp. $\underline{K} = \mathrm{GL}_2(\mathcal{O})$) to be a maximal compact subgroup of G (resp. $\underline{G}$). By the Iwasawa decomposition, we have $G = TNK$ (resp. $\underline{G} = \underline{TNK}$). Our object of study is the central extension $\mathtt{G}$ of G by μ_n,

$$0 \longrightarrow \mu_n \xrightarrow{\check{i}} \mathtt{G} \xrightarrow{\check{p}} G \longrightarrow 0, \tag{2.1}$$

where $\check{i}$ and $\check{p}$ are natural injection and projection maps, respectively. The group $\mathtt{G}$, which we call the n-fold covering group of G, is constructed explicitly by Kubota [15]. In order to describe Kubota's construction, we need knowledge of the n-th Hilbert symbol $(\cdot,\cdot)_n \colon F^\times \times F^\times \to \mu_n$. Under our assumption on n, the n-th Hilbert symbol is given via $(a,b)_n = \overline{c}^{(q-1)/n}$, where $c = (-1)^{\mathrm{val}(a)\,\mathrm{val}(b)} a^{\mathrm{val}(b)}/b^{\mathrm{val}(a)}$, and $\overline{c}$ is the image of c in $\kappa^\times$. We benefit from the properties of the n-th Hilbert symbol, which can be found in [29, Ch XIV]. In particular, we benefit extensively from the following fact: $(a,b)_n = 1$ for all $a \in F^\times$ if and only if $b \in F^{\times n}$.

Define the map $\beta\colon G \times G \to \mu_n$ by

$$\beta(\mathbf{g}_1,\mathbf{g}_2) = \Big(\frac{X(\mathbf{g}_1\mathbf{g}_2)}{X(\mathbf{g}_1)}, \frac{X(\mathbf{g}_1\mathbf{g}_2)}{X(\mathbf{g}_2)}\Big)_n, \text{ where } X\Big(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\Big) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0.\end{cases}$$

In [15] Kubota proved that β is a non-trivial 2-cocycle in the continuous second cohomology group of G with coefficients in μ_n; whence, $\mathtt{G} = G \times \mu_n$ as a set, with the multiplication given via $(\mathbf{g}_1,\zeta_1)(\mathbf{g}_2,\zeta_2) = (\mathbf{g}_1\mathbf{g}_2, \beta(\mathbf{g}_1,\mathbf{g}_2)\zeta_1\zeta_2)$, for all $\mathbf{g}_1,\mathbf{g}_2 \in G$ and $\zeta_1,\zeta_2 \in \mu_n$. In [16, Thm. 1], Kubota extends the map β to a 2-cocycle β' for $\underline{G}$, which defines the n-fold covering group $\underline{\mathtt{G}} \cong F^\times \ltimes \mathtt{G}$ of $\underline{G}$. The covering group $\underline{\mathtt{G}}$ fits into the exact sequence

$$0 \longrightarrow \mu_n \xrightarrow{\check{i}} \underline{\mathtt{G}} \xrightarrow{\check{p}} \underline{G} \longrightarrow 0.$$

Our notational convention is to use upper case (underlined) letters in roman font (for example B, $\underline{B}$) for subgroups of the linear group G ($\underline{G}$), lower case letters in boldface font (for example $\mathbf{g}$) for elements of G ($\underline{G}$), upper case (underlined) letters in typewriter font (for example $\mathtt{B}, \underline{\mathtt{B}}$) for subgroups of the covering groups $\mathtt{G}$ ($\underline{\mathtt{G}}$), lower case letters in typewriter font (for example $\mathtt{g}$) for elements of the covering groups $\mathtt{G}$ and $\underline{\mathtt{G}}$, and lower case letters in roman fonts (for example b) for the elements of the base field F.

For all $t, s \in F^\times$, set $\mathrm{dg}(t) = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right) \in T$, $\mathrm{dg}(t,s) = \left(\begin{smallmatrix} t & 0 \\ 0 & s \end{smallmatrix}\right) \in \underline{T}$, and $\iota(t) = (\mathrm{dg}(t), 1) \in \mathtt{T}$. Set $\mathbf{w} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and $\mathtt{w} = (w, 1) \in \mathtt{G}$. Moreover, for matrices $\mathbf{x}$ and $\mathbf{y}$, with $\mathbf{y}$ invertible, set $\mathbf{x}^{\mathbf{y}} := \mathbf{y}^{-1}\mathbf{x}\mathbf{y}$ and ${}^{\mathbf{y}}\mathbf{x} := \mathbf{y}\mathbf{x}\mathbf{y}^{-1}$.

3 Structure Theory

For any subgroup H of G, the inverse image $\mathtt{H} := \check{p}^{-1}(H)$ is a subgroup of $\mathtt{G}$. In particular, we are interested in the subgroups $\mathtt{T}$, $\mathtt{B}$, and $\mathtt{K}$ of $\mathtt{G}$. We say the central extension splits over the subgroup H of G if there exists an isomorphism that yields $\check{p}(H)^{-1} \cong H \times \mu_n$.

It is not difficult to see that $\mathtt{T}$ is not commutative, and hence, the central extension does not split over T (and therefore not over B). The commutator subgroup $[\mathtt{T}, \mathtt{T}] \cong \mu_{\underline{n}}$ is central in (2.1), which implies that $\mathtt{T}$ is a two-step nilpotent group, also known as a Heisenberg group. Clearly, $\mu_n \subset Z(\mathtt{T})$, indeed, under proper identifications, one can see that

$$Z(\mathtt{T}) = \{(\mathrm{dg}(t), \zeta) \mid t \in F^{\times \underline{n}}, \zeta \in \mu_n\} \cong \mathcal{O}^{\times \underline{n}} \times \underline{n}\mathbb{Z} \times \mu_n.$$

When $\underline{n}$ is odd, $(\varpi, \varpi)_n = -1$, in which case, the isomorphism map is not trivial. A straightforward application of the Hensel's lemma shows that $[\mathtt{T} : Z(\mathtt{T})] = \underline{n}^2$.

In order to construct principal series representations of $\mathtt{G}$ in Section 4, we need to construct irreducible representations of the Heisenberg group $\mathtt{T}$. To do so, we need to identify a maximal abelian subgroup of $\mathtt{T}$. Let $\mathtt{A} = \mathrm{C}_{\mathtt{T}}(\mathtt{T} \cap \mathtt{K})$ be the centralizer of $\mathtt{T} \cap \mathtt{K}$ in $\mathtt{T}$. It is not difficult to calculate that

$$\mathtt{A} = \{(\mathrm{dg}(a), \zeta) \mid a \in F^\times, \underline{n} \,|\, \mathrm{val}(a), \zeta \in \mu_n\},$$

which, under proper identifications, is isomorphic to $\mathcal{O}^\times \times \underline{n}\mathbb{Z} \times \mu_n$ (similar to the case of $Z(\mathtt{T})$, the isomorphism map is not trivial when $\underline{n}$ is odd), and that $\mathtt{A}$ is abelian. Observe that $\mathtt{T} \cap \mathtt{K} \subset \mathtt{A}$ implies that $\mathtt{A}$ is a maximal abelian subgroup. Note that $[\mathtt{T} : \mathtt{A}] = [\mathbb{Z} : \underline{n}\mathbb{Z}] = \underline{n}$.

Let N be the unipotent radical of B. It follows directly from the Kubota's formula for β that $\beta|_N$ is trivial, so $N \times \{1\}$ is a subgroup of $\mathtt{G}$. We identify N with $N \times \{1\}$. Under this identification, we have the covering analogue of the Levi decomposition: $\mathtt{B} = \mathtt{T} \ltimes N$.

Next, we describe a family of compact open subgroups of $\mathtt{G}$. Under the assumption $n | q - 1$, it was proved in [16, Thm. 2] that

$$\mathtt{K} \longrightarrow K \times \mu_n, \tag{3.1}$$

$$(\mathbf{k}, \zeta) \longmapsto (\mathbf{k}, s(\mathbf{k})\zeta), \text{ where } s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} (c, d)_n & \text{if } 0 < \mathrm{val}(c) < \infty, \\ 1 & \text{otherwise,} \end{cases}$$

is an isomorphism. The image of K in $\mathtt{K}$ under the isomorphism (3.1) is the subgroup $\mathtt{K}_0 := \{(\mathbf{k}, s(\mathbf{k})^{-1}) \mid \mathbf{k} \in K\}$ of $\mathtt{K}$. Consider the compact open congruent subgroups $K_j := \{\mathbf{g} \in K \mid \mathbf{g} \equiv \mathrm{I}_2 \mod \mathfrak{p}^j\}$, for $j \geq 1$, of K. Noticing that $1 + \mathfrak{p} \subset \mathcal{O}^{\times n}$, it is a straightforward consequence of the properties of the n-th Hilbert symbol that the central extension (2.1) splits trivially over each of the subgroups K_j, $j \geq 1$, $T \cap K$, and $B \cap K$.

We identify $K_j \cong K_j \times \{1\}$, $j \geq 1$, $B \cap K \cong (B \cap K) \times \{1\}$, and $T \cap K \cong (T \cap K) \times \{1\}$ as subgroups of $\mathtt{K}$. Observe that $\mathtt{T} \times \mathtt{K} \cong \mathcal{O}^\times \times \mu_n$ via the trivial map.

In a similar way, we define the subgroups $\underline{\mathtt{T}}$, $\underline{\mathtt{B}}$, and $\underline{\mathtt{K}}$ of $\underline{\mathtt{G}}$ to be the inverse images of the standard torus, Borel, and the maximal compact $\underline{K} = \mathrm{GL}(\mathcal{O})$ subgroups of $\underline{G}$, respectively. The central extension $\underline{\mathtt{G}}$ does not split over $\underline{T}$. Moreover, $\underline{\mathtt{T}}$ is a Heisenberg group. It is not difficult to see that

$$Z(\underline{\mathtt{T}}) = \left\{ (\mathrm{dg}(s,t), \zeta) \mid s, t \in F^{\times n}, \zeta \in \mu_n \right\},$$

and $[\underline{\mathtt{T}} : Z(\underline{\mathtt{T}})] = n^4$. Moreover, set

$$\underline{\mathtt{A}} = \mathrm{C}_{\underline{\mathtt{T}}}(\underline{\mathtt{T}} \cap \underline{\mathtt{K}}) = \left\{ (\mathrm{dg}(s,t), \zeta) \,\Big|\, s, t \in F^\times, n | \mathrm{val}(s), n | \mathrm{val}(t), \zeta \in \mu_n \right\}.$$

Then $\underline{\mathtt{A}}$ is a maximal abelian subgroup of $\underline{\mathtt{T}}$, and $[\underline{\mathtt{T}} : \underline{\mathtt{A}}] = n^2$. Regardless of parity of n, $Z(\underline{\mathtt{T}})$ and $\underline{\mathtt{A}}$ are trivially isomorphic to $F^{\times n} \times F^{\times n} \times \mu_n$ and $\mathcal{O}^\times \times \mathcal{O}^\times \times n\mathbb{Z} \times n\mathbb{Z} \times \mu_n$, respectively. In addition, $\beta'|_{\underline{N}}$ is trivial, hence, we can identify $\underline{N}$ with $\underline{N} \times \{1\}$. Under this identification, we have the Levi decomposition: $\underline{\mathtt{B}} = \underline{\mathtt{T}} \ltimes \underline{N}$. It is shown in [16] that the central extension $\underline{\mathtt{G}}$ splits over $\underline{K}$. For $j \geq 1$, let $\underline{K}_j$ denote the family of compact open congruent subgroups $\{\mathbf{g} \in \underline{K} \mid \mathbf{g} \equiv \mathrm{I}_2 \mod \mathfrak{p}^j\}$ of $\underline{K}$. The central extension $\underline{\mathtt{G}}$ splits over $\underline{K}_j$, $\underline{T} \cap \underline{K}$, and $\underline{B} \cap \underline{K}$.

4 Branching Rules for $\mathtt{G}$

First, we present the construction of the principal series representations of $\mathtt{G}$ following [19]. Fix a faithful character $\epsilon\colon \mu_n \to \mathbb{C}^\times$. A representation of $\mathtt{G}$ is genuine if the central subgroup μ_n acts by ϵ. Such representations do not factor through representations of G. The construction of principal series representations of $\mathtt{G}$ is based on the essential fact that $\mathtt{T}$ is a Heisenberg subgroup, and hence its representations are governed by the Stone-von Neumann theorem, which we state here. See [19] for the proof.

Stone–von Neumann Theorem *Let H be a Heisenberg group with center $Z(H)$ such that $H/Z(H)$ is finite, and let χ be a character of $Z(H)$. Suppose that* $\ker(\chi) \cap [H,H] = \{1\}$. *Then there is a unique (up to isomorphism) irreducible representation π of H with central character χ. Let A be any maximal abelian subgroup of H and let χ_0 be any extension of χ to A. Then $\pi \cong \mathrm{Ind}_A^H \chi_0$.*

Note that $[\mathtt{T} : Z(\mathtt{T})] = \underline{n}^2 < \infty$. Let χ be a genuine character of $Z(\mathtt{T})$, so that $\chi|_{\mu_n} = \epsilon$. Thus, $\ker(\chi) \cap [\mathtt{T}, \mathtt{T}]$ is trivial. Hence, Stone–von Neumann Theorem applies: genuine irreducible smooth representations ρ of $\mathtt{T}$ are classified by genuine smooth characters of $Z(\mathtt{T})$. Moreover, $\dim(\rho) = [\mathtt{T} : \mathtt{A}] = \underline{n}$.

Let χ_0 be a fixed extension of χ to $\mathtt{A}$, so that $(\rho, \mathrm{Ind}_{\mathtt{A}}^{\mathtt{T}} \chi_0)$ is the unique smooth genuine irreducible representation of $\mathtt{T}$ with central character χ. Let us again write ρ for the genuine smooth irreducible representation of $\mathtt{T}$, with central character χ, extended trivially over N to a representation of $\mathtt{B} = \mathtt{T} \ltimes N$. Then the genuine principal series representation of $\mathtt{G}$ associated with ρ is $(\pi, \mathrm{Ind}_{\mathtt{B}}^{\mathtt{G}} \rho)$, where Ind denotes the smooth (non-normalized) induction. In the rest of this section, we decompose $\mathrm{Res}_{\mathtt{K}} \mathrm{Ind}_{\mathtt{B}}^{\mathtt{G}} \rho$ into irreducible constituents. We drop the adjective "genuine" for brevity.

Every element $\mathtt{K}$ can be decomposed as $\mathtt{k}(\mathrm{I}_2, \zeta)$ for some $\mathtt{k} \in \mathtt{K}_0$ and $\zeta \in \mu_n$. Define $\widetilde{\mathbf{1}}\colon \mathtt{K} \to \mathbb{C}$ to be the character of $\mathtt{K}$ given by $\widetilde{\mathbf{1}}(\mathtt{k}(\mathrm{I}_2, \zeta)) = \epsilon(\zeta)$. Define the character

$$\vartheta\colon F^\times \longrightarrow \mathbb{C}^\times, \quad a \longmapsto \epsilon\big((\varpi, a)_n\big). \tag{4.1}$$

Observe that ϑ is ramified of degree one. Observe that a typical element of $\mathtt{A}$ can be written as (a, r, ζ), and a typical element of $\mathtt{T} \cap \mathtt{K}$ can be written as (a, ζ), where $a \in \mathcal{O}^\times$, $r \in \mathbb{Z}$, and $\zeta \in \mu_n$. We can express every character χ of $Z(\mathtt{T})$ as $\chi = \tau \otimes \nu_s \otimes \epsilon$, where τ is a character of $\mathcal{O}^{\times \underline{n}}$ and $\nu_s\colon \underline{n}\mathbb{Z} \to \mathbb{C}$, $\nu_s(\underline{n}r) = q^{-\underline{n}r}$, $s \in \mathbb{C}$. Then the fixed extension χ_0 of χ to $\mathtt{A}$ is of the form $\tau_0 \otimes \nu_s \otimes \epsilon$, where τ_0 is a fixed extension of τ to $\mathcal{O}^\times$. For brevity, set $\tau_{0\,\mathcal{O}^{\times 2}} := \tau_0|_{\mathcal{O}^{\times 2}}$ and set $\vartheta_{\mathcal{O}^{\times 2}} := \vartheta|_{\mathcal{O}^{\times 2}}$.

Lemma 4.1 *Let ρ be the unique irreducible representation of $\mathtt{T}$ with central character χ. Then $\mathrm{Res}_{\mathtt{A}} \rho \cong \bigoplus_{i=0}^{\underline{n}-1} \chi_i$, where $\chi_i = \tau_i \otimes \nu_s \otimes \epsilon$, and the τ_i's are $\underline{n}$ distinct characters of $\mathcal{O}^\times$ defined by $\tau_i(a) = \tau_0(a)\vartheta^{2i}(a)$, for all $a \in \mathcal{O}^\times$, and $0 \le i < \underline{n}$.*

Proof By the Stone–von Neumann Theorem, $\rho \cong \mathrm{Ind}_{\mathtt{A}}^{\mathtt{T}} \chi_0$. By Mackey theory,

$$\mathrm{Res}_{\mathtt{A}} \mathrm{Ind}_{\mathtt{A}}^{\mathtt{T}} \chi_0 = \bigoplus_{\mathtt{r} \in S_{\underline{n}}} \mathrm{Ind}_{\mathtt{A} \cap {}^{\mathtt{r}}\mathtt{A}}^{\mathtt{A}} {\chi_0}^{\mathtt{r}},$$

where $S_{\underline{n}}$ is a complete set of coset representatives for $\mathtt{A}\backslash\mathtt{T}/\mathtt{A}$. We can choose $S_{\underline{n}} = \{(\mathrm{dg}(\varpi^i), 1) | 0 \le i < \underline{n}\}$. Since $\mathtt{A}$ is stable under conjugation by $S_{\underline{n}}$, $\mathrm{Ind}_{\mathtt{A} \cap {}^{\mathtt{r}}\mathtt{A}}^{\mathtt{A}} {\chi_0}^{\mathtt{r}} = {\chi_0}^{\mathtt{r}}$. Let $\mathtt{a} = (a, k, \zeta) \in \mathtt{A}$ and $\mathtt{r} = (\mathrm{dg}(\varpi^i), 1) \in S_{\underline{n}}$. Note that $\mathtt{r}^{-1} = (\mathrm{dg}(\varpi^{-i}), (\varpi^i, \varpi^i)_n)$. A simple calculation shows that $\mathtt{r}^{-1}\mathtt{a}\mathtt{r} = (a, k, (\varpi, a)_n^{2i}\zeta)$. Hence,

$$\begin{aligned} {\chi_0}^{\mathtt{r}}(a, k, \zeta) &= \chi_0\big(a, k, (\varpi, a)_n^{2i}\zeta\big) = \tau_0(a)\nu_s(k)\epsilon\big((\varpi, a)_n^{2i}\zeta\big) \\ &= \tau(a)\vartheta^{2i}(a)\nu_s(k)\epsilon(\zeta). \end{aligned}$$

Denote this character by χ_i. To show that the χ_i, $0 \le i < \underline{n}$, are distinct, it is enough to show that $\vartheta^{2i}|_{\mathcal{O}^\times} = 1$ if and only if $i = 0$. Observe that $\vartheta^{2i}(a) = \overline{a^{-1}}^{(q-1)2i/n}$, which is equal to one for all $a \in \mathcal{O}^\times$ if and only if $n | 2i$. ∎

The characters χ_i defined in Lemma 4.1 are clearly distinct when restricted to $\mathtt{T} \cap \mathtt{K} \cong \mathcal{O}^\times \times \mu_n$ and, again writing χ_i for these restrictions,

$$\mathrm{Res}_{\mathtt{T} \cap \mathtt{K}} \rho = \bigoplus_{i=0}^{\underline{n}-1} \chi_i. \tag{4.2}$$

Proposition 4.2 *Let χ_i, $0 \le i < \underline{n}$, also denote the trivial extension of the characters in (4.2) to $\mathtt{B} \cap \mathtt{K}$. Then*

$$\mathrm{Res}_{\mathtt{K}} \mathrm{Ind}_{\mathtt{B}}^{\mathtt{G}} \rho \cong \bigoplus_{i=0}^{\underline{n}-1} \mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i.$$

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Proof By Mackey theory, we have $\mathrm{Res}_{\mathtt{K}} \mathrm{Ind}_{\mathtt{B}}^{\mathtt{G}} \rho \cong \bigoplus_{\mathtt{x} \in X} \mathrm{Ind}_{\mathtt{B}^{\mathtt{x}^{-1}} \cap \mathtt{K}}^{\mathtt{K}} \mathrm{Res}_{\mathtt{B}^{\mathtt{x}^{-1}} \cap \mathtt{K}} \rho^{\mathtt{x}}$, where X is a complete set of double coset representatives of $\mathtt{K}$ and $\mathtt{B}$ in $\mathtt{G}$. The Iwasawa decomposition $\mathtt{KB} = \mathtt{G}$ implies that $X = \{(\mathrm{I}_2, 1)\}$, and hence $\mathrm{Res}_{\mathtt{K}} \mathrm{Ind}_{\mathtt{B}}^{\mathtt{G}} \rho = \mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \mathrm{Res}_{\mathtt{B} \cap \mathtt{K}} \rho$. The result follows from (4.2). ∎

Hence, in order to calculate the K-types, it is enough to decompose each $\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i$, $0 \le i < \underline{n}$, into irreducible representations. Note that $\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i$ is smooth and admissible. Fix $i \in \{0, \ldots, \underline{n} - 1\}$. The smoothness of $\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i$ implies that

$$\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i = \bigcup_{l \ge 1} (\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i)^{K_l}.$$

By admissibility, $(\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i)^{K_l}$ is finite-dimensional for every $l \ge 1$, and since K_l is normal in $\mathtt{K}$, it is $\mathtt{K}$-invariant. Hence, to decompose $\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i$ into irreducible constituents, it is enough to decompose each $(\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i)^{K_l}$ into irreducible constituents.

For any character γ of any subgroup $\mathtt{D}$ of $\mathtt{T}$, we say γ is primitive mod m if m is the smallest strictly positive integer for which $\mathrm{Res}_{\mathtt{D} \cap K_m} \gamma = 1$. From now on, let $m \ge 1$ be a positive integer such that χ is primitive mod m. Because $1 + \mathfrak{p} \subset F^{\times n}$, $Z(\mathtt{T}) \cap K_m = \mathtt{T} \cap K_m$, for all $m \ge 1$. Note that since $\chi_i|_{Z(\mathtt{T})} = \chi$, $\chi_i|_{\mathtt{T} \cap K_m} = \chi|_{Z(\mathtt{T}) \cap K_m}$. Hence, χ is primitive mod m if and only if the χ_i for $0 \le i < \underline{n}$ are primitive mod m. Set $\mathtt{B}_l := (\mathtt{B} \cap \mathtt{K}) K_l$.

Lemma 4.3 *For every* $0 \le i < \underline{n}$*,*

$$\left(\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i \right)^{K_l} = \begin{cases} \{0\} & \text{if } 0 < l < m, \\ \mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_i & \text{otherwise.} \end{cases}$$

Proof Suppose $0 < l < m$, and that f is a vector in $(\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i)^{K_l}$. Because $\chi_i|_{\mathtt{B} \cap K_l} \ne 1$ for $l < m$, we can choose $\mathtt{b} \in \mathtt{B} \cap K_l$ such that $\chi_i(\mathtt{b}) \ne 1$. Let $\mathtt{g} \in \mathtt{K}$. Note that K_l is normal in $\mathtt{K}$, and hence $\mathtt{g}^{-1}\mathtt{b}\mathtt{g} \in K_l$. On the one hand, $f(\mathtt{bg}) = \chi_i(\mathtt{b}) f(\mathtt{g})$; on the other hand, $f(\mathtt{bg}) = f(\mathtt{g}\mathtt{g}^{-1}\mathtt{b}\mathtt{g}) = (\mathtt{g}^{-1}\mathtt{b}\mathtt{g}) \cdot f(\mathtt{g}) = f(\mathtt{g})$, since f is fixed by K_l. It follows that $\chi_i(\mathtt{b}) f(\mathtt{g}) = f(\mathtt{g})$. Our choice of $\mathtt{b}$ implies that $f(\mathtt{g}) = 0$, and because $\mathtt{g}$ is arbitrary, $f = 0$. However, if $l \ge m$, then $\chi_i|_{K_l} = 1$ and because K_l is normal in $\mathtt{K}$, it is not difficult to see that every K_l-fixed vector f translates on the left by $\mathtt{B}_l$ and vice-versa. Hence, the result follows. ∎

Lemma 4.3 tells us that, in order to decompose $(\mathrm{Ind}_{\mathtt{B} \cap \mathtt{K}}^{\mathtt{K}} \chi_i)^{K_l}$ into irreducible constituents, it is enough to decompose $\mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_i$. Hence, we are interested in counting the dimension of $\mathrm{Hom}_{\mathtt{K}}(\mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_i, \mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_i)$. By Frobenius reciprocity, this latter space is isomorphic to $\mathrm{Hom}_{\mathtt{B}_l}(\mathrm{Res}_{\mathtt{B}_l} \mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_i, \chi_i)$. It follows from Mackey theory that

$$\mathrm{Res}_{\mathtt{B}_l} \mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_i \cong \bigoplus_{\mathtt{x} \in S} \mathrm{Ind}_{\mathtt{B}_l^{\mathtt{x}^{-1}} \cap \mathtt{B}_l}^{\mathtt{B}_l} \chi_i^{\mathtt{x}},$$

where S is a set of double coset representatives of $\mathtt{B}_l \backslash \mathtt{K} / \mathtt{B}_l$. The set S is a lift to the covering group $\mathtt{K}$ of a similar set of double coset representatives calculated in [23, Eqn (4.1)]. Using the latter set, and because $\mu_n \subset \mathtt{B}_l$, it is easy to see that

$$S = \left\{ (\mathrm{I}_2, 1), \mathtt{w}, \mathrm{lt}(x \varpi^r) \mid x \in \{1, \varepsilon\}, 1 \le r < l \right\}, \tag{4.3}$$

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where $\mathrm{lt}(x\varpi^r) = \left(\left(\begin{smallmatrix} 1 & 0 \\ x\varpi^r & 1 \end{smallmatrix}\right), 1\right)$, and ε is a fixed non-square. For $0 \le i, j < \underline{n}$, let $\mathcal{H}_{i,j}$ be the Hecke algebra

$$\begin{aligned} \mathcal{H}_{i,j} &:= \mathcal{H}(\mathtt{B}_l \backslash \mathtt{K} / \mathtt{B}_l, \chi_i, \chi_j) \\ &= \left\{ f : \mathtt{K} \to \mathbb{C} \mid f(lgh) = \chi_i(l) f(g) \chi_j(h), l, h \in \mathtt{B}_l, g \in \mathtt{K} \right\}. \end{aligned}$$

Proposition 4.4 *Let* $0 \le i, j < \underline{n}$. *Then* $\dim \mathrm{Hom}_{\mathtt{K}}(\mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_i, \mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_j) = \dim \mathcal{H}_{i,j}$.

Proof On the one hand, observe that

$$\mathrm{Hom}_{\mathtt{K}}(\mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_i, \mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_j) = \bigoplus_{\mathtt{x} \in S} \mathrm{Hom}_{\mathtt{B}_l}(\mathrm{Ind}_{\mathtt{B}_l^{\mathtt{x}^{-1}} \cap \mathtt{B}_l}^{\mathtt{B}_l} \chi_i^{\mathtt{x}}, \chi_j),$$

which by Frobenius reciprocity is equal to $\bigoplus_{\mathtt{x} \in S} \mathrm{Hom}_{\mathtt{B}_l^{\mathtt{x}^{-1}} \cap \mathtt{B}_l}(\chi_i^{\mathtt{x}}, \chi_j)$. Let $S_{i,j}$ be the set of all $\mathtt{x} \in S$ such that $\chi_i(\mathtt{g}) = \chi_j(\mathtt{h})$, whenever $\mathtt{h}, \mathtt{g} \in \mathtt{B}_l$ and $\mathtt{x}\mathtt{g}\mathtt{x}^{-1} = \mathtt{h}$. Then $\dim \mathrm{Hom}_{\mathtt{K}}(\mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_i, \mathrm{Ind}_{\mathtt{B}_l}^{\mathtt{K}} \chi_j) = |S_{i,j}|$. On the other hand, observe that for every $\mathtt{x} \in S$, there exists a function $f \in \mathcal{H}_{i,j}$ with support on the double coset represented by $\mathtt{x}$ if and only if $\mathtt{h} = \mathtt{x}\mathtt{g}\mathtt{x}^{-1}$ implies $\chi_i(\mathtt{g}) = \chi_j(\mathtt{h})$ for all $\mathtt{h}, \mathtt{g} \in \mathtt{B}_l$. Moreover, the basis of $\mathcal{H}_{i,j}$ is parametrized by such double coset representatives. Hence, $\dim \mathcal{H}_{i,j} = |S_{i,j}|$. ∎

Set $(T \cap K)^2 := \{\mathrm{dg}(t^2) \mid t \in \mathcal{O}^\times\}$, $\mathtt{T}_l := \{\iota(t) \mid t \in \mathcal{O}^\times(1 + \mathfrak{p}^l)\}$, and $(\mathtt{T}_l)^2 := \{\iota(t^2) \mid t \in \mathcal{O}^\times(1 + \mathfrak{p}^l)\}$. It is not difficult to see that $\mathtt{T}_l$ and $(\mathtt{T}_l)^2$ are subgroups of $(\mathtt{T} \cap \mathtt{K})K_l$.

Proposition 4.5 *Let* $l \ge m$ *and* $0 \le i < \underline{n}$. *Then*

$$\dim \mathcal{H}_{i,i} = \begin{cases} 1 + 2(l - m) & \textit{if } \chi_i|_{\mathcal{O}^{\times 2}} \neq 1, \\ 2l & \textit{otherwise.} \end{cases}$$

Proof Assume $l \ge m$. Note that $f(\mathtt{b}\mathtt{k}\mathtt{b}') = \chi_i(\mathtt{b}) f(\mathtt{k}) \chi_i(\mathtt{b}')$ for all $f \in \mathcal{H}_{i,i}$, $\mathtt{b}, \mathtt{b}' \in \mathtt{B}_l$, and $\mathtt{k} \in \mathtt{K}$. Hence, for every double coset representative $\mathtt{x}$ in (4.3), there exists a function $f \in \mathcal{H}_{i,i}$, with support on the double coset represented by $\mathtt{x}$ if and only if $\mathtt{b}\mathtt{x}\mathtt{b}' = \mathtt{x}$ implies that $\chi_i(\mathtt{b}\mathtt{b}') = 1$ for all $\mathtt{b}, \mathtt{b}' \in \mathtt{B}_l$. The set of such double cosets parameterizes a basis for $\mathcal{H}_{i,i}$. We now determine these double cosets. Let $\mathtt{b} = (\mathbf{b}, \zeta) = \left(\left(\begin{smallmatrix} t & s \\ 0 & t^{-1} \end{smallmatrix}\right), \zeta\right)$ and $\mathtt{b}' = (\mathbf{b}', \zeta') = \left(\left(\begin{smallmatrix} t' & s' \\ 0 & t'^{-1} \end{smallmatrix}\right), \zeta'\right)$, where $t, t' \in \mathcal{O}^\times(1 + \mathfrak{p}^l)$, $s, s' \in \mathfrak{p}^l$ and $\zeta, \zeta' \in \mu_n$, denote arbitrary elements of $\mathtt{B}_l$.

A function $f \in \mathcal{H}_{i,i}$ has support on the identity coset $\mathtt{B}_l$ if and only if $f(\mathtt{b}) = \chi_i(\mathtt{b})$, for all $\mathtt{b} \in \mathtt{B}_l$. So there is always a function with support on the identity coset, namely $f = \chi_i$.

Next, we consider the coset of $\mathtt{w}$. For $\mathtt{b}$ and $\mathtt{b}'$ in $\mathtt{B}_l$, $\mathtt{b}\mathtt{w}\mathtt{b}' = \mathtt{w}$ implies, via a quick calculation, that $\mathbf{b} = \mathbf{b}' = \mathrm{dg}(t)$, for some $t \in \mathcal{O}^\times(1 + \mathfrak{p}^l)$ and $\zeta' = \zeta^{-1}$. Therefore, $\chi_i(\mathtt{b}\mathtt{b}') = \chi_i((\mathrm{dg}(t), \zeta)(\mathrm{dg}(t), \zeta^{-1})) = \chi_i(\mathrm{dg}(t^2), (t, t)_n) = \chi_i(\mathrm{dg}(t^2), 1)$. So, $\mathcal{H}_{i,i}$ contains a function with support on this coset if and only if $\chi_i(\iota(t^2)) = 1$ for all $t \in \mathcal{O}^\times(1 + \mathfrak{p}^l)$; that is if and only if $\chi_i|_{(\mathtt{T}_l)^2} = 1$. Observe that for $0 \le i < \underline{n}$, $\chi_i|_{(\mathtt{T}_l)^2} = 1$, where $l \ge m$, if and only if $\chi_i|_{\mathcal{O}^{\times 2}} = 1$. Suppose that $\chi_i|_{\mathcal{O}^{\times 2}} = 1$ for some $0 \le i < \underline{n}$. We show that in this case, $m = 1$. Suppose that $\alpha \in 1 + \mathfrak{p}$, and consider $f(X) = X^2 - \alpha$. Observe that $f(1) = 0 \mod \mathfrak{p}$, and $f'(1) = 2(1) \neq 0 \mod p$. By Hensel's lemma,

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$f(X)$ has a root in $\mathcal{O}$; that is, $\alpha \in \mathcal{O}^{\times 2}$. Therefore, $1+\mathfrak{p} \subset \mathcal{O}^{\times 2}$, which implies $\chi_i|_{\mathrm{T}\cap K_1} = 1$, so $m = 1$.

Finally, for $\mathtt{b}$ and $\mathtt{b}'$ in B_l, $\mathtt{b}\mathrm{lt}(x\varpi^r)\mathtt{b}' = \mathrm{lt}(x\varpi^r)$ implies that $tt' \in 1+\mathfrak{p}^r$ and $\zeta = \zeta'^{-1}$. Therefore,

$$\chi_i(\mathtt{bb}') = \chi_i(\mathbf{bb}', 1) = \chi_i\left(\begin{pmatrix} tt' & ts' + st'^{-1} \\ 0 & t^{-1}t'^{-1} \end{pmatrix}, 1\right).$$

Note that $\left(\begin{smallmatrix} tt' & ts'+st'^{-1} \\ 0 & t^{-1}t'^{-1} \end{smallmatrix}\right) \in \mathrm{B}\cap K_r$. Hence, $\chi_i(\mathtt{bb}') = 1$ if and only if $\mathrm{B}\cap K_r \subseteq \ker(\chi_i)$. The latter holds if and only if $r \geq m$, since χ_i is primitive mod m. Now, let us summarize our result. There is always one function with support on the identity coset, and $2(l-m)$ functions on cosets represented by $\mathrm{lt}(x\varpi^r)$, $x \in \{1, \varepsilon\}$, $m \leq r < l$. If $\chi_i\big|_{\mathcal{O}^{\times 2}} \neq 1$, no function in $\mathcal{H}_{i,i}$ has support on the double coset represented by $\mathtt{w}$; otherwise, there exists an additional function in $\mathcal{H}_{i,i}$ with support on the double coset represented by $\mathtt{w}$. ∎

We will also calculate the $\dim \mathcal{H}_{k,i}$, when $i \neq k$, in Proposition 4.10. The next two lemmas elaborate on the condition $\chi_i|_{\mathcal{O}^{\times 2}} = 1$ that appears in Proposition 4.5.

Lemma 4.6 *For each $0 \leq i < \underline{n}$, $\chi_i|_{\mathcal{O}^{\times 2}} = 1$ if and only if $\tau_{0\,\mathcal{O}^{\times 2}} = \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$.*

Proof Let $s \in \mathcal{O}^{\times 2}$. By Lemma 4.1, $\chi_i(s, 0, 1) = \tau_0(s)\vartheta^{2i}(s)$, which is equal to one if and only if $\tau_{0\,\mathcal{O}^{\times 2}} = \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$. ∎

Lemma 4.7 *If $4 \nmid n$, then the characters $\vartheta^{-2i}_{\mathcal{O}^{\times 2}}$, $0 \leq i < \underline{n}$ are distinct. Otherwise, the $\vartheta^{-2i}_{\mathcal{O}^{\times 2}}$, $0 \leq i < n/4$, are distinct, and for $\frac{n}{4} \leq i < \frac{n}{2}$, $\vartheta^{-2i}_{\mathcal{O}^{\times 2}} = \vartheta^{-2(i-\frac{n}{4})}_{\mathcal{O}^{\times 2}}$.*

Proof By definition of ϑ in (4.1), $\vartheta^{-2i}(s) = 1$ for all $s \in \mathcal{O}^{\times 2}$ if and only if $\overline{t^2}^{(q-1)2i/n} = 1$ for all $t \in \mathcal{O}^{\times}$, or equivalently when $n|4i$. Therefore, the equality holds only for $i = 0$ unless $4|n$, in which case the equality holds for both $i = 0$ and $i = \frac{n}{4}$. ∎

For $0 \leq i < \underline{n}$, set $\mathtt{V}_i := \mathrm{Ind}^{\mathrm{K}}_{\mathrm{B}\cap\mathrm{K}}\,\chi_i$. Moreover, for $l > m$, let the $\mathtt{W}_{i,l}$ denote the level-l representations $\mathtt{V}_i^{K_l}/\mathtt{V}_i^{K_{l-1}}$.

Proposition 4.8 *Assume $l \geq m$. We can decompose $\mathrm{Res}_{\mathrm{K}}\,\mathrm{Ind}^{\mathrm{G}}_{\mathrm{B}}\,\rho$ as follows:*

$$\mathrm{Res}_{\mathrm{K}}\,\mathrm{Ind}^{\mathrm{G}}_{\mathrm{B}}\,\rho \cong \bigoplus_{i=0}^{\underline{n}-1}\Big(\mathtt{V}_i^{K_m} \oplus \bigoplus_{l>m}(\mathtt{W}^+_{i,l} \oplus \mathtt{W}^-_{i,l})\Big),$$

where $\mathtt{W}^+_{i,l} \oplus \mathtt{W}^-_{i,l} \cong \mathtt{W}_{i,l}$. All the pieces are irreducible, except when $m = 1$ and $\chi_0|_{\mathcal{O}^{\times 2}} = \vartheta^{-2i}_{\mathcal{O}^{\times 2}}$ for some $0 \leq i < \underline{n}$, in which case, we are in one of the following situations:

(i) *If $4 \nmid n$, then there is exactly one $0 \leq i < \underline{n}$ for which $\mathtt{V}_i^{K_1}$ decomposes into two irreducible constituents. All other constituents are irreducible.*

(ii) *If $4|n$, then there are exactly two $0 \leq i, k < \underline{n}$, $|i-k| = \frac{n}{4}$ for which $\mathtt{V}_i^{K_1}$ decomposes into two irreducible constituents. All other constituents are irreducible.*

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Proof It follows from Lemma 4.3 and Proposition 4.5 that for $l > m$, $\dim \operatorname{Hom}(\mathtt{W}_{i,l}, \mathtt{W}_{i,l}) = 2$. Hence, $\mathtt{W}_{i,l}$ decomposes into two inequivalent irreducible subrepresentations. Moreover,

$$\dim \operatorname{Hom}(\mathtt{V}_i^{K_m}, \mathtt{V}_i^{K_m}) = \begin{cases} 1 & \text{if } \chi_i\big|_{\mathcal{O}^{\times 2}} \neq 1, \\ 2 & \text{otherwise.} \end{cases}$$

It follows from Lemma 4.6 that $\mathtt{V}_i^{K_m}$ is irreducible except when $m = 1$ and $\chi_0|_{\mathcal{O}^{\times 2}} = \vartheta_{\mathcal{O}^{\times 2}}^{-2i}$, where it decomposes into two irreducible constituents. If the latter is the case, situations (i) and (ii) follow from Lemma 4.7. ∎

The characters χ_i for which $\chi_i|_{\mathcal{O}^{\times 2}} = 1$ are exactly of the form $\mathbf{1}\otimes\nu_s\otimes\epsilon$ or $\operatorname{sgn}\otimes\nu_s\otimes\epsilon$ for some $s \in \mathbb{C}$. Proposition 4.8 tells us that when $4\nmid n$, only one of these two choices occurs among χ_i's. Whereas, when $4|n$, both characters $\mathbf{1}\otimes\nu_s\otimes\epsilon$ and $\operatorname{sgn}\otimes\nu_s\otimes\epsilon$ occur among χ_i's.

Note that in the light of the isomorphism (3.1), the $\mathtt{K}$-spaces $\mathtt{V}_i^{K_1}$ can be though of as K_1-invariant representations of $K\times\mu_n$, or equivalent as representations of $K/K_1\times\mu_n \cong \mathrm{SL}_2(\kappa)\times\mu_n$. Conversely, one can lift every genuine representation of $\mathrm{SL}_2(\kappa)\times\mu_n$ to a K_1-invariant representation of $\mathtt{K}$, by letting K_1 act trivially. Let $\widetilde{\mathbf{St}}$ denote such a lift of $\mathbf{St}\otimes\epsilon$, where $\mathbf{St}$ is the q-dimensional Steinberg representation of the finite group of Lie type $\mathrm{SL}_2(\kappa)$.

Lemma 4.9 *If $\chi_i = \mathbf{1}\otimes\nu_s\otimes\epsilon$, then $\mathtt{V}_i^{K_1} \cong \widetilde{\mathbf{1}}\oplus\widetilde{\mathbf{St}}$. If $\chi_i = \operatorname{sgn}\otimes\nu_s\otimes\epsilon$, then $\mathtt{V}_i^{K_1} \cong \Xi^+\oplus\Xi^-$, where $\Xi^\pm$ are two inequivalent irreducible constituents of the same degree.*

Proof The result follows from observing that $\mathtt{V}_i^{K_1} \cong (\operatorname{Ind}_{B\cap K}^{K}\tau_i)^{K_1}\otimes\epsilon$, and identifying $(\operatorname{Ind}_{B\cap K}^{K}\tau_i)^{K_1}$ with the corresponding representation of the finite group of Lie type $\mathrm{SL}_2(\kappa)$, whose representation theory is well understood and can be found in [7]. ∎

Next, we determine the multiplicity of each constituent in the decomposition in Proposition 4.8. To do so, we count the dimension of $\operatorname{Hom}_{\mathtt{K}}(\operatorname{Ind}_{\mathtt{B}_l}^{\mathtt{K}}\chi_k, \operatorname{Ind}_{\mathtt{B}_l}^{\mathtt{K}}\chi_i)$, which is equal to the dimension of the Hecke algebra $\mathcal{H}_{k,i} = \mathcal{H}(\mathtt{B}_l\backslash\mathtt{K}/\mathtt{B}_l, \chi_k, \chi_i)$.

Proposition 4.10 *Let $l \geq m$, $0 \leq k, i < \underline{n}$, and $i \neq k$. Then*

$$\dim\mathcal{H}_{k,i} = \begin{cases} 2l-1 & \textit{if } \tau_{0\,\mathcal{O}^{\times 2}} = \vartheta_{\mathcal{O}^{\times 2}}^{-(k+i)}, \\ 2(l-m) & \textit{otherwise.} \end{cases}$$

Proof Similar to the proof of Proposition 4.5, we determine which double cosets in $\mathtt{B}_l\backslash\mathtt{K}/\mathtt{B}_l$ support a function in $\mathcal{H}_{k,i}$. For every double coset representative $\mathtt{x}$ in (4.3), there exists a function $f \in \mathcal{H}_{k,i}$ with support on the double coset represented by $\mathtt{x}$ if and only if $\mathtt{bxb}' = \mathtt{x}$, $\mathtt{b}, \mathtt{b}' \in \mathtt{B}_l$, implies that $\chi_k(\mathtt{b})\chi_i(\mathtt{b}') = 1$. Let $t, t' \in \mathcal{O}^\times(1+\mathfrak{p}^l)$, $s, s' \in \mathfrak{p}^l$, and $\zeta, \zeta' \in \mu_n$, so that

$$\mathtt{b} = (\mathbf{b}, \zeta) = \left(\begin{pmatrix} t & s \\ 0 & t^{-1}\end{pmatrix}, \zeta\right) \quad\text{and}\quad \mathtt{b}' = (\mathbf{b}', \zeta') = \left(\begin{pmatrix} t' & s' \\ 0 & t'^{-1}\end{pmatrix}, \zeta'\right)$$

are arbitrary elements of $\mathtt{B}_l$.

Because $\chi_k \neq \chi_i$, there is no function in $\mathcal{H}_{k,i}$ with support on the identity double coset. For the double coset of $\mathtt{w}$, $\mathtt{bwb}' = \mathtt{w}$ implies that $\mathbf{b} = \mathbf{b}' = \mathrm{dg}(t)$ for some $t \in \mathcal{O}^\times(1+\mathfrak{p}^l)$ and $\zeta' = \zeta^{-1}$. Therefore,

$$\begin{aligned}\chi_k(\mathtt{b})\chi_i(\mathtt{b}') &= \chi_k(t,0,\zeta)\chi_i(t,0,\zeta^{-1}) = \tau_0(t)\vartheta^{2k}(t)\epsilon(\zeta)\tau_0(t)\vartheta^{2i}(t)\epsilon(\zeta^{-1})\\ &= \tau_0(t^2)\vartheta^{2(k+i)}(t) = \tau_0(t^2)\vartheta^{k+i}(t^2).\end{aligned}$$

Therefore, because $l \geq m$, $\chi_k(\mathtt{b})\chi_i(\mathtt{b}') = 1$ if and only if $\tau_{0\,\mathcal{O}^{\times 2}} = \vartheta^{-(k+i)}_{\mathcal{O}^{\times 2}}$. In this case, $m = 1$ and $\mathtt{w}$ supports a function in $\mathcal{H}_{k,i}$. Finally, for the double cosets represented by $\mathrm{lt}(x\varpi^r)$, $x \in \{1,\varepsilon\}$, $1 \leq r < l$, $\mathtt{b}\mathrm{lt}(x\varpi^r)\mathtt{b}' = \mathrm{lt}(x\varpi^r)$ implies that $\zeta' = \zeta^{-1}$, and $t + s\varpi^r = t'^{-1} \mod \mathfrak{p}^l$, or equivalently, $t = t'^{-1} \mod \mathfrak{p}^r$, and $t^{-1}\varpi^r = \varpi^r t'^{-1} \mod \mathfrak{p}^l$, or equivalently $t^{-1} = t'^{-1} \mod \mathfrak{p}^{l-r}$. Observe that, in general, $\chi_k(\mathtt{b})\chi_i(\mathtt{b}')$ is equal to

$$\begin{aligned}\chi_k(t,0,\zeta)\chi_i(t',0,\zeta') &= \tau_0(t)\vartheta^{2k}(t)\epsilon(\zeta)\tau_0(t')\vartheta^{2i}(t')\epsilon(\zeta')\\ &= \tau_0(tt')\vartheta^{2k}(t)\vartheta^{2i}(t')\epsilon(\zeta\zeta').\end{aligned} \tag{4.4}$$

Note that ϑ is primitive mod one. Observe that $r \geq 1$ and $l - r \geq 1$. Therefore, $t = t'^{-1} \mod \mathfrak{p}$ and $t = t' \mod \mathfrak{p}$, which implies that $t = t' = \alpha \mod \mathfrak{p}$ where $\alpha \in \{\pm 1\}$. Hence, $\vartheta^2(t) = \vartheta^2(t') = 1$, and (4.4) simplifies to $\tau_0(tt')\epsilon(\zeta\zeta')$. We are in one of the following situations:

Case 1: Suppose $r \geq m$. Then we have $\zeta' = \zeta^{-1}$, and $t = t'^{-1} \mod \mathfrak{p}^m$; that is $tt' \in 1 + \mathfrak{p}^m$. Hence, $\tau_0(tt')\epsilon(\zeta\zeta') = \tau_0(tt') = 1$, because χ_0, and hence τ_0, is primitive mod m. Therefore, in this case, there is always a function in $\mathcal{H}_{k,i}$ with support on these double cosets.

Case 2: Suppose $r < m$. Then $\zeta' = \zeta^{-1}$, so $\tau_0(tt')\epsilon(\zeta\zeta') = \tau_0(tt')$, which equals one if and only if $tt' \in 1 + \mathfrak{p}^m$, which is not the case in general. Hence, in this case, there is no function in $\mathcal{H}_{k,i}$ with support on these double cosets.

To summarize the result, the coset represented by $\mathtt{w}$ supports a function in $\mathcal{H}_{k,i}$ if and only if $\tau_{0\,\mathcal{O}^{\times 2}} = \vartheta^{-(k+i)}_{\mathcal{O}^{\times 2}}$. If $r \geq m$, then the cosets represented by $\mathrm{lt}(x\varpi^r)$ support a function in $\mathcal{H}_{k,i}$. Otherwise, there is no function in $\mathcal{H}_{k,i}$ with support on these double cosets. ∎

Corollary 4.11 *Assume the decomposition of* $\mathrm{Res}_{\mathtt{K}}\,\mathrm{Ind}_{\mathtt{B}}^{\mathtt{G}}\,\rho$ *given in Proposition 4.8.*

(i) *For each* $0 \leq i < \underline{n}$ *and* $l > m$, *there exists a way of decomposing* $\mathtt{W}_{i,l}$ *as* $\mathtt{W}^+_{i,l} \oplus \mathtt{W}^-_{i,l}$ *such that for* $l > m$, $\mathtt{W}^+_{i,l} \cong \mathtt{W}^+_{j,l}$ *and* $\mathtt{W}^-_{i,l} \cong \mathtt{W}^-_{j,l}$ *for all* $0 \leq i, j < \underline{n}$.

(ii) *For* $l = m$, $\{\mathtt{V}_i^{K_m} \mid 0 \leq i < \underline{n}\}$ *consists of mutually inequivalent representations, except when* $m = 1$ *and* $\tau_0|_{\mathcal{O}^{\times 2}} = \vartheta^{-j}_{\mathcal{O}^{\times 2}}$, *for some* $0 \leq j < \underline{n}$, *where* $\mathtt{V}_i^{K_1} \cong \mathtt{V}_k^{K_1}$, *exactly when* $i + k \equiv j \mod \underline{n}$.

Proof It follows from Proposition 4.10 that for $l > m$, $\dim \mathrm{Hom}_{\mathtt{K}}(\mathtt{W}_{i,l}, \mathtt{W}_{k,l}) = 2$, and when $i + k \equiv j \mod \underline{n}$

$$\dim \mathrm{Hom}_{\mathtt{K}}(\mathtt{V}_i^{K_m}, \mathtt{V}_k^{K_m}) = \begin{cases} 1 & \text{if } \tau_{0\,\mathcal{O}^{\times 2}} = \vartheta^{-j}_{\mathcal{O}^{\times 2}},\\ 0 & \text{otherwise,}\end{cases}$$

and hence the result. ∎

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In order to further investigate the irreducible spaces $\mathtt{W}^+_{i,l}$ and $\mathtt{W}^-_{i,l}$, we will show that $\mathtt{W}_{i,l}$, $0 \le i < \underline{n}$, is the restriction to $\mathtt{K}$ of an irreducible representation of the maximal compact subgroup $\underline{\mathtt{K}}$ of the covering group $\underline{\mathtt{G}}$ of $\mathrm{GL}_2(F)$.

5 Branching Rules for $\underline{\mathtt{G}}$

We define the genuine principal series representations of $\underline{\mathtt{G}}$ similarly, by starting with a genuine smooth irreducible representation ρ' of $\underline{\mathtt{T}}$ with the central character χ', which is constructed via the Stone–von Neumann theorem. Observe that $\dim \rho' = [\underline{\mathtt{T}}:\underline{\mathtt{A}}] = n^2$. Then, after extending ρ' trivially over $\underline{N}$, the genuine principal series representation π' of $\underline{\mathtt{G}}$ is $\mathrm{Ind}_{\underline{\mathtt{B}}}^{\underline{\mathtt{G}}}\rho'$. Applying a similar machinery as in Section 4, we obtain the K-type decomposition for $\mathrm{Res}_{\underline{\mathtt{K}}}\,\pi'$. Since the argument in Section 4 goes through almost exactly, here we only overview the main steps and point out the differences. For detailed calculations, see [13].

Similar to Lemma 4.1, it follows that $\mathrm{Res}_{\underline{\mathtt{A}}}\,\rho' \cong \bigoplus_{i,j=0}^{n-1}\chi'_{i,j}$, where the $\chi'_{i,j}$ denote n^2 distinct characters of $\underline{\mathtt{A}}$, defined by

$$\chi'_{i,j}\big(\,\mathrm{dg}(a\varpi^{nu}, b\varpi^{nv}), \zeta\big) = \chi'_0\big(\,\mathrm{dg}(a\varpi^{nu}, b\varpi^{nv}), \zeta\big)\,\vartheta^{-j}(a)\vartheta^{-i}(b),$$

where $a, b \in \mathcal{O}^\times$, $u, v \in \mathbb{Z}$ and $\zeta \in \mu_n$ and $\vartheta(a) = \epsilon((\varpi, a)_n)$ was defined in (4.1), and χ'_0 is a fixed extension of χ' to $\underline{\mathtt{A}}$. The $\chi'_{i,j}$ remain distinct when restricted to $\underline{\mathtt{T}} \cap \underline{\mathtt{K}}$, and again writing $\chi'_{i,j}$ for there restrictions, $\mathrm{Res}_{\underline{\mathtt{T}}\cap\underline{\mathtt{K}}}\,\rho' \cong \bigoplus_{i,j=0}^{n-1}\chi'_{i,j}$. Then similar to Proposition 4.2, we have $\mathrm{Res}_{\underline{\mathtt{K}}}(\mathrm{Ind}_{\underline{\mathtt{B}}}^{\underline{\mathtt{G}}}\rho') \cong \bigoplus_{i,j=0}^{n-1}\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\chi'_{i,j}$, which reduces the problem of decomposing the K-type to the one of decomposing each $\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\chi'_{i,j}$, which by smoothness can be written as the union of its $\underline{K}_l$, $l \ge 1$, fixed points.

Suppose χ' is primitive mod m. It follows that the $\chi'_{i,j}$ are also primitive mod m. Set $\underline{\mathtt{B}}_l = (\underline{\mathtt{B}} \cap \underline{\mathtt{K}})\underline{K}_l$. It can be seen that each level l representation $(\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\chi'_{i,j})^{\underline{K}_l} = \mathrm{Ind}_{\underline{\mathtt{B}}_l}^{\underline{\mathtt{K}}}\chi'_{i,j}$ if $l \ge m$, and is zero if $l < m$. Similar to Proposition 4.4, one can see that

$$\dim \mathrm{Hom}_{\underline{\mathtt{K}}}(\mathrm{Ind}_{\underline{\mathtt{B}}_l}^{\underline{\mathtt{K}}}\chi'_{i,j}, \mathrm{Ind}_{\underline{\mathtt{B}}_l}^{\underline{\mathtt{K}}}\chi'_{i,j}) = \dim \mathcal{H}'_{i,j}(\underline{\mathtt{B}}_l\backslash\underline{\mathtt{K}}/\underline{\mathtt{B}}_l, \chi'_{i,j}, \chi'_{i,j}).$$

To count the dimension of $\mathcal{H}'_{i,j}$, we need to calculate a set of double coset representatives of $\underline{\mathtt{B}}_l$ in $\underline{\mathtt{K}}$.

Lemma 5.1 *A complete set of double coset representatives of $\underline{\mathtt{B}}_l$ in $\underline{\mathtt{K}}$ is given by*

$$\big\{\,(\mathrm{I}_2, 1), \mathtt{w}, \mathrm{lt}(\varpi^r) \mid 1 \le r < l\big\}.$$

Proof Note that this set is a subset of the set S in (4.3). Observe that under the isomorphism

$$F^\times \ltimes \mathtt{G} \cong \underline{\mathtt{G}}, \quad \big(y, (\mathbf{g}, \zeta)\big) \longmapsto \big(\,\mathrm{dg}(1, y)\mathbf{g}, \zeta\big), \tag{5.1}$$

$\mathcal{O}^\times \times \mathtt{K}$ maps to $\underline{\mathtt{K}}$ and $\mathcal{O}^\times \times \mathtt{B}_l$ maps to $\underline{\mathtt{B}}_l$. For every $\mathtt{k}' \in \underline{\mathtt{K}}$, let $(y, \mathtt{k})$ be the inverse image of $\mathtt{k}'$ under the isomorphism (5.1), and let $\mathtt{b}_1, \mathtt{b}_2 \in \mathtt{B}_l$ be such that $\mathtt{b}_1\mathtt{x}\mathtt{b}_2 = \mathtt{k}$, for some $\mathtt{x} \in S$. Let $\mathtt{b}'_1$ and $\mathtt{b}'_2$ be the image of $(y, \mathtt{b}_1)$ and $(y, \mathtt{b}_2)$ under (5.1), respectively. It follows from the multiplication of $F^\times \ltimes \mathtt{G}$ and the isomorphism map (5.1), that $\mathtt{b}'_1\mathtt{x}\mathtt{b}'_2 =$

$\mathtt{k}'$. Thus, $\underline{\mathtt{K}} = \bigcup_{\mathtt{x}\in S} \underline{\mathtt{B}}_l \mathtt{x} \underline{\mathtt{B}}_l$. A short calculation shows that

$$\big(\operatorname{dg}(\varepsilon^{-1},1),1\big)\operatorname{lt}(\varpi^r)\big(\operatorname{dg}(\varepsilon,1),1\big) = \big(\operatorname{lt}(\varepsilon\varpi^r),(\varpi^r,\varepsilon)_n(\varepsilon,\varpi^r)_n\big) = \operatorname{lt}(\varepsilon\varpi^r),$$

where ε is a fixed non-square and $1 \le r < l$. It is not difficult to see that other cosets of S remain distinct in $\underline{\mathtt{K}}$. ■

The following proposition can be proved similarly to Proposition 4.5.

Proposition 5.2 *Let* $l \ge m$. *Then*

$$\dim \mathcal{H}'_{i,j} = \begin{cases} 1 + (l-m) & \text{if } \chi'_{i,j}\big|_{T\cap K} \ne 1, \\ 2 + (l-m) & \text{otherwise.} \end{cases}$$

It follows from the definition of $\chi'_{i,j}$ that for $0 \le i, j < n$, $\chi'_{i,j}|_{T\cap K} = 1$ if and only if $\chi'_{0,0}|_{T\cap K} = \vartheta^{j-i}$. For $l > m$, let $\mathtt{W}'_{i,j,l}$ denote the level-l quotient representation $(\operatorname{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}} \chi'_{i,j})^{\underline{K}_l} / (\operatorname{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}} \chi'_{i,j})^{\underline{K}_{l-1}}$. The K-type decomposition $\operatorname{Res}_{\underline{\mathtt{K}}}(\operatorname{Ind}_{\underline{\mathtt{B}}}^{\underline{\mathtt{G}}} \rho')$ is given in the following corollary.

Corollary 5.3 *We can decompose* $\operatorname{Res}_{\underline{\mathtt{K}}}(\operatorname{Ind}_{\underline{\mathtt{B}}}^{\underline{\mathtt{G}}} \rho')$ *as follows:*

$$\operatorname{Res}_{\underline{\mathtt{K}}}(\operatorname{Ind}_{\underline{\mathtt{B}}}^{\underline{\mathtt{G}}} \rho') \cong \bigoplus_{i,j=0}^{n-1} \Big((\operatorname{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}} \chi'_{i,j})^{\underline{K}_m} \oplus \bigoplus_{l>m} \mathtt{W}'_{i,j,l} \Big).$$

If $\chi'_{0,0}|_{T\cap K} \ne \vartheta^k|_{\mathcal{O}^\times}$ *for all* $0 \le k < n$, *then all the pieces are irreducible. Otherwise, there are exactly* n *pairs* (i,j), $0 \le i, j < n$, *such that* $j - i \equiv k \mod n$, *and* $(\operatorname{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}} \chi'_{i,j})^{\underline{K}_m}$ *decomposes into two irreducible constituents. The rest of the constituents are irreducible.*

Proof It follows from Proposition 5.2, and the fact that the kernel of the map $(i,j) \to j - i \mod n$ is of size n. ■

5.1 Restriction of $\operatorname{Ind}_{\underline{\mathtt{B}}}^{\underline{\mathtt{G}}} \rho'$ to $\mathtt{K}$

Fix a genuine irreducible representation ρ of $\mathtt{T}$ with central character χ, where χ is primitive mod m. Let $\mathtt{W}_{k,l}$, $\mathtt{W}^+_{k,l}$, and $\mathtt{W}^-_{k,l}$ be the representations of $\mathtt{K}$ that appear in the K-type decomposition of $\operatorname{Res}_{\mathtt{K}} \operatorname{Ind}_{\mathtt{B}}^{\mathtt{G}} \rho$ in Proposition 4.8. In this section, we show that, for each $0 \le k < \underline{n}$, $\mathtt{W}_{k,l} \cong \operatorname{Res}_{\mathtt{K}} W'$, where W' is some irreducible representation of $\underline{\mathtt{K}}$. We deduce that $\mathtt{W}^+_{k,l}$ and $\mathtt{W}^-_{k,l}$ have the same dimension.

Let ρ' be a genuine irreducible representation of $\underline{\mathtt{T}}$ with central character χ' such that depth of χ' is equal to depth of χ, and that ρ appears in $\operatorname{Res}_{\mathtt{T}} \rho'$. Let $\chi'_{i,j}$, $0 \le i, j < n$ be all possible extensions of χ' to $\underline{\mathtt{A}}$. To find W', we consider the restriction of the principal series representation $\operatorname{Ind}_{\underline{\mathtt{B}}}^{\underline{\mathtt{G}}} \rho'$ to $\mathtt{K}$. Because the structure of $\mathtt{T}$ depends on the parity of n, we consider the cases for even and odd n separately.

5.1.1 n Odd

Observe that when n is odd, $Z(\underline{\mathtt{T}}) \cap \mathtt{T} = Z(\mathtt{T})$ and $\underline{\mathtt{A}} \cap \mathtt{T} = \mathtt{A}$.

We compute $\operatorname{Res}_{\mathrm{K}} \operatorname{Res}_{\underline{\mathrm{K}}} \operatorname{Ind}_{\underline{\mathrm{B}}}^{\underline{\mathrm{G}}} \rho'$, where the decomposition of $\operatorname{Res}_{\underline{\mathrm{K}}} \operatorname{Ind}_{\underline{\mathrm{B}}}^{\underline{\mathrm{G}}} \rho'$ is given in Corollary 5.3. The assumption that ρ appears in $\operatorname{Res}_{\mathrm{T}} \rho'$ implies $\chi'|_{Z(\mathrm{T})} = \chi$. We further assume that the choice of χ_0 is such that $\operatorname{Res}_A \chi'_0 = \chi_0$. So that, for $0 \le i, j < n$, and $k \in \{0, \cdots, n-1\}$ such that $k \equiv \frac{i-j}{2} \mod n$, we have $\operatorname{Res}_{\mathrm{T} \cap \mathrm{K}} \chi'_{i,j} = \chi_k$. Note for each k, there are exactly n distinct characters $\chi'_{i,j}$ of $\underline{\mathrm{T}} \cap \underline{\mathrm{K}}$ that restrict to χ_k on $\mathrm{T} \cap \mathrm{K}$.

Lemma 5.4 *Assume n is odd. Let i, j, and k be in $\{0, \dots, n-1\}$ such that $\chi'_{i,j}|_{\mathrm{T}\cap\mathrm{K}} = \chi_k$. Then, for all $l \ge m$,*

$$\operatorname{Res}_{\mathrm{K}} \big(\operatorname{Ind}_{\underline{\mathrm{B}} \cap \underline{\mathrm{K}}}^{\underline{\mathrm{K}}} \chi'_{i,j} \big)^{K_l} \cong \big(\operatorname{Ind}_{\mathrm{B} \cap \mathrm{K}}^{\mathrm{K}} \chi_k \big)^{K_l}.$$

Proof It is enough to show that $\operatorname{Res}_{\mathrm{K}} \operatorname{Ind}_{\underline{\mathrm{B}}_l}^{\underline{\mathrm{K}}} \chi'_{i,j} \cong \operatorname{Ind}_{\mathrm{B}_l}^{\mathrm{K}} \chi_k$, which follows from the Mackey theory, and the choice of i, j, and k. ∎

5.1.2 n Even

Observe that for even n,

$$Z(\underline{\mathrm{T}}) \cap \mathrm{T} \cong F^{\times n} \times \mu_n \subset F^{\times \underline{n}} \times \mu_n \quad \text{and} \quad \underline{\mathrm{A}} \cap \mathrm{T} \cong \mathcal{O}^\times \times n\mathbb{Z} \times \mu_n \subset \mathcal{O}^\times \times \underline{n}\mathbb{Z} \times \mu_n.$$

Unlike the case for odd n, the centre $Z(\underline{\mathrm{T}})$ and the maximal abelian subgroup $\underline{\mathrm{A}}$ of $\underline{\mathrm{T}}$ do not restrict to those of T upon restriction to T. In fact,

$$[Z(\mathrm{T}) : Z(\underline{\mathrm{T}}) \cap \mathrm{T}] = 4, \quad [\mathrm{A} : \underline{\mathrm{A}} \cap \mathrm{T}] = 2.$$

This mismatch makes the computation of $\operatorname{Res}_{\mathrm{K}} \operatorname{Ind}_{\underline{\mathrm{B}}}^{\underline{\mathrm{G}}} \rho'$ more delicate. Indeed, our assumption that ρ appears in $\operatorname{Res}_{\mathrm{T}} \rho'$ does not imply that ρ' is ρ isotypic, upon restriction to T. We show that ρ is one of the four distinct irreducible representations of T that appear in $\operatorname{Res}_{\mathrm{T}} \rho'$.

Set $\underline{\chi} := \operatorname{Res}_{Z(\underline{\mathrm{T}}) \cap \mathrm{T}} \chi'$. Note that $|\underline{n}\mathbb{Z}/n\mathbb{Z}| = |\mathcal{O}^{\times \underline{n}}/\mathcal{O}^{\times n}| = 2$. We denote the coset representatives of the former by $\{e, o\}$. Let L denote the set of coset representatives for $Z(\mathrm{T})/(Z(\underline{\mathrm{T}}) \cap \mathrm{T})$, so $|L| = 4$. The representation $\operatorname{Ind}_{Z(\underline{\mathrm{T}}) \cap \mathrm{T}}^{Z(\mathrm{T})} \underline{\chi}$ decomposes into four distinct characters ${}_\ell\chi$:

$$\operatorname{Ind}_{Z(\underline{\mathrm{T}}) \cap \mathrm{T}}^{Z(\mathrm{T})} \underline{\chi} = \bigoplus_{\ell \in L} {}_\ell\chi. \tag{5.2}$$

We denote the irreducible genuine representation of T with central character ${}_\ell\chi$ by ρ_ℓ.

Proposition 5.5 *Assume n is even. Let ${}_\ell\chi$, $\ell \in L$ be as in (5.2). Then $\operatorname{Res}_{\mathrm{T}} \rho' = \bigoplus_{\ell \in L} [(\rho_\ell)^{\oplus n/2}]$, where ρ_ℓ are mutually inequivalent and $\rho \cong \rho_\ell$ for some $\ell \in L$.*

Proof Note that $X = \{(\operatorname{dg}(1, \varpi^j), 1) \mid 0 \le j < n\}$ is a system of coset representatives for $\mathrm{T} \backslash \underline{\mathrm{T}} / \underline{\mathrm{A}}$, and that $\underline{\mathrm{A}}$ is stable under conjugation by $\mathrm{x} \in X$. Moreover, it is not difficult to see that for $\mathrm{x} = (\operatorname{dg}(1, \varpi^j), 1)$, $\chi_0'^{\mathrm{x}} = \chi'_{0,j}$. Therefore, by Mackey theory,

$$\operatorname{Res}_{\mathrm{T}} \rho' = \bigoplus_{\mathrm{x} \in X} \big(\operatorname{Ind}_{(\mathrm{T} \cap \underline{\mathrm{A}}^{\mathrm{x}})}^{\mathrm{T}} \chi_0'^{\mathrm{x}} \big) = \bigoplus_{j=0}^{n-1} \operatorname{Ind}_{\mathrm{A}}^{\mathrm{T}} \big(\operatorname{Ind}_{\mathrm{T} \cap \underline{\mathrm{A}}}^{\mathrm{A}} \chi'_{0,j} \big).$$

Observe that $[\mathtt{A}:\mathtt{T}\cap\underline{\mathtt{A}}]=2$, with coset representatives $\{e,o\}$. Therefore, for every $0\le j<n$, $\mathrm{Ind}_{\mathtt{T}\cap\underline{\mathtt{A}}}^{\mathtt{A}}\chi'_{0,j}$ is a two-dimensional representation of the abelian group $\mathtt{A}$ and hence decomposes into direct sum of two characters: ${}_e\chi'_j\oplus{}_o\chi'_j$.

Note that for $0\le j<n$, $\mathrm{Res}_{\mathtt{T}\cap\underline{\mathtt{A}}}\mathrm{Ind}_{\mathtt{T}\cap\underline{\mathtt{A}}}^{\mathtt{A}}\chi'_{0,j}\cong\chi'_{0,j}\oplus\chi'_{0,j}$. Suppose $0\le i,j<n$, by Frobenius reciprocity

$$\begin{aligned}\mathrm{Hom}_{\mathtt{A}}\big(\mathrm{Ind}_{\mathtt{T}\cap\underline{\mathtt{A}}}^{\mathtt{A}}\chi'_{0,j},\mathrm{Ind}_{\mathtt{T}\cap\underline{\mathtt{A}}}^{\mathtt{A}}\chi'_{0,i}\big)&=\mathrm{Hom}_{\mathtt{T}\cap\underline{\mathtt{A}}}\big(\mathrm{Res}_{\mathtt{T}\cap\underline{\mathtt{A}}}\mathrm{Ind}_{\mathtt{T}\cap\underline{\mathtt{A}}}^{\mathtt{A}}\chi'_{0,j},\chi'_{0,i}\big)\\&=\mathrm{Hom}_{\mathtt{T}\cap\underline{\mathtt{A}}}(\chi'_{0,j}\oplus\chi'_{0,j},\chi'_{0,i}).\end{aligned}$$

We can easily see that $\chi'_{0,j}$ and $\chi'_{0,i}$ coincide on $\mathtt{T}\cap\underline{\mathtt{A}}$ if and only if $i=j$. Whence,

$$\dim\mathrm{Hom}_{\mathtt{A}}\big(\mathrm{Ind}_{\mathtt{T}\cap\underline{\mathtt{A}}}^{\mathtt{A}}\chi'_{0,j},\mathrm{Ind}_{\mathtt{T}\cap\underline{\mathtt{A}}}^{\mathtt{A}}\chi'_{0,i}\big)=\begin{cases}2&\text{if } i=j,\\0&\text{otherwise.}\end{cases}$$

Therefore, the elements of $\{{}_e\chi'_j,{}_o\chi'_j\mid 0\le j<n\}$ are $2n$ distinct characters of $\mathtt{A}$, which, because $[\mathtt{A}:Z(\mathtt{T})]=n/2$, implies that they restrict to at least four distinct characters upon restriction to $Z(\mathtt{T})$. Moreover, because ρ appears in $\mathrm{Res}_{\mathtt{T}}\rho'$, at least one of these four central characters is χ. Observe that for $0\le j<n$, and $\alpha\in\{e,o\}$, $\mathrm{Res}_{Z(\underline{\mathtt{T}})\cap\mathtt{T}}\,{}_\alpha\chi'_j=\underline{\chi}$.

Consider

$$\mathrm{Ind}_{Z(\underline{\mathtt{T}})\cap\mathtt{T}}^{\mathtt{A}}\underline{\chi}=\mathrm{Ind}_{Z(\mathtt{T})}^{\mathtt{A}}\mathrm{Ind}_{Z(\underline{\mathtt{T}})\cap\mathtt{T}}^{Z(\mathtt{T})}\underline{\chi}=\mathrm{Ind}_{Z(\mathtt{T})}^{\mathtt{A}}\bigoplus_{\ell\in L}{}_\ell\chi=\bigoplus_{\ell\in L,0\le k<n/2}{}_\ell\chi_k.$$

Observe that the ${}_\ell\chi_k$ are $2n$ distinct characters that restrict to $\underline{\chi}$ on $Z(\underline{\mathtt{T}})\cap\mathtt{T}$ and exhaust every such character. Hence, the sets $\{{}_e\chi'_{0,j},{}_o\chi'_{0,j}\mid 0\le j<n\}$ and $\{{}_\ell\chi_k\mid\ell\in L,0\le k<n/2\}$ are equal. In particular,

$$\mathrm{Res}_{\mathtt{T}}\rho'\cong\mathrm{Ind}_{\mathtt{A}}^{\mathtt{T}}\bigoplus_{0\le j<n}{}_e\chi'_j\oplus{}_o\chi'_j=\mathrm{Ind}_{\mathtt{A}}^{\mathtt{T}}\Big(\bigoplus_{\ell\in L,0\le k<n/2}{}_\ell\chi_k\Big)\cong\bigoplus_{\ell\in L}\rho_\ell^{\oplus\frac{n}{2}}.\qquad\blacksquare$$

We compute $\mathrm{Res}_{\mathtt{K}}\mathrm{Res}_{\underline{\mathtt{K}}}\mathrm{Ind}_{\underline{\mathtt{B}}}^{\mathtt{G}}\rho'$. First, we need to study $\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\chi'_{i,j}$. Note that

$$\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\chi'_{i,j}\big(\mathrm{dg}(t),\zeta\big)=\chi'_{0,0}\big(\mathrm{dg}(t),\zeta\big)\,\vartheta^{i-j}(t),$$

for all $(\mathrm{dg}(t),\zeta)\in\mathtt{T}\cap\mathtt{K}$. Therefore, $\{\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\chi'_{i,j}\mid 0\le i,j<n\}$ consists of n distinct characters of $\mathtt{T}\cap\mathtt{K}$. In the next lemma and proposition, we realize these characters as characters of $\mathtt{T}\cap\mathtt{K}$ that come from central characters ${}_\ell\chi$, $\ell\in L$, of $Z(\mathtt{T})$.

Lemma 5.6 *Each* $\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\chi'_{i,j}$ *appears exactly twice in* $\bigoplus_{\ell\in L,0\le k<\frac{n}{2}}{}_\ell\chi_k$.

Proof Note that $\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\mathrm{Ind}_{Z(\underline{\mathtt{T}})\cap\mathtt{T}}^{\mathtt{A}}\underline{\chi}=\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}(\bigoplus_{\ell\in L,0\le k<\frac{n}{2}}{}_\ell\chi_k)$. Consider

$$\mathrm{Hom}_{\mathtt{T}\cap\mathtt{K}}\big(\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\chi'_{i,j},\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\mathrm{Ind}_{Z(\underline{\mathtt{T}})\cap\mathtt{T}}^{\mathtt{A}}\underline{\chi}\big).\tag{5.3}$$

Observe that $\mathtt{T}\cap\mathtt{K}\backslash\mathtt{A}/Z(\underline{\mathtt{T}})\cap\mathtt{T}\cong\underline{n}\mathbb{Z}/n\mathbb{Z}$. So, by Mackey theory and Frobenius reciprocity, (5.3) is

$$\mathrm{Hom}_{\mathtt{T}\cap\mathtt{K}}\big(\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\chi'_{i,j},(\mathrm{Ind}_{Z(\underline{\mathtt{T}})\cap\mathtt{K}}^{\mathtt{T}\cap\mathtt{K}}\underline{\chi})^{\oplus2}\big)\cong\mathrm{Hom}_{Z(\underline{\mathtt{T}})\cap\mathtt{K}}(\mathrm{Res}_{Z(\underline{\mathtt{T}})\cap\mathtt{K}}\chi'_{i,j},\mathrm{Res}_{Z(\underline{\mathtt{T}})\cap\mathtt{K}}\underline{\chi}^{\oplus2}).$$

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Because $\mathrm{Res}_{Z(\mathtt{T})}\,\chi' = \underline{\chi}$, for all $0 \le i, j < n$, $\mathrm{Res}_{Z(\underline{\mathtt{T}})\cap\mathtt{K}}\,\chi'_{i,j} = \mathrm{Res}_{Z(\underline{\mathtt{T}})\cap\mathtt{K}}\,\underline{\chi}$, and hence, (5.3) is two-dimensional, which shows that $\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\,\chi'_{i,j}$ appears exactly twice in $\bigoplus_{\ell\in L, 0\le k<\underline{n}}\, {}_{\ell}\chi_k$. ■

It is easy to see that $\{\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\,\chi'_{0,j} \mid 0 \le j < n\}$ consists of n distinct characters of $\mathtt{T}\cap\mathtt{K}$, each appearing exactly twice in $\bigoplus_{\ell\in L,0\le k<\underline{n}}\, {}_{\ell}\chi_k$ by Lemma 5.6. By a simple counting argument, we deduce that for every $0 \le k < \underline{n}$ and $\ell \in L$, there exists a $0 \le j < n$, such that $\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\,\chi'_{0,j} = {}_{\ell}\chi_k$. Similar to Lemma 5.4, we see that, for n even, if $0 \le j < n$, $0 \le k < \underline{n}$ and $\ell \in L$ are such that $\mathrm{Res}_{\mathtt{T}\cap\mathtt{K}}\,\chi'_{0,j} = {}_{\ell}\chi_k$, then, for all $l \ge m$,

$$(\mathrm{Ind}_{\mathtt{B}\cap\mathtt{K}}^{\mathtt{K}}\, {}_{\ell}\chi_k)^{K_l} \cong \mathrm{Res}_{\mathtt{K}}(\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\,\chi'_{0,j})^{\underline{K_l}}.$$

The following proposition and corollary sum up the results in this section.

Proposition 5.7 *Let ρ and ρ' be irreducible representations of $\mathtt{T}$ and $\underline{\mathtt{T}}$ with central characters χ and χ', primitive mod m, respectively, such that ρ appears in $\mathrm{Res}_{\mathtt{T}}\,\rho'$. For $l > m$, $0 \le k < \underline{n}$, $0 \le i, j < n$, let $\mathtt{W}_{k,l} = \mathtt{W}^-_{k,l} \oplus \mathtt{W}^+_{k,l}$ and $\mathtt{W}'_{i,j,l}$ be the quotient spaces that appear in the decompositions in Proposition 4.8 and Corollary 5.3, respectively. Then, for each $0 \le k < \underline{n}$, $l > m$, $\mathtt{W}_{k,l} = \mathrm{Res}_{\mathtt{K}}\,\mathtt{W}'_{i,j,l}$, for some $0 \le i, j < n$.*

Proof If n is odd, it follows from Lemma 5.4 that for a given k and l there exists $0 \le i, j < n$ such that $(\mathrm{Ind}_{\mathtt{B}\cap\mathtt{K}}^{\mathtt{K}}\,\chi_k)^{K_l} \cong \mathrm{Res}_{\mathtt{K}}(\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\,\chi'_{i,j})^{\underline{K_l}}$. Without loss of generality, we can assume $i = 0$. For n even, it follows from Proposition 5.5 that $\chi = {}_{\ell}\chi$ for some $\ell \in L$, where ${}_{\ell}\chi$ are defined in (5.2). It is a consequence of Lemma 5.6 that, for a given k and l, there exists $0 \le j < n$ such that $(\mathrm{Ind}_{\mathtt{B}\cap\mathtt{K}}^{\mathtt{K}}\, {}_{\ell}\chi_k)^{K_l} \cong \mathrm{Res}_{\mathtt{K}}(\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\,\chi'_{0,j})^{\underline{K_l}}$. Consider $\mathtt{W}'_{0,j,l} = (\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\,\chi'_{0,j})^{\underline{K_l}} / (\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\,\chi'_{0,j})^{\underline{K_{l-1}}}$.

Observe that

$$\begin{aligned}\mathrm{Res}_{\mathtt{K}}\,\mathtt{W}'_{0,j,l} &= \mathrm{Res}_{\mathtt{K}}\left[(\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\,\chi'_{0,j})^{\underline{K_l}}\right] / \mathrm{Res}_{\mathtt{K}}\left[(\mathrm{Ind}_{\underline{\mathtt{B}}\cap\underline{\mathtt{K}}}^{\underline{\mathtt{K}}}\,\chi'_{0,j})^{\underline{K_{l-1}}}\right]\\ &= (\mathrm{Ind}_{\mathtt{B}\cap\mathtt{K}}^{\mathtt{K}}\,\chi_k)^{K_l} / (\mathrm{Ind}_{\mathtt{B}\cap\mathtt{K}}^{\mathtt{K}}\,\chi_k)^{K_{l-1}} = \mathtt{W}^-_{k,l} \oplus \mathtt{W}^+_{k,l}.\end{aligned}$$ ■

Corollary 5.8 *The inequivalent irreducible representations $\mathtt{W}^-_{k,l}$ and $\mathtt{W}^+_{k,l}$, $0 \le k < \underline{n}$, $l > m$, that appear in the K-type decomposition $\mathrm{Res}_{\mathtt{K}}\,\mathrm{Ind}_{\mathtt{B}}^{\mathtt{G}}\,\rho$ in Proposition 4.8 are of the same dimension.*

Proof By Proposition 5.7, for any $0 \le k < \underline{n}$, $l > m$, $\mathtt{W}_{k,l} = \mathtt{W}^-_{k,l} \oplus \mathtt{W}^+_{k,l}$, is restriction of some irreducible representation $\mathtt{W}'_{i,j}$ of $\underline{\mathtt{K}}$, for some $0 \le i, j < n$. Hence, there exists an element of $\underline{\mathtt{K}} \setminus \mathtt{K}$ that maps $\mathtt{W}^-_{k,l}$ to $\mathtt{W}^+_{k,l}$ bijectively. ■

Remark 5.9 Note that the isomorphism (3.1) implies that, for any $0 \le k < \underline{n}$, $l > m$, $\mathtt{W}_{k,l}$ is isomorphic to a representation of K, of the same dimension as $\mathtt{W}_{k,l}$, tensored with the faithful character ϵ. The dimension of $\mathtt{W}_{k,l}$ can be calculated directly. One can then explicitly identify these representations using the classification of irreducible representations of K given by Shalika [30] as done in [23, Sec.5].

6 Main Result

Finally, we put all of our results together to obtain the main result of this paper.

Theorem 6.1 *Let ρ be a genuine irreducible representation of $\mathtt{T}$ with central character χ, primitive mod m, and let χ_k, $0 \le k < \underline{n}$, be all the possible extensions of χ to $\mathtt{A}$. Then*

$$\operatorname{Res}_{\mathtt{K}} \operatorname{Ind}_{\mathtt{B}}^{\mathtt{G}} \rho \cong \bigoplus_{k=0}^{\underline{n}-1} (\mathtt{V}_k^{K_m}) \oplus \bigoplus_{l>m} (\mathtt{W}_{0,l}^{+} \oplus \mathtt{W}_{0,l}^{-})^{\oplus \underline{n}},$$

where $\mathtt{V}_k = \operatorname{Ind}_{\mathtt{B}\cap\mathtt{K}}^{\mathtt{K}} \chi_k$, and $\mathtt{W}_{0,l}^{+}$ and $\mathtt{W}_{0,l}^{-}$ are two inequivalent irreducible representations of $\mathtt{K}$ with the same dimension, and $(\mathtt{W}_{0,l}^{+} \oplus \mathtt{W}_{0,l}^{-}) \cong \mathtt{V}_0^{K_l}/\mathtt{V}_0^{K_{l-1}}$. The level-$m$ representations $\mathtt{V}_k^{K_m}$, where $0 \le k < \underline{n}$, are irreducible and mutually inequivalent, except when $m = 1$ and for some k, $\chi_k|_{\mathcal{O}^\times}$ is a quadratic character.

Proof The decomposition and irreducibility results follow from Proposition 4.8. The multiplicity results follow from Corollary 4.11 and the fact that $\mathtt{W}_{0,l}^{+}$ and $\mathtt{W}_{0,l}^{-}$ have the same degree is proved in Corollary 5.8. ■

The next corollary states what happens when $\chi_k|_{\mathcal{O}^\times}$ is a quadratic character for some $0 \le k < \underline{n}$. Up to relabelling, we can assume that $\chi_0|_{\mathcal{O}^\times}$ is a quadratic character. In this case, indeed, both the irreducibility and being multiplicity-free fails.

Corollary 6.2 *In the setting of Theorem 6.1, suppose that $m = 1$, and $\chi_0|_{\mathcal{O}^\times}$ is a quadratic character. Then we are in one of the following situations:*

(i) *If $4 \nmid n$, then $\mathtt{V}_k^{K_1}$ is reducible if and only if $k = 0$, in which case, $\mathtt{V}_0^{K_1} \cong \widetilde{\mathbf{1}} \oplus \widetilde{\mathbf{St}}$ if $\chi_0|_{\mathcal{O}^\times} = \mathbf{1}$, and $\mathtt{V}_0^{K_1} \cong \Xi^{+} \oplus \Xi^{-}$ otherwise. Moreover, $\mathtt{V}_i^{K_1} \cong \mathtt{V}_k^{K_1}$ exactly when $i + k = \underline{n}$.*

(ii) *If $4 \mid n$, then $\mathtt{V}_k^{K_1}$ is reducible exactly when $k = 0$ or $k = \frac{n}{4}$, in which case, $\mathtt{V}_0^{K_1} \oplus \mathtt{V}_{n/4}^{K_1} \cong \widetilde{\mathbf{1}} \oplus \widetilde{\mathbf{St}} \oplus \Xi^{+} \oplus \Xi^{-}$. Moreover, $\mathtt{V}_i^{K_1} \cong \mathtt{V}_k^{K_1}$ exactly when $i + k = \underline{n}$.*

Proof It follows from Proposition 4.8, Lemma 4.9, and Corollary 4.11. ■

Remark 6.3 The fact that when $4 \mid n$, all four of $\widetilde{\mathbf{1}}$, $\widetilde{\mathbf{St}}$, Ξ^{+} and Ξ^{-} appear in the top piece of the K-type decomposition is curious; its roots lies in the fact that in this case, $\mathbf{1}$ and sgn are both characters of $\mathcal{O}^\times/\mathcal{O}^{\times \underline{n}}$.

Remark 6.4 Most of the information about the principal series representation is encoded in the level-m piece. The tail piece in Theorem 6.1 (the level-l pieces for $l > m$) does not see the twists of the extension of the central character to $\mathtt{A}$ by characters of $\mathcal{O}^\times/\mathcal{O}^{\times \underline{n}}$, and hence fails to see the non-triviality of the cover; thus, it behaves exactly like its counterpart in the K-type decomposition of the linear group. In particular, for $l \ge 2m$ the tail only depends on $\tau_0(-1)$ ([23, Proposition 4.5]).

Remark 6.5 If one can generalizes the local Langlands correspondence to the covering groups, representations of the covering analogue of the Weil–Deligne group are associated with L-packets of representations of $\mathtt{G}$. The analogy with the linear case suggests that the restriction to the inertia subgroup of these representations is parametrized, up to isomorphism, by the Bernstein components. Inertial Langlands correspondence relates such representations to representations of $\mathtt{K}$. To establish such a correspondence, one needs to understand the Bernstein support of a representation of $\mathtt{G}$ from its restriction to $\mathtt{K}$. To that end, we are interested in classifying all irreducible representations of $\mathtt{K}$ whose occurrence in $\mathrm{Res}_{\mathtt{K}}\,\pi$, for a representation π of $\mathtt{G}$, guarantees a given cuspidal support; such representations are called typical representations. Our result in this paper suggests that the irreducible pieces in the top level of the K-type decomposition are typical.

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