ON IMMERSION OF MANIFOLDS

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1. Introduction. In **(3)** R. Lashof and S. Smale proved among other things the following theorem. *If the compact oriented manifold M is immersed into the oriented manifold M', with* dim $M' \geqslant$ dim $M + 2$, *then the normal degree of the immersion is equal to the Euler-Poincaré characteristic χ of M reduced module the characteristic χ' of M'. If M' is not compact, χ' is replaced by* 0. "Manifold" always means C^∞-manifold. An immersion is a differentiable (that is, C^∞) map f whose differential df is non-singular throughout. The normal degree is defined in a certain fashion using the normal bundle of M in M', derived from f, and injecting it into the tangent bundle of M'.

It is our purpose to give an elementary proof, using vector fields, of this theorem, and at the same time to identify the homology class that represents the normal degree (Theorem I), and to give a proof, using the theory of Morse, for the special case M' = Euclidean space (Theorem II). The proof of Theorem II consists of a slight addition to arguments due to Chern and Lashof **(1; 2)**. We introduce some notation. If x is a point of M, we write M_x for the tangent space of M at x; if g is a (C^∞) map of M, we write dg for the differential of g, that is, the induced map of the tangent vectors; and we write g_* for the induced map of the homology group $H_*(M)$ (homology is always meant as singular homology with integral coefficients); these conventions apply to all manifolds. Let T' be the bundle of non-zero tangent vectors of M', and let S' be the direction sphere of T' at some point $q \in M'$, that is, the unit sphere of M_q', with respect to some Euclidean metric in M_q'. Let s' denote the element of the homology group $H_*(T')$, respresented by the basic cycle of S' with the given orientation of M'.

Given an auxiliary Riemannian metric in a neighbourhood of $f(M)$ in M', the normal bundle B_ν of M under f consists of all pairs (x, v) where $x \in M$, and v is a unit tangent vector of M' at $f(x)$, orthogonal to $df(M_x)$. We write ν for the map of B_ν into T', given by $(x, v) \to v$; this is the normal map or Gauss map. The manifold B_ν receives a definite orientation from the orientations of M and M'; let b_ν be the corresponding basic homology class; the dimension of B_ν is equal to that of S'. The image of b_ν in $H_*(T')$ under the homology map ν_* induced by ν is called the normal degree of f. Our result then takes the following form.

THEOREM I. *The normal degree of f is $\chi \,.\, s'$ (in $H_*(T')$).*

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It is well known that in case of compact M' the element s' is of order χ' in $H_*(T')$; this is the theorem that the sum of the indices of a vector field equals the characteristic. On the other hand, if M' is not compact, then s' generates an infinite cyclic group in $H_*(T')$. The theorem therefore allows us to identify the normal degree with the integer χ reduced mod χ' (resp. mod 0); this is the result of Lashof and Smale. The proof of Theorem I appears in sections 2 and 3 below.

2. A construction in vector bundles. We begin with the prototype of immersion. Let M be as above a compact oriented C^∞ manifold, and let E be an oriented C^∞ vector bundle over M, with fibre a vector space of some (finite) dimension; we write p for the projection $E \to M$. We introduce in E an auxiliary Riemannian metric which on each fiber is translation invariant, so that each fibre carries a Euclidean metric. Without loss of generality we may assume that the 0-section of E, which we identify with M, is orthogonal to all the fibres. The metric defines the unit ball bundle A and the unit sphere bundle B: a point y of E belongs to A (resp. B) if the norm $|y|$ of y, computed in the fibre of y, is $\leqslant 1$ (resp. $=1$). A is a bounded manifold, with B as boundary, with natural orientations induced from E. On B we consider a vector field N, the "exterior normal" which assigns to each point y of B the tangent of the curve (line) $\{ty\colon t \in \text{reals}\}$ at y ,that is, for $t = 1$. We propose to extend N over all of A with a single singularity. To this end we choose in M ($=$0-section of E) a vector field F with a single singularity, that is, a continuous vector field that vanishes at a single point x. We extend F to a vector field F_1 on A by defining $F_1(y)$ as the vector orthogonal to the fibre of y, that projects into $F(p(y))$, multiplied by $1 - |y|$; we note explicitly that F_1 vanishes on B. As a second step we extend N to a vector field F_2 on A by using exactly the definition of N: $F_2(y)$ is the derivative vector (*not* the unit tangent vector) of the curve $\{ty\colon t \in \text{reals}\}$ for $t = 1$. Then $F_2(y)$ is always tangent to the fibre of y and vanishes for $|y| = 0$, that is, on M.

We now define a vector field G on A by $G = F_1 + F_2$, meaning $G(y) = F_1(y) + F_2(y)$ for $y \in A$. One verifies immediately from the properties given above that G is an extension of N, and that G vanishes only at the point x.

The main point of our argument is now the following contention.

(1) *The indices of the vector fields F and G at their singular point x are equal.*

This is purely a local matter. We take a neighbourhood V of x in M, homeomorphic to Euclidean space, with x corresponding to the origin. We may assume that in terms of this Euclidean structure the directions of the vectors of the field F are constant along rays from the origin; if necessary we deform F a bit. From the local product structure of the bundle E and the definition $G = F_1 + F_2$ one verifies then that the map of a sphere around x in E into the unit sphere S of the tangent space E_x, derived from G and defining the index of G at x, is homotopic to the join of the two corresponding maps for

F_1 and F_2. The degree for maps of spheres behaves multiplicatively under forming joins, and the index of F_2 is clearly equal to 1. This proves (1).

The index of F at x is well known to equal the characteristic χ of M. We shall interpret all this in homology language:

Let T_A denote the restriction to A of the bundle of all non-zero tangent vectors of E (we could use unit vectors instead). Let s denote the element of $H_*(T_A)$ represented by the basic cycle on the positively oriented unit sphere S in E_x, and let b denote the basic homology class of B in the given orientation: both s and b are of infinite order. The vector field N is a map of B into T_A, in fact a section over B.

Then the statement about the index of G can be phrased as follows:

$$N_*(b) = \chi \,.\, s \text{ (in } H_*(T_A)). \tag{2}$$

To prove this we change our point of view of the field G. Instead of having it vanish at x, we deform it so that it is radially constant near x. We construct the bounded orientable manifold $\tilde{A}$ obtained from A by replacing x by S, with the topology so defined that a neighbourhood of a unit vector v at x consists of v and all points of $A - \{x\}$ near x in a cone around v (cf. the construction of $\mathscr{T}$ in **(3)**. The boundary of $\tilde{A}$ is $B - S$ in the usual notation. The vector field G defines a map of $\tilde{A}$ into T_A, coinciding with N on B, and mapping S into itself with degree equal to the index of G at x. This clearly proves (2).

Next we note that there is a natural map I of B into T_A, mapping y into the *unit* vector tangent to $\{ty\}$ for $t = 0$; and the two maps I and N are homotopic (the image vector sliding on the ray of y from the origin to y). From (2) we get

$$I_*(b) = \chi \,.\, s \text{ in } H_*(T_A). \tag{3}$$

Finally let T_0 be the restriction of T_A to M; clearly T_0 is a strong deformation retract of T_A, by "radial contraction"; from $I(B) \subset T_0$ and (3) we get

$$I_*(b) = \chi \,.\, s \text{ in } H_*(T_0), \tag{4}$$

with the obvious meaning of s.

3. Application to immersion. Suppose now M is immersed into the manifold M' by a map f, as in the introduction ($\dim M' > \dim M$). In addition to the notation and concepts already defined there we consider the normal (vector) bundle E_ν, consisting of *all* pairs (x, v) with $x \in M$ and $v \in M_{f(x)}'$, orthogonal to $df(M_x)$; we use the metric of $M_{f(x)}'$ for v. We apply to E_ν the considerations of (2), using a subscript ν where applicable (thus B_ν is the normal unit sphere bundle, etc.).

Let h be the exponential map of E_ν into M', constructed by means of the Riemannian metric in M'; if the metric is defined only in a neighbourhood of $f(M)$, then h is defined in a neighbourhood of M in E_ν. The differential of h

maps T_0 (the tangent vectors of E_ν at points of M) into T' in a *non-degenerate* fashion. This implies $dh_*(s_\nu) = s'$. Further, the composition of the map I, defined above, and dh is just the normal map ν. Applying dh_* to (4) we obtain:

$$\nu_*(b_\nu) = \chi \, . \, s' \text{ in } H_*(T'),$$

thus proving Theorem *I*.

It should be noted that Theorem I, as stated, holds also in the case $\dim M' = \dim M + 1$; but in this case the customary concept of normal degree, in particular in the case M' = Euclidean space, is somewhat different. The reason is that B_ν here consists of *two* copies of M.

4. Immersion in Euclidean space. Suppose we have again the situation of Theorem I, but that now M' is a Euclidean space E^k, with a fixed orientation and Euclidean metric. We keep the same notation, but make use of the usual identification of E^k with its various tangent spaces. E_ν is the normal bundle, the pairs (x, v) with $x \in M$ and $v \in E^k$, orthogonal to the subspace $df(M_x)$. Requiring $|v| = 1$ we get B_ν. The map $(x, v) \to x$ is the projection p. The map $(x, v) \to v$, still called ν, is now regarded as a map of B_ν into the unit sphere S^{k-1} of E^k. Since orientations are fixed on these two manifolds, the degree of ν is well defined; this is again called the normal degree of f; we write n_f for it now.

THEOREM II. $n_f = \chi$.

Remark. In the case $k = \dim M + 1$ (and odd k) the normal degree, as defined here, is twice the usual normal degree, since B_ν consists of two copies of M, one to either side of M; for even k both integers are 0; the contributions of the two parts of B_ν to n_f cancel out.

For a detailed description of the facts used below see **(1)**, particularly pp. 310–312 and **(2**, p. 8**)**; all we add to the arguments given there is our relation (5).

By Sard's theorem there exists a vector $v_0 \in S^{k-1}$ such that at each point $y \in B_\nu$ with $\nu(y) = v_0$ the differential of the map ν (which maps the tangent space of B_ν at y into the tangent space of S^{k-1} at v_0) is non-degenerate; there are only a finite number of such points y. Let the function ϕ on E be defined by $\phi(v) = v \, . \, v_0$ (inner product of v and v_0), and let ψ be the function $\phi \circ f$, induced on M.

Then the following two statements hold:

(A) *The set D of points y of B_ν with $\nu(y) = v_0$ and the set C of critical points of ψ in M are in one-one correspondence under the projection p.*

(B) *All critical points of ψ are non-degenerate. If $y = (x, v_0)$ is a point of D, then the local degree $d(y)$ of ν at y and the index $j(x)$ of ψ at x are related by*

$$d(y) = (-1)^{n+j(x)}. \qquad (5)$$

To prove (A), we note that x is critical for ψ if $df(M_x)$ is orthogonal to v_0, that is, if among the points $y \in p^{-1}(x)$ there is one with ν-image v_0. The proof of (B) is subtler. Let $y = (x, v_0)$ be a point of D. The determinant $J(y)$ of the differential of ν at y (with respect to the given orientations of B_ν and S^{k-1}) can be interpreted as follows. The matter being local, we restrict to a small neighbourhood V of x in M. Let $L(v_0)$ be the subspace of E^k spanned by $f(M_x)$ and v_0, and oriented accordingly. We follow the map f of V into E^k by the orthogonal projection into $L(v_0)$, obtaining an immersed manifold V', with x' corresponding to x. Then $J(y)$ is the ordinary Gauss-Kronecker curvature of the hypersurface V' of the Euclidean space $L(v_0)$ at x'; of course v_0 is automatically the positive normal of V' at x'. With respect to a suitable co-ordinate system in $L(v_0)$, with the origin at x' and with v_0 as the last co-ordinate vector, V' is described by a function g, in the form

$$t_{n+1} = g(t_1, \ldots, t_n).$$

It is clear from the construction that g is essentially the function ψ; in detail, if we write f' for the map from V to V', we have $\psi(z) = g(t_1(f'(z)), \ldots t_n(f'(z)))$ for every z in V. Moreover, since $df(M_x)$ is the tangent space to V' at x', the Taylor expansion of g begins with the quadratic terms. The Gauss-Kronecker curvature is $(-1)^n.4$ times the determinant of the quadratic form; the non-degeneracy implies that it is not zero. Let the quadratic form be diagonalized, so that

$$g(t_1, \ldots, t_n) = -\lambda_1 t_1^2 - \ldots - \lambda_j t_j^2 + \lambda_{j+1} t_{j+1}^2 + \ldots + \lambda_n t_n^2 + \ldots$$

with all the $\lambda_i > 0$. Then the curvature is $(-1)^{n+j}\, 4\, \lambda_1 \ldots \lambda_n$; this is then also $J(y)$. The degree $d(y)$ of ν at y is the sign of $J(y)$:

$$d(y) = (-1)^{n+j}. \tag{6}$$

On the other hand, since g is just ψ, the above form of g shows that the critical point x of ψ is non-degenerate and that its index is j:

$$j(x) = j. \tag{7}$$

Together, (6) and (7) prove (B).

8. We apply the theory of Morse **(4)**. The sum $\sum (-1)^{j(x)}$, extended over all critical points of ψ is the alternating sum of the type numbers, and therefore equal to the characteristic χ of M. By (A) and (B) we have then $\chi = \sum (-1)^n d(y) = (-1)^n \cdot \sum d(y)$, where the sum extends over all points y with $\nu(y) = v_0$; but by well-known principles the sum $\sum d(y)$ is exactly the degree of the map ν, that is, the normal degree n_f of f. We have then

$$\chi = (-1)^n\, n_f.$$

This proves Theorem II, since for odd n one knows that $\chi = 0$.

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