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Equivariant Tamagawa Numbers, Fitting Ideals and Iwasawa Theory

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Abstract. Let L/K be a finite Galois extension of number fields of group G. In [4] the second named author used complexes arising from étale cohomology of the constant sheaf $\mathbb{Z}$ to define a canonical element $T\Omega(L/K)$ of the relative algebraic K-group $K_0(\mathbb{Z}[G], \mathbb{R})$. It was shown that the Stark and Strong Stark Conjectures for L/K can be reinterpreted in terms of $T\Omega(L/K)$, and that the Equivariant Tamagawa Number Conjecture for the $\mathbb{Q}[G]$-equivariant motive h^0(Spec L) is equivalent to the vanishing of $T\Omega(L/K)$. In this paper we give a natural description of $T\Omega(L/K)$ in terms of finite G-modules and also, when G is Abelian, in terms of (first) Fitting ideals. By combining this description with techniques of Iwasawa theory we prove that $T\Omega(L/\mathbb{Q})$ vanishes for an interesting class of Abelian extensions $L/\mathbb{Q}$.

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1. Introduction

In the seminal paper [1] Bloch and Kato give a conjectural formula for the leading coefficient (up to sign) in the Laurent expansion at $s = 0$ of L-functions associated to certain motives defined over number fields in terms of motivic realisations and motivic cohomology groups. This 'Tamagawa Number Conjecture' of Bloch and Kato has been reformulated and extended by Fontaine and Perrin-Riou in [13] and by Kato in [18, 19] to the setting of mixed motives with commutative coefficients. In [5–7] Flach and the second named author extended and reworked the conjectures of [1, 13, 18, 19] to make clearer the consequences concerning equivariant structure of lattices in the associated de Rham, Betti, and motivic cohomology spaces. In particular, if L/K is a finite Abelian extension of number fields, then it was shown that by considering h^0(Spec L) as a motive defined over K and with coefficients $\mathbb{Q}[\mathrm{Gal}(L/K)]$ one obtains in this way a considerable strengthening of the conjecture formulated by Chinburg in [10]. In [3] the 'Equivariant Tamagawa Number Conjecture' of [7] is generalized to the case of motives which are defined over number fields and have coefficients which are not necessarily commutative.

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Now let L/K be any finite Galois extension of number fields of group G. For any integral domain R and any field extension E of the quotient field of R, let $K_0(R[G], E)$ denote the Grothendieck group of the fibre category of the functor $-\otimes_R E$ from the category of finitely generated projective left $R[G]$-modules to the category of finitely generated left $E[G]$-spaces.

In [4] the second named author used complexes arising from étale cohomology with compact support of the constant sheaf $\mathbb{Z}$ on open subshemes of Spec $\mathcal{O}_L$ to define a canonical element $T\Omega(L/K)$ of $K_0(\mathbb{Z}[G], \mathbb{R})$. It was shown that the Stark, respectively Strong Stark, Conjecture for L/K is equivalent to asserting that $T\Omega(L/K)$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$, respectively to the torsion subgroup of $K_0(\mathbb{Z}[G], \mathbb{Q})$, and in addition that the central conjecture of [3] is for the $\mathbb{Q}[G]$-equivariant motive $h^0(\text{Spec } L)$ equivalent to the equality $T\Omega(L/K) = 0$.

In this paper we give a natural description of $T\Omega(L/K)$ in terms of finite G-modules and also, when G is Abelian, in terms of (first) Fitting ideals. We then restrict exclusively to the case that G is Abelian, and show how this description of $T\Omega(L/K)$ is amenable to investigation using Iwasawa theory. By means of an explicit example, we combine this approach with refinements of arguments used in [7] and [16] and prove the following result.

For each element $x \in K_0(\mathbb{Z}[G], \mathbb{Q})$ and each prime p we write x_p for the component of x in $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p)$ according to the canonical decomposition

$$K_0(\mathbb{Z}[G], \mathbb{Q}) \simeq \coprod_p K_0(\mathbb{Z}_p[G], \mathbb{Q}_p).$$

For any natural number n we let ζ_n denote a choice of complex primitive n-th root of unity. We write $\mathbb{Q}(n)$ in place of $\mathbb{Q}(\zeta_n)$ and let $\mathbb{Q}(n)^+$ denote its maximal real subfield.

THEOREM 1.1. *Let l_1 and l_2 be distinct odd primes, a and b positive integers, and L any subfield of $\mathbb{Q}(l_1^a)^+\mathbb{Q}(l_2^b)^+$. If neither l_1 or l_2 splits in $\mathbb{Q}(l_1^a)^+\mathbb{Q}(l_2^b)^+/\mathbb{Q}$, then $T\Omega(L/\mathbb{Q})_p = 0$ for each odd prime p. In particular therefore, if $L/\mathbb{Q}$ is also of odd degree, then $T\Omega(L/\mathbb{Q}) = 0$.*

Remarks 1.2. (i) If G is Abelian, then the equality $T\Omega(L/K) = 0$ implies the vanishing of Chinburg's invariant $\Omega(L/K, 3)$ (see also Remark 2.9(iii)). Theorem 1.1 therefore proves that the conjecture formulated by Chinburg in [10] is valid for the extensions $L/\mathbb{Q}$ under consideration.

(ii) In [26] Ritter and Weiss proved the Strong Stark Conjecture for all Abelian extensions $L/\mathbb{Q}$ of odd conductor. Hence one knows that $T\Omega(L/\mathbb{Q})$ belongs to the torsion subgroup of $K_0(\mathbb{Z}[\text{Gal}(L/\mathbb{Q})], \mathbb{Q})$ for all such extensions $L/\mathbb{Q}$ (cf. Proposition 2.1(iv)).

At the moment there are still very few explicit examples in which the equivariant Tamagawa number conjecture (for any motive) has been completely verified, and Theorem 1.1 represents a considerable improvement of what is known for motives of the form $h^0(\mathrm{Spec}\ L)$. In addition, by combining Iwasawa theoretic techniques with a good deal of explicit computation, our proof of Theorem 1.1 provides much new insight into the rather subtle nature of the general conjecture.

2. Tamagawa Numbers and Fitting Ideals

Throughout this paper all modules are considered as left modules.

In this section we describe the equivariant Tamagawa number $T\Omega(L/K)$ of [4] in terms of finite G-modules and also, in the case that G is Abelian, in terms of (first) Fitting ideals. To do this we must first quickly review the Euler characteristic construction introduced in [3]. For further details the reader is referred to [3] or to the review given in [4].

Let R be an integral domain and E a field extension of the quotient field of R. Let Γ be any finite group such that $E[\Gamma]$ is semisimple, and write $K_0(R[\Gamma], E)$ for the Grothendieck group of the fibre category of the functor $-\otimes_R E$ from the category of finitely generated projective $R[\Gamma]$-modules to the category of finitely generated $E[\Gamma]$-spaces (cf. [30, p. 215]). Recall that this group lies in a long exact sequence of relative K-theory

$$K_1(R[\Gamma]) \longrightarrow K_1(E[\Gamma]) \xrightarrow{\partial_{R[\Gamma],E}} K_0(R[\Gamma], E) \longrightarrow K_0(R[\Gamma]) \longrightarrow K_0(E[\Gamma]) \qquad (1)$$

(cf. *loc. cit.*, Th. 15.5). If $R = \mathbb{Z}$ and $E = \mathbb{R}$, resp. $R = \mathbb{Z}$ and $E = \mathbb{Q}$, resp. $R = \mathbb{Z}_p$ and $E = \mathbb{Q}_p$, then we shall abbreviate $\partial_{R[\Gamma],E}$ to ∂_Γ, resp. $\partial_{\Gamma,\mathbb{Q}}$, resp. $\partial_{\Gamma,p}$.

For any finitely generated $E[\Gamma]$-spaces V and W we write $\mathrm{Is}_{E[\Gamma]}(V, W)$ for the set of $E[\Gamma]$-equivariant isomorphisms from V to W. If $\mathrm{Is}_{E[\Gamma]}(V, W)$ is not empty, then there exists a canonical rank one space $\delta_{E[\Gamma]}(V, W)$ over the centre $\zeta(E[\Gamma])$ of $E[\Gamma]$, and to each $\phi \in \mathrm{Is}_{E[\Gamma]}(V, W)$ there is associated a canonical 'reduced determinant' $\det_{E[\Gamma]}(\phi)$ which belongs to $\delta_{E[\Gamma]}(V, W)$. If Γ is Abelian, then

$$\delta_{E[\Gamma]}(V, W) = \mathrm{Hom}_{E[\Gamma]}(\det_{E[\Gamma]} V, \det_{E[\Gamma]} W)$$

and $\det_{E[\Gamma]}(\phi)$ is equal to the $E[\Gamma]$-determinant of ϕ, but in general both $\delta_{E[\Gamma]}(V, W)$ and $\det_{E[\Gamma]}(\phi)$ are defined via Galois descent. We set

$$\delta^+_{E[\Gamma]}(V, W) := \{\det_{E[\Gamma]}(\phi) : \phi \in \mathrm{Is}_{E[\Gamma]}(V, W)\} \subset \delta_{E[\Gamma]}(V, W)$$

and refer to elements of this subset as 'trivialisations'. If $V = W$, then $\delta_{E[\Gamma]}(V, V)$ naturally identifies with $\zeta(E[\Gamma])$ and, with respect to this identification, $\delta^+_{E[\Gamma]}(V, V)$ is equal to the subset $\zeta(E[\Gamma])^{\times +}$ of $\zeta(E[\Gamma])^\times$ consisting of those elements which are reduced norms of units of the semisimple E-algebra $E[\Gamma]$. Recall that taking reduced norms induces an isomorphism $K_1(E[\Gamma]) \xrightarrow{\sim} \zeta(E[\Gamma])^{\times +}$ (cf. [11, §45A]).

For any R-module X and any ring extension Λ of R we write X_Λ for the associated $\Lambda[\Gamma]$-module $X \otimes_R \Lambda$. If $\phi: X \longrightarrow Y$ is a morphism of $R[\Gamma]$-modules, then we write ϕ_Λ

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for the induced morphism $X_\Lambda \longrightarrow Y_\Lambda$ of $\Lambda[\Gamma]$-modules. If X and Y are finitely generated projective $R[\Gamma]$-modules and $\phi \in \mathrm{Is}_{E[\Gamma]}(X_E, Y_E)$, then the element $[X, \phi, Y]$ of $K_0(R[\Gamma], E)$ depends only upon the trivialisation $\tau = \det_{E[\Gamma]}(\phi)$ (cf. [4, Lem. 1.1.1]), and so will occasionally be written $[X, \tau, Y]$.

For each complex $C^\bullet$ we write C^o, resp. $H^o(C^\bullet)$, for the direct sum of C^i, resp. $H^i(C^\bullet)$, over all odd integers i, and C^e, resp. $H^e(C^\bullet)$, for the direct sum of C^i, resp. $H^i(C^\bullet)$, over all even integers i.

In [3] it is shown that to each pair consisting of a perfect complex of $R[\Gamma]$-modules $C^\bullet$ and an element τ of $\delta^+_{E[\Gamma]}(H^o(C^\bullet)_E, H^e(C^\bullet)_E)$, one can associate a canonical element $\chi_{R[\Gamma],E}(C^\bullet, \tau)$ of $K_0(R[\Gamma], E)$. To be more precise let $P^\bullet$ be a bounded complex of finitely generated projective $R[\Gamma]$-modules and $\alpha\colon P^\bullet \to C^\bullet$ an $R[\Gamma]$-equivariant quasi-isomorphism. Then α induces a bijection

$$\delta^+_{E[\Gamma]}(H^o(C^\bullet)_E, H^e(C^\bullet)_E) \xrightarrow{\sim} \delta^+_{E[\Gamma]}(H^o(P^\bullet)_E, H^e(P^\bullet)_E)$$

and we let τ_α denote the image of τ under this bijection, and choose $\phi \in Is_{E[\Gamma]}(H^o(P^\bullet)_E, H^e(P^\bullet)_E)$ such that $\det_{E[\Gamma]}(\phi) = \tau_\alpha$. In each degree i we let $B^i(P^\bullet)$, resp. $Z^i(P^\bullet)$, denote the module of coboundaries, resp. cocycles, of $P^\bullet$. After choosing in each degree $E[\Gamma]$-splittings of the canonical short exact sequences

$$0 \longrightarrow Z^i(P^\bullet)_E \longrightarrow P^i_E \longrightarrow B^{i+1}(P^\bullet)_E \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow B^i(P^\bullet)_E \longrightarrow Z^i(P^\bullet)_E \longrightarrow H^i(P^\bullet)_E \longrightarrow 0, \tag{3}$$

one obtains a composite isomorphism

$$\begin{aligned} P^o_E &\longrightarrow \bigoplus_{i\in\mathbb{Z}} (B^{2i+1}(P^\bullet)_E \oplus H^{2i+1}(P^\bullet)_E \oplus B^{2i+2}(P^\bullet)_E) \\ &\longrightarrow \left(\bigoplus_{i\in\mathbb{Z}} B^i(P^\bullet)_E\right) \oplus H^o(P^\bullet)_E \\ &\xrightarrow{(1,\phi)} \left(\bigoplus_{i\in\mathbb{Z}} B^i(P^\bullet)_E\right) \oplus H^e(P^\bullet)_E \\ &\longrightarrow \bigoplus_{i\in\mathbb{Z}} (B^{2i}(P^\bullet)_E \oplus H^{2i}(P^\bullet)_E \oplus B^{2i+1}(P^\bullet)_E) \\ &\longrightarrow P^e_E \end{aligned}$$

(where the second and fourth listed isomorphisms are the obvious ones). The reduced determinant $\tau_\alpha(P^\bullet_E)$ of this isomorphism is independent of the chosen splittings of each sequence (2) and (3) and of the precise choice of ϕ and one sets

$$\chi_{R[\Gamma],E}(C^\bullet, \tau) := [P^o, \tau_\alpha(P^\bullet_E), P^e] \in K_0(R[\Gamma], E).$$

It can be shown that this element is indeed independent of the precise choice of $P^\bullet$ and α. In effect, this 'refined Euler characteristic' construction can be used as a natural replacement for the Grothendieck–Knudsen–Mumford determinant functor (which is used systematically in [7]) in the case that Γ is not Abelian.

For any ring Λ, we shall say that a Λ-module is 'perfect' if it is both finitely generated and of finite projective dimension. For any such module N and any integer m we write $N[m]$ for the perfect complex which consists of N placed in degree $-m$ and $\{0\}$ placed in all other degrees. We recall that if R is either $\mathbb{Z}$ or $\mathbb{Z}_p$ for some prime p, then a finitely generated $R[\Gamma]$-module is perfect if and only if its projective dimension is either 0 or 1.

We now let L/K be a finite Galois extension of number fields and set $G := \mathrm{Gal}(L/K)$. For any finite set S of places of K which contains the set S_∞ of archimedean places of K we write $S(L)$ for the set of places of L which lie above the places in S, and write $\mathcal{O}_{L,S}$ for the ring of $S(L)$-integers in L (in the case $S = S_\infty$ we write $\mathcal{O}_L$ rather than $\mathcal{O}_{L,S_\infty}$). We say that any such set S is 'admissible' if it contains all places which ramify in L/K and is also sufficiently large to ensure that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$.

We let Y_S denote the free Abelian group on the set $S(L)$ and write X_S for the kernel of the augmentation map $Y_S \to \mathbb{Z}$. We also let U_S denote the group $\mathcal{O}_{L,S}^\times$ of $S(L)$-units of L. Each of Y_S, X_S and U_S has a natural structure as G-module.

For any admissible set S there exist perfect $\mathbb{Z}[G]$-modules Ψ_S^0 and Ψ_S^1 and an exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow U_S \longrightarrow \Psi_S^0 \xrightarrow{d} \Psi_S^1 \longrightarrow X_S \longrightarrow 0 \tag{4}$$

which represents the canonical element $c_S(L/K)$ of $\mathrm{Ext}_G^2(X_S, U_S)$ defined by Tate in [31]. We let $\Psi_S^\bullet$ denote the complex $\Psi_S^0 \xrightarrow{d} \Psi_S^1$ where the modules are placed in degrees 0 and 1, and the cohomology is computed via the exact sequence (4). It is shown in [6] that the most natural interpretation of $\Psi_S^\bullet$ is in terms of the étale cohomology with compact support of the constant sheaf $\mathbb{Z}$ on Spec $\mathcal{O}_{L,S}$, but we shall use no details of this aspect of the theory here.

For each place w of L we let $|-|_w$ denote the absolute value of w which is normalised as in [32, Chap. 0, 0.2]. We let $R_S\colon U_{S,\mathbb{R}} \longrightarrow X_{S,\mathbb{R}}$ denote the $\mathbb{R}[G]$-equivariant isomorphism given by $R_S(u) = -\sum_{w\in S(L)} \log|u|_w \cdot w$ for each $u \in U_S$.

We let $L_S(s)$ denote the S-truncated L-function which is associated to the motive $h^0(\mathrm{Spec}\, L)$, considered as defined over K and with coefficients $\mathbb{Q}[G]$ (cf. [12, 2.12] or [4, §2] for a more explicit description). We write $L_S^*(0)$ for the leading coefficient in the Laurent expansion of $L_S(s)$ at $s = 0$. Then $L_S^*(0) \in \zeta(\mathbb{R}[G])^\times$ and we choose an element $\lambda \in \zeta(\mathbb{Q}[G])^\times$ such that

$$\lambda \cdot L_S^*(0)^\# \in \zeta(\mathbb{R}[G])^{\times +}, \tag{5}$$

where here # denotes the $\mathbb{R}$-linear involution of $\zeta(\mathbb{R}[G])$ induced by $g \mapsto g^{-1}$ for each $g \in G$. (The existence of such an element λ is guaranteed by the Weak Approximation Theorem.)

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For simplicity we assume henceforth that S is sufficiently large so that each element of $\zeta(\mathbb{R}[G])^{\times +}$ is the reduced norm of a unit of $\mathrm{End}_{\mathbb{R}[G]}(X_{S,\mathbb{R}})$. Then (5) implies that $\tau_S(\lambda) := \lambda L_S^*(0)^{\#} \cdot \det_{\mathbb{R}[G]}(R_S^{-1})$ belongs to $\delta^+_{\mathbb{R}[G]}(X_{S,\mathbb{R}}, U_{S,\mathbb{R}})$. Following [4], we set

$$T\Omega(L/K) := \psi_G^*(\chi_{\mathbb{Z}[G],\mathbb{R}}(\Psi_S^\bullet, \tau_S(\lambda)) - \sum_p \hat{\partial}_{G,p}(\lambda_p)) \in K_0(\mathbb{Z}[G], \mathbb{R}),$$

where here ψ_G^* denotes the involution of $K_0(\mathbb{Z}[G], \mathbb{R})$ which is induced by the linear duality functor $R\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ and, for each prime p, $\hat{\partial}_{G,p}$ denotes the composite morphism $\zeta(\mathbb{Q}_p[G])^\times \xleftarrow{\sim} K_1(\mathbb{Q}_p[G]) \xrightarrow{\partial_{G,p}} K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \subset K_0(\mathbb{Z}[G], \mathbb{Q})$ and λ_p denotes the image of λ in $\zeta(\mathbb{Q}_p[G])^\times$. Note that the summation term in the formula for $T\Omega(L/K)$ makes sense since for almost all p one has $\lambda_p \in \zeta(\mathbb{Z}_p[G])^\times = \ker(\hat{\partial}_{G,p})$.

We now recall some basic properties of $T\Omega(L/K)$. For any subgroup, resp. normal subgroup, H we let

$$\rho_{G,H}: K_0(\mathbb{Z}[G], \mathbb{R}) \longrightarrow K_0(\mathbb{Z}[H], \mathbb{R}),$$
$$\text{resp.}\pi_{G,G/H}: K_0(\mathbb{Z}[G], \mathbb{R}) \longrightarrow K_0(\mathbb{Z}[G/H], \mathbb{R})$$

denote the natural restriction, resp. deflation, homomorphism. Following the usual conventions, we shall refer to the conjecture [32, Ch. I, 5.1] as the 'Stark Conjecture', and to [10, Conj. 2.2] as the 'Strong Stark Conjecture'.

PROPOSITION 2.1. (cf. [4], or [7] if G is Abelian.). (i) *$T\Omega(L/K)$ depends only upon the extension L/K.*

(ii) *For each subgroup H of G one has $\rho_{G,H} T\Omega(L/K) = T\Omega(L/L^H)$. For each normal subgroup H of G one has $\pi_{G,G/H} T\Omega(L/K) = T\Omega(L^H/K)$.*

(iii) *$T\Omega(L/K)$ belongs to $K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if the Stark Conjecture is true for L/K.*

(iv) *$T\Omega(L/K)$ belongs to the torsion subgroup of $K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if the Strong Stark Conjecture is true for L/K.* □

Let $K_0T(\mathbb{Z}[G])$ denote the Grothendieck group of the category of finite perfect $\mathbb{Z}[G]$-modules (with relations given by short exact sequences). Each finite perfect $\mathbb{Z}[G]$-module X has a resolution $0 \to P^{-1} \xrightarrow{\psi} P^0 \to X \to 0$ in which P^{-1} and P^0 are finitely generated projective $\mathbb{Z}[G]$-modules, and the association $X \mapsto [P^{-1}, \psi_{\mathbb{Q}}, P^0]$ induces a well defined isomorphism of Grothendieck groups $t_G: K_0T(\mathbb{Z}[G]) \xrightarrow{\sim} K_0(\mathbb{Z}[G], \mathbb{Q})$. We now use t_G to describe $T\Omega(L/K)$ in terms of finite G-modules.

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Recall first that the standard method of constructing an explicit extension of the form (4) is to take a resolution of X_S

$$0 \longrightarrow X_S(-2) \longrightarrow F^0 \longrightarrow F^1 \longrightarrow X_S \longrightarrow 0, \tag{6}$$

in which F^0 and F^1 are $\mathbb{Z}[G]$-projective, and then consider the pushout of this sequence along a G-morphism $\varphi_S: X_S(-2) \to U_S$ which represents $c_S(L/K)$ when $\mathrm{Ext}^2_G(X_S, U_S)$ is computed by means of (6). We shall use the following two lemmas to refine this construction.

LEMMA 2.2. *For all sufficiently large (admissible) sets S there exists an exact sequence*

$$0 \longrightarrow X_S(-2) \longrightarrow F \xrightarrow{d'} F \longrightarrow X_S \longrightarrow 0 \tag{7}$$

in which F is a finitely generated free $\mathbb{Z}[G]$-module.

Proof. For any (admissible) set S' we take a truncated free resolution

$$0 \longrightarrow \ker(\theta) \xrightarrow{\subseteq} \mathbb{Z}[G]^s \xrightarrow{\theta} \mathbb{Z}[G]^r \xrightarrow{\pi} X_{S'} \longrightarrow 0$$

of $X_{S'}$ and without loss of generality we assume that $s > r$. Then

$$0 \longrightarrow \ker(\theta) \xrightarrow{\subseteq} \mathbb{Z}[G]^s \xrightarrow{(\theta,0)} \mathbb{Z}[G]^r \oplus \mathbb{Z}[G]^{s-r} \xrightarrow{(\pi,\mathrm{id})} X_{S'} \oplus \mathbb{Z}[G]^{s-r} \longrightarrow 0$$

is a truncated free resolution of $X_{S'} \oplus \mathbb{Z}[G]^{s-r}$. But if S'' is any set of $s-r$ places of K which do not belong to S' and are fully split in L/K, then $X_{S'} \oplus \mathbb{Z}[G]^{s-r} \simeq X_S$ with $S = S' \cup S''$. □

Remarks 2.3. (i) Proposition 2.1(i) implies that $T\Omega(L/K)$ is in particular independent of the choice of admissible set S and, hence, it can always be computed by using a resolution of the form (7).

(ii) For any given (admissible) set S a resolution of the form (7) need not exist. However, for comparatively small sets S there often exists such a resolution as a consequence of a 'generation theorem' of Swan (cf. [25, Lem. 7] and [17, 7.3]).

LEMMA 2.4. *(Assuming that S is sufficiently large) fix a resolution of X_S as in (7), and use this to compute* $\mathrm{Ext}^2_G(X_S, U_S)$*. Then for each element $\alpha \in \mathrm{Ext}^2_G(X_S, U_S)$ there exists an injective G-morphism $\varphi: X_S(-2) \to U_S$ which represents α.*

Proof. The resolution (7) implies that the $\mathbb{Q}[G]$-spaces $X_S(-2)_{\mathbb{Q}}$ and $X_{S,\mathbb{Q}}$ are isomorphic, and hence that there exists an injective G-morphism $\psi: X_S(-2) \to U_S$. If now $\varphi: X_S(-2) \to U_S$ is any G-morphism which represents α, then $\varphi + N \cdot \psi$ also represents α if N is any integer which is divisible by $|G|$. If in addition $-N$ is not an eigenvalue of $(\psi^{-1} \circ \varphi) \otimes \mathbb{Q}$, then it follows that $\varphi + N\psi$ is injective. $\square$

Assuming S to be sufficiently large we now fix a resolution of X_S of the form (7) and an injective G-morphism $\varphi_S: X_S(-2) \to U_S$ which represents $c_S(L/K)$, and we consider the following commutative diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & \\
0 \arrow[r] & X_S(-2) \arrow[r, "\iota_S"] \arrow[d, "\varphi_S"'] & F \arrow[r, "d'"] \arrow[d] & F \arrow[r, "\pi"] \arrow[d, "="'] & X_S \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & U_S \arrow[r] \arrow[d] & \Psi_S^0 \arrow[r, "d"] \arrow[d] & \Psi_S^1 \arrow[r] & X_S \arrow[r] & 0 \\
 & \mathrm{cok}(\varphi_S) \arrow[r, "="] \arrow[d] & \mathrm{cok}(\varphi_S) \arrow[d] & & & \\
 & 0 & 0. & & &
\end{tikzcd} \tag{8}$$

In this diagram Ψ_S^0 denotes the pushout of ι_S and φ_S, $\Psi_S^1 = F$, d is the morphism which is induced by d' and the central row is an extension of the form (4). Furthermore, since φ_S represents $c_S(L/K)$ it induces isomorphisms in all dimensions of Tate cohomology, and so $\mathrm{cok}(\varphi_S)$ is both finite and $\mathbb{Z}[G]$-perfect. Let $F^\bullet$ denote the complex $F \xrightarrow{d'} F$ given by the top row of diagram (8), with the modules placed in degrees 0 and 1. Then diagram (8) induces a G-equivariant distinguished triangle of perfect complexes

$$F^\bullet \longrightarrow \Psi_S^\bullet \longrightarrow \mathrm{cok}(\varphi_S)[0]. \tag{9}$$

Since $\mathrm{cok}(\varphi_S)[0]_\mathbb{R}$ is acyclic this triangle induces a quasi-isomorphism $F_\mathbb{R}^\bullet \longrightarrow \Psi_{S,\mathbb{R}}^\bullet$ and, hence, $\tau_S(\lambda)$ induces a trivialisation $\tau_S(\lambda)_{\varphi_S} \in \delta_{\mathbb{R}[G]}^+(H^1(F^\bullet)_\mathbb{R}, H^0(F^\bullet)_\mathbb{R})$. We set $\hat{\tau}_S(\lambda)_{\varphi_S} := \tau_S(\lambda)_{\varphi_S}(F_\mathbb{R}^\bullet) \in \delta_{\mathbb{R}[G]}^+(F_\mathbb{R}, F_\mathbb{R}) \xrightarrow{\sim} \zeta(\mathbb{R}[G])^{\times +}$. More explicitly therefore, if $\phi \in \mathrm{Is}_{\mathbb{R}[G]}(X_{S,\mathbb{R}}, U_{S,\mathbb{R}})$ satisfies $\det_{\mathbb{R}[G]}(\phi) = \tau_S(\lambda)$, then $\hat{\tau}_S(\lambda)_{\varphi_S}$ is equal to the reduced determinant of the composite isomorphism (reading from left to right)

$$F_\mathbb{R} \xleftarrow{(\mu_\mathbb{R}, \subseteq)} X_{S,\mathbb{R}} \oplus B_\mathbb{R} \xrightarrow{(\varphi_{S,\mathbb{R}}^{-1} \circ \phi, \mathrm{id})} X_S(-2)_\mathbb{R} \oplus B_\mathbb{R} \xrightarrow{(\subseteq, \sigma_\mathbb{R})} F_\mathbb{R} \tag{10}$$

where here μ is any choice of $\mathbb{Q}[G]$-equivariant section to the natural projection $F_\mathbb{Q} \to H^1(F^\bullet)_\mathbb{Q} = X_{S,\mathbb{Q}}$, $B := \ker(\pi) = \mathrm{Im}(d')$ and σ is any choice of $\mathbb{Q}[G]$- equivariant section to the differential $d'_\mathbb{Q}: F_\mathbb{Q} \to B_\mathbb{Q}$.

If $E=\mathbb{R}$, resp. $E=\mathbb{Q}$, then we write $\hat{\partial}_G$, resp. $\hat{\partial}_{G,\mathbb{Q}}$, for the composite morphism

$$\zeta(E[G])^{\times +} \xleftarrow{\sim} K_1(E[G]) \xrightarrow{\partial_{G,E}} K_0(\mathbb{Z}[G], E).$$

PROPOSITION 2.5. *Let S be admissible and sufficiently large that X_S has a resolution of the form (7). Fix such a resolution of X_S and let $\varphi_S: X_S(-2) \to U_S$ be an injective G-morphism which represents $c_S(L/K)$ when $\mathrm{Ext}^2_G(X_S, U_S)$ is computed via the chosen resolution. Then $\mathrm{cok}(\varphi_S)$ is a finite perfect $\mathbb{Z}[G]$-module and if λ is any element of $\zeta(\mathbb{Q}[G])^\times$ which satisfies (5), then*

$$\psi^*_G(T\Omega(L/K)) = t_G(\mathrm{cok}(\varphi_S)) + \hat{\partial}_G(\hat{\tau}_S(\lambda)_{\varphi_S}) - \sum_p \hat{\partial}_{G,p}(\lambda_p) \in K_0(\mathbb{Z}[G], \mathbb{R}).$$

Proof. We have already seen that the choice of φ_S ensures that $\mathrm{cok}(\varphi_S)$ is both finite and $\mathbb{Z}[G]$-perfect. In addition, the definition of $\tau_S(\lambda)_{\varphi_S}$ ensures that, in the language of [3, Def. 1.2.6], the triangle (9) underlies a distinguished triangle of perfect trivialised complexes $(F^\bullet, \tau_S(\lambda)_{\varphi_S}) \longrightarrow (\Psi^\bullet_S, \tau_S(\lambda)) \longrightarrow (\mathrm{cok}(\varphi_S)[0], 1)$, and so [3, Th. 1.2.7] implies that

$$\chi_{\mathbb{Z}[G],\mathbb{R}}(\Psi^\bullet_S, \tau_S(\lambda)) = \chi_{\mathbb{Z}[G],\mathbb{R}}(F^\bullet, \tau_S(\lambda)_{\varphi_S}) + \chi_{\mathbb{Z}[G],\mathbb{R}}(\mathrm{cok}(\varphi_S)[0], 1). \tag{11}$$

Now if $\phi \in \mathrm{Aut}_{\mathbb{R}[G]}(F_\mathbb{R})$ satisfies $\det_{\mathbb{R}[G]}(\phi) = \hat{\tau}_S(\lambda)_{\varphi_S}$, then

$$\begin{aligned}\chi_{\mathbb{Z}[G],\mathbb{R}}(F^\bullet, \tau_S(\lambda)_{\varphi_S}) &= [F, \phi, F]\\ &= \partial_G([F_\mathbb{R}, \phi])\\ &= \hat{\partial}_G(\hat{\tau}_S(\lambda)_{\varphi_S}),\end{aligned}$$

and for any finite perfect $\mathbb{Z}[G]$-module N one has $t_G(N) = \chi_{\mathbb{Z}[G],\mathbb{R}}(N[0], 1)$. Hence, (11) implies that

$$\begin{aligned}\psi^*_G(T\Omega(L/K)) &= \chi_{\mathbb{Z}[G],\mathbb{R}}(\Psi^\bullet_S, \tau_S(\lambda)) - \sum_p \hat{\partial}_{G,p}(\lambda_p)\\ &= t_G(\mathrm{cok}(\varphi_S)) + \hat{\partial}_G(\hat{\tau}_S(\lambda)_{\varphi_S}) - \sum_p \hat{\partial}_{G,p}(\lambda_p).\end{aligned}$$ □

In the remainder of this section we assume (often without explicit comment) that G is Abelian. We will show that in this case Proposition can be rephrased in terms of determinants and (first) Fitting ideals, and in later sections we shall find that this description renders $T\Omega(L/K)$ more amenable to investigation using techniques of Iwasawa theory.

Let R be any commutative ring. Recall that a graded invertible R-module is a pair (L, α) consisting of an invertible (that is, rank one projective) R-module L and a locally-constant function $\alpha: \mathrm{Spec} R \to \mathbb{Z}$. The category $\mathcal{P}(R)$ of graded invertible R-modules and isomorphisms of such is a symmetric monoidal category with tensor product $(L, \alpha) \otimes (M, \beta) = (L \otimes_R M, \alpha + \beta)$, unit object $(R, 0)$, the usual associativity

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constraint and a commutativity constraint specified via the 'Koszul rule' (cf. [7, (4)]). For each finitely generated projective R-module P one sets

$$\mathrm{Det}_R(P) := (\det_R(P), \mathrm{rk}_R(P))$$

where $\det_R(P)$ denotes the highest exterior power of the R-module P and $\mathrm{rk}_R(P)$ is the locally constant function given by the R-rank of P. For each $(L, \alpha) \in \mathrm{Ob}(\mathcal{P}(R))$ one sets $L^{-1} := \mathrm{Hom}_R(L, R)$ and $(L, \alpha)^{-1} := (L^{-1}, -\alpha) \in \mathrm{Ob}(\mathcal{P}(R))$. For each bounded complex $P^\bullet$ of finitely generated projective R-modules one defines

$$\mathrm{Det}_R P^\bullet := \bigotimes_{i \in \mathbb{Z}} \mathrm{Det}_R(P^i)^{(-1)^{i+1}}.$$

(Here we use the normalisation of [5–7] rather than [20].) Let $\mathcal{D}^b(R)$ denote the derived category of the homotopy category of bounded complexes of R-modules. In [20] it is shown that Det_R extends to give a well defined functor from the subcategory of $\mathcal{D}^b(R)$ consisting of perfect complexes and where morphisms are restricted to quasi-isomorphisms, to the category $\mathcal{P}(R)$.

If N is any perfect R-module, then we set $\mathrm{Det}_R(N) := \mathrm{Det}_R N[-1]$. If each cohomology module of a perfect complex $P^\bullet$ is itself a perfect R-module, then there is a canonical morphism in $\mathcal{P}(\mathcal{R})$ (cf. [20, Rem. b) following Th. 2])

$$\mathrm{Det}_R P^\bullet \xrightarrow{\sim} \bigotimes_{i \in \mathbb{Z}} \mathrm{Det}_R(H^i(P^\bullet))^{(-1)^{i+1}}. \tag{12}$$

Any isomorphism $\theta: P_1 \xrightarrow{\sim} P_2$ of finitely generated projective R-modules induces a canonical morphism in $\mathcal{P}(R)$

$$\theta_{\mathrm{triv}}: \mathrm{Det}_R(P_1) \otimes \mathrm{Det}_R(P_2)^{-1} \xrightarrow{\sim} (R, 0)$$

given by

$$\theta_{\mathrm{triv}}((p_1 \otimes \phi_2, 0)) = (\phi_2(\det_R(\theta)(p_1)), 0)$$

for each $p_1 \in \det_R(P_1)$ and $\phi_2 \in \det_R(P_2)^{-1}$. On occasion, we shall identify graded invertible modules of the form $(L, 0)$ with the underlying invertible R-module L.

The following lemma describes explicitly the link between elements of $K_0(\mathbb{Z}[G], E)$, determinants and Fitting ideals.

LEMMA 2.6. *Let G be Abelian.* (i) *Let X and Y be finitely generated projective $\mathbb{Z}[G]$-modules and $\lambda: X_E \xrightarrow{\sim} Y_E$ an isomorphism of $E[G]$-spaces for some extension field E of $\mathbb{Q}$. Then the association $[X, \lambda, Y] \mapsto \lambda_{\mathrm{triv}}(\mathrm{Det}_{\mathbb{Z}[G]}(X) \otimes \mathrm{Det}_{\mathbb{Z}[G]}(Y)^{-1})$ induces an isomorphism $\iota_{G,E}$ between $K_0(\mathbb{Z}[G], E)$ and the group of invertible $\mathbb{Z}[G]$-submodules of $E[G]$.*

(ii) *If $[X, \lambda, Y] \in K_0(\mathbb{Z}[G], \mathbb{Q})$, then*

$$\iota_{G,\mathbb{Q}}([X, \lambda, Y]) = \mathrm{Fitt}_{\mathbb{Z}[G]}(Y/U) \cdot \mathrm{Fitt}_{\mathbb{Z}[G]}(\lambda(X)/U)^{-1},$$

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where here U is any full projective $\mathbb{Z}[G]$-sublattice of $Y \cap \lambda(X)$. In particular, if M is any finite perfect $\mathbb{Z}[G]$-module, then $\iota_{G,\mathbb{Q}}(t_G(M))$ is equal to the (first) Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}[G]}(M)$ *of M.*

Proof. For any finitely generated projective $\mathbb{Z}[G]$-module X we abbreviate $\det_{\mathbb{Z}[G]}(X)$ to $\mathcal{D}(X)$, and for any isomorphism $\lambda\colon V \xrightarrow{\sim} W$ of finitely generated $E[G]$-spaces we abbreviate $\det_{E[G]}(\lambda)$ to $\mathcal{D}(\lambda)$.

To prove (i) we recall that every element of $K_0(\mathbb{Z}[G], E)$ is of the form $[X, \lambda, Y]$ with X and Y finitely generated projective $\mathbb{Z}[G]$-modules and $\lambda\colon X_E \xrightarrow{\sim} Y_E$ an isomorphism of $E[G]$-spaces (cf. [30, Lem. 15.6]), and that

$$[X, \lambda, Y] = [\mathcal{D}(X), \mathcal{D}(\lambda), \mathcal{D}(Y)] \in K_0(\mathbb{Z}[G], E) \tag{13}$$

(cf. [7, Lem. 1(c)]). This implies in particular that in $K_0(\mathbb{Z}[G], E)$

$$\begin{aligned}&[X_1, \lambda_1, Y_1] + [X_2, \lambda_2, Y_2]\\ &\quad = [X_1 \oplus X_2, \lambda_1 \oplus \lambda_2, Y_1 \oplus Y_2]\\ &\quad = [\mathcal{D}(X_1 \oplus X_2), \mathcal{D}(\lambda_1 \oplus \lambda_2), \mathcal{D}(Y_1 \oplus Y_2)]\\ &\quad = [\mathcal{D}(X_1) \otimes_{\mathbb{Z}[G]} \mathcal{D}(X_2), \mathcal{D}(\lambda_1) \otimes_{E[G]} \mathcal{D}(\lambda_2), \mathcal{D}(Y_1) \otimes_{\mathbb{Z}[G]} \mathcal{D}(Y_2)].\end{aligned} \tag{14}$$

Now $[\mathcal{D}(Y)^{-1}, 1, \mathcal{D}(Y)^{-1}] = 0 \in K_0(\mathbb{Z}[G], E)$ and hence one has

$$\begin{aligned}[X, \lambda, Y] &\overset{(13)}{=} [\mathcal{D}(X), \mathcal{D}(\lambda), \mathcal{D}(Y)] + [\mathcal{D}(Y)^{-1}, 1, \mathcal{D}(Y)^{-1}]\\ &\overset{(14)}{=} [\mathcal{D}(X) \otimes_{\mathbb{Z}[G]} \mathcal{D}(Y)^{-1}, \mathcal{D}(\lambda) \otimes_{E[G]} 1, \mathcal{D}(Y) \otimes_{\mathbb{Z}[G]} \mathcal{D}(Y)^{-1}]\\ &= [\lambda_{\mathrm{triv}}\big(\mathrm{Det}_{\mathbb{Z}[G]}(X) \otimes \mathrm{Det}_{\mathbb{Z}[G]}(Y)^{-1}\big), 1, \mathbb{Z}[G]],\end{aligned}$$

where the last equality follows because $\lambda_{\mathrm{triv}} = \varepsilon_{Y,\mathrm{triv}} \circ (\mathcal{D}(\lambda) \otimes_{E[G]} 1)$ and $\varepsilon_{Y,\mathrm{triv}}(\mathrm{Det}_{\mathbb{Z}[G]}(Y) \otimes \mathrm{Det}_{\mathbb{Z}[G]}(Y)^{-1}) = \mathbb{Z}[G]$ with ε_Y equal to the identity automorphism of $\mathcal{D}(Y)_{\mathbb{Q}}$. This shows that every element of $K_0(\mathbb{Z}[G], E)$ is of the form $[L, 1, \mathbb{Z}[G]]$ with L an invertible $\mathbb{Z}[G]$-sublattice of $E[G]$. Furthermore, [7, Lem. 1(b)] implies $[L, 1, \mathbb{Z}[G]] = 0 \in K_0(\mathbb{Z}[G], E) \Leftrightarrow L = \mathbb{Z}[G]$, and if L_1 and L_2 are any invertible $\mathbb{Z}[G]$-sublattices of $E[G]$, then (14) implies that

$$[L_1, 1, \mathbb{Z}[G]] + [L_2, 1, \mathbb{Z}[G]] = [L_1 L_2, 1, \mathbb{Z}[G]].$$

Claim (i) is now clear.

To prove (ii) we note that in $\mathbb{Q}[G]$

$$\begin{aligned}\iota_{G,\mathbb{Q}}([X, \lambda, Y]) &= \lambda_{\mathrm{triv}}\big(\mathrm{Det}_{\mathbb{Z}[G]}(X) \otimes \mathrm{Det}_{\mathbb{Z}[G]}(Y)^{-1}\big)\\ &= \varepsilon_{Y,\mathrm{triv}}\big(\mathrm{Det}_{\mathbb{Z}[G]}(\lambda(X)) \otimes \mathrm{Det}_{\mathbb{Z}[G]}(Y)^{-1}\big)\\ &= \mathrm{Det}_{\mathbb{Z}[G]}(Y/U)^{-1} \otimes \mathrm{Det}_{\mathbb{Z}[G]}(\lambda(X)/U).\end{aligned}$$

Using these equalities both assertions of (ii) follow from the fact that $\mathrm{Det}_{\mathbb{Z}[G]}(N) = (\mathrm{Fitt}_{\mathbb{Z}[G]}(N), 0)^{-1}$ for any finite perfect $\mathbb{Z}[G]$-module N. To prove this last equality we let $\pi\colon \mathbb{Z}[G]^s \to N$ be a G-epimorphism and set $Q := \ker(\pi)$. Since

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N is $\mathbb{Z}[G]$-perfect Q is $\mathbb{Z}[G]$-projective. Hence one has

$$\mathrm{Det}_{\mathbb{Z}[G]}(N) = \mathrm{Det}_{\mathbb{Z}[G]}(Q)^{-1} \otimes \mathrm{Det}_{\mathbb{Z}[G]}(\mathbb{Z}[G]^s) = \mathrm{Det}_{\mathbb{Z}[G]}(Q)^{-1} \subseteq (\mathbb{Z}[G], 0),$$

and this is equal to $(\mathrm{Fitt}_{\mathbb{Z}[G]}(N), 0)^{-1}$ as an easy consequence of the definition of first Fitting ideal (cf. [23, App. 4]). □

Since G is Abelian one has $\zeta(\mathbb{R}[G])^{\times +} = \zeta(\mathbb{R}[G])^{\times}$ and so the condition (5) is satisfied by setting $\lambda = 1$. We write τ_S and $\hat{\tau}_S$ in place of $\tau_S(1)$ and $\hat{\tau}_S(1)$ respectively, and note that τ_S is equal to the canonical isomorphism

$$L_S^*(0)^\# \cdot \det_{\mathbb{R}[G]}(R_S^{-1})\colon \det_{\mathbb{R}[G]}(X_{S,\mathbb{R}}) \xrightarrow{\sim} \det_{\mathbb{R}[G]}(U_{S,\mathbb{R}}).$$

We write $F^\bullet[1]$ for the left shift of $F^\bullet$ and let

$$\vartheta(\varphi_S)\colon (\mathrm{Det}_{\mathbb{Z}[G]} F^\bullet[1])_{\mathbb{R}} \xrightarrow{\sim} (\mathbb{R}[G], 0)$$

denote the composite morphism

$$\begin{aligned}
&(\mathrm{Det}_{\mathbb{Z}[G]} F^\bullet[1])_{\mathbb{R}} \\
&\xrightarrow{(12)} \mathrm{Det}_{\mathbb{R}[G]}(X_S(-2)_{\mathbb{R}}) \otimes \mathrm{Det}_{\mathbb{R}[G]}(X_{S,\mathbb{R}})^{-1} \\
&\xrightarrow{\det_{\mathbb{R}[G]}(\varphi_{S,\mathbb{R}})\otimes 1} \mathrm{Det}_{\mathbb{R}[G]}(U_{S,\mathbb{R}}) \otimes \mathrm{Det}_{\mathbb{R}[G]}(X_{S,\mathbb{R}})^{-1} \\
&\xrightarrow{(\tau_S^{-1})_{\mathrm{triv}}} (\mathbb{R}[G], 0).
\end{aligned}$$

THEOREM 2.7. *Let G be Abelian. Then with the same assumptions and notation of Proposition 2.5 one has*

$$\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{cok}(\varphi_S)) = \vartheta(\varphi_S)(\mathrm{Det}_{\mathbb{Z}[G]} F^\bullet[1]) \cdot \iota_{G,\mathbb{R}}(\psi_G^*(T\Omega(L/K))) \subseteq \mathbb{Z}[G].$$

Proof. Choose $\phi \in Aut_{\mathbb{R}[G]}(F_{\mathbb{R}})$ such that $\det_{\mathbb{R}[G]}(\phi) = \tau_{S,\varphi_S}^{-1}(F^\bullet[1])$. Then Lemma 2.6(i) implies

$$\begin{aligned}
\hat{\partial}_G(\hat{\tau}_{S,\varphi_S}^{-1}) &= \partial_G[F_{\mathbb{R}}, \phi] \\
&= [F, \phi, F] \\
&= [\phi_{\mathrm{triv}}(\mathrm{Det}_{\mathbb{Z}[G]} F^\bullet[1]), 1, \mathbb{Z}[G]]
\end{aligned}$$

so that

$$-\iota_{G,\mathbb{R}}(\hat{\partial}_G(\hat{\tau}_{S,\varphi_S})) = \phi_{\mathrm{triv}}(\mathrm{Det}_{\mathbb{Z}[G]} F^\bullet[1]). \tag{15}$$

In addition, after unwinding the explicit construction of (12) one finds that $\phi_{\mathrm{triv}} = \vartheta(\varphi_S)$. The theorem therefore follows upon combining equality (15) with the formula of Proposition 2.5 (with $\lambda = 1$) and the final assertion of Lemma 2.6(ii). □

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COROLLARY 2.8. *If G is Abelian, then $T\Omega(L/K) = 0$ if and only if*

$$\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{cok}(\varphi_S)) = \vartheta(\varphi_S)(\mathrm{Det}_{\mathbb{Z}[G]}F^\bullet[1]) \subseteq \mathbb{Z}[G].$$

Proof. This is an immediate consequence of Theorem 2.7 and the fact that

$$\begin{aligned}\iota_{G,\mathbb{R}}(\psi_G^*(T\Omega(L/K))) = \mathbb{Z}[G] &\iff \psi_G^*(T\Omega(L/K)) = 0 \\ &\iff T\Omega(L/K) = 0.\end{aligned}$$ □

Remarks 2.9. (i) The 'Equivariant Tamagawa Number Conjecture' discussed in [3] and [4] predicts that $T\Omega(L/K) = 0$ for all finite Galois extensions L/K.

(ii) If Stark's Conjecture is true for L/K, then Corollary 2.8 can be rephrased completely in terms of Fitting ideals by using Proposition 2.1(iii) and Lemma 2.6(ii).

(iii) Since F is $\mathbb{Z}[G]$-free $\vartheta(\varphi_S)(\mathrm{Det}_{\mathbb{Z}[G]}F^\bullet[1])$ is a principal ideal of $\mathbb{Z}[G]$. The formula of Corollary 2.8 therefore predicts that $\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{cok}(\varphi_S))$ is a principal ideal of $\mathbb{Z}[G]$, and it is not difficult to show that (if G is Abelian, then) this is equivalent to the original conjecture formulated by Chinburg in [10].

3. Abelian Fields

In this section we use the approach described in Section 2 to prove Theorem 1.1. In particular, we show how the description of Theorem 2.7 renders $T\Omega(L/K)$ more amenable to investigation using Iwasawa theory. It is possible that our approach could be used to compute $T\Omega(L/\mathbb{Q})$ for Abelian extensions $L/\mathbb{Q}$ which are more general than those in Theorem 1.1, but we do not pursue this point further here.

3.1. REVIEW OF KNOWN RESULTS

In this subsection we quickly review some known results concerning $T\Omega(L/\mathbb{Q})$ for finite Abelian extensions $L/\mathbb{Q}$.

Since Stark's Conjecture is known to be valid for all such extensions (cf. [32]) Proposition 2.1(iii) implies that $T\Omega(L/\mathbb{Q})$ belongs to $K_0(\mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})], \mathbb{Q})$. In addition, under a very mild ramification restriction, Ritter and Weiss have verified the Strong Stark Conjecture for $L/\mathbb{Q}$. More specifically, in conjunction with Proposition 2.1(iv) the result of [26, Th. A] implies the following.

PROPOSITION 3.1. *If L is a finite Abelian extension of $\mathbb{Q}$ of odd conductor, then $T\Omega(L/\mathbb{Q})$ belongs to the torsion subgroup of $K_0(\mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})], \mathbb{Q})$. In particular, if $p \nmid [L : \mathbb{Q}]$ then $T\Omega(L/\mathbb{Q})_p = 0$.* □

At the moment, finer results have only been proved for much more restricted classes of extensions $L/\mathbb{Q}$. For example, if the ideal class group $\mathrm{cl}(L)$ of L is

$\mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})]$-perfect, then the necessary computations are greatly simplified. Such extensions $L/\mathbb{Q}$ have been systematically studied in [14], [15] and [2] and motivated by these earlier results the following is proved in [7, §8].

PROPOSITION 3.2. *Let $L/\mathbb{Q}$ be an Abelian extension of odd prime power degree. If $L/\mathbb{Q}$ is of prime power conductor or is the compositum of two linearly disjoint extensions of prime power conductor and both primes which ramify in $L/\mathbb{Q}$ have decomposition subgroup equal to* $\mathrm{Gal}(L/\mathbb{Q})$, *then* $T\Omega(L/\mathbb{Q}) = 0$. □

If one relaxes the degree restriction on any extension $L/\mathbb{Q}$ in Proposition 3.2, then $\mathrm{cl}(L)$ need no longer be $\mathbb{Z}[\mathrm{Gal}(L/\mathbb{Q})]$-perfect, and the situation becomes considerably more complicated.

To discuss an explicit example we now take L to be $\mathbb{Q}(l^a)^+$ for any prime l and any exponent a. Then one can choose a set $S' = \{p_1, \ldots, p_{s'}\}$ consisting of rational primes which are fully split in $L/\mathbb{Q}$ and such that $S = \{\infty, l, p_1, \ldots, p_{s'}\}$ is admissible. Let $\mathfrak{l}$ be the unique prime of L above l and choose a prime $\mathfrak{p}_i$ above p_i for each i with $1 \leqslant i \leqslant s'$. Let $\mathfrak{p}_\infty$ be the archimedean place of L which is induced by sending ζ_{l^a} to $\exp(2\pi i/l^a)$. Then

$$X_S = \mathbb{Z}[G](\mathfrak{p}_\infty - \mathfrak{l}) \oplus \bigoplus_{i=1}^{s'} \mathbb{Z}[G](\mathfrak{p}_i - \mathfrak{l})$$

is a free $\mathbb{Z}[G]$-module so that $\mathrm{Ext}^2_G(X_S, U_S) = 0$. This means that one can take any exact sequence of the form

$$0 \longrightarrow X_S \xrightarrow{\sim} \mathbb{Z}[G]^{s'+1} \xrightarrow{0} \mathbb{Z}[G]^{s'+1} \xrightarrow{\sim} X_S \longrightarrow 0$$

for the resolution (7) and any injective G-morphism $\varphi_S\colon X_S \to U_S$ for the map in Theorem 2.7 (with $K = \mathbb{Q}$). Following [16], we specify φ_S by means of the conditions

$$\varphi_S(\mathfrak{p}_\infty - \mathfrak{l}) = (1 - \zeta_{l^a})(1 - \zeta_{l^a}^{-1}),$$
$$\varphi_S(\mathfrak{p}_i - \mathfrak{l}) = x_i, \quad i = 1, \ldots, s',$$

where here x_i is a choice of generator of the principal $\mathcal{O}_L$-ideal $\mathfrak{p}_i^h$ with h equal to the class number of L. By using Iwasawa theoretic techniques Greither has computed that $\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{cok}(\varphi_S)) = (2h^{s'})$ (cf. *loc. cit.*, Th. 6.1), and on the other hand it is a straightforward exercise to verify that

$$\vartheta(\varphi_S)(\mathrm{Det}_{\mathbb{Z}[G]}(\mathbb{Z}[G]^{s'+1}) \otimes \mathrm{Det}_{\mathbb{Z}[G]}(\mathbb{Z}[G]^{s'+1})^{-1}) = (2h^{s'}).$$

The next result follows directly by combining these computations with Corollary 2.8 and Proposition 2.1(ii).

PROPOSITION 3.3. *If L is a real absolutely Abelian field of prime power conductor, then* $T\Omega(L/\mathbb{Q}) = 0$. □

There are two further results which should be mentioned in this context. Let $E/\mathbb{Q}$ be a tamely ramified Abelian extension of odd prime degree. If the conductor of $E/\mathbb{Q}$ is equal to the product of two (necessarily distinct) primes, then [27, §1, Th.] implies that $T\Omega(E/\mathbb{Q}) = 0$ (cf. [4, Prop. 2.3.3]). Also, Greither and Kucera have very recently generalised this result by removing the restriction on the number of primes which divide the conductor of $E/\mathbb{Q}$.

3.2. PROOF OF THEOREM 1.1

In this section we reduce the proof of Theorem 1.1 to an explicit computation of Fitting ideals. The necessary computation is then made in subsequent sections by using Iwasawa theory.

For any Abelian group H and $\mathbb{Q}_p^c$-valued character ψ of H we write $\mathbb{Z}_p(\psi)$ for the ring extension of $\mathbb{Z}_p$ generated by the values of ψ. For a $\mathbb{Z}_p[H]$-module M and an Abelian character ψ we set $M_\psi := \mathbb{Z}_p(\psi) \otimes_{\mathbb{Z}_p[H]} M$ where H acts on $\mathbb{Z}_p(\psi)$ via ψ. In general this is a quotient of $\mathbb{Z}_p(\psi) \otimes_{\mathbb{Z}_p} M$ but if $p \nmid |H|$ and e_ψ denotes the idempotent $|H|^{-1} \sum_{h \in H} \psi(h^{-1})h$ of $\mathbb{Z}_p(\psi)[H]$, then M_ψ naturally identifies with the direct summand $e_\psi(\mathbb{Z}_p(\psi) \otimes_{\mathbb{Z}_p} M)$ of $\mathbb{Z}_p(\psi) \otimes_{\mathbb{Z}_p} M$.

Let l_1, l_2, a and b be as in Theorem 1.1. Taking into account Proposition 2.1(ii) it suffices to consider the field $L := \mathbb{Q}(l_1^a)^+\mathbb{Q}(l_2^b)^+$. We set $G := \mathrm{Gal}(L/\mathbb{Q})$. Proposition 3.1 implies it is enough to consider primes p which divide $|G|$, and for any such p we write $G = G_p \times G'$ with G_p equal to the Sylow p-subgroup of G. We fix a set of representatives Υ of the orbits of the action of $\mathrm{Gal}(\mathbb{Q}_p^c/\mathbb{Q}_p)$ on $\mathrm{Hom}(G', \mathbb{Q}_p^{c\times})$. The group ring $\mathbb{Z}_p[G]$ decomposes canonically as $\mathbb{Z}_p[G] = \bigoplus_{\xi \in \Upsilon} \mathbb{Z}_p(\xi)[G_p]$ and this induces in turn a canonical isomorphism

$$K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \simeq \bigoplus_{\xi \in \Upsilon} K_0(\mathbb{Z}_p(\xi)[G_p], \mathbb{Q}_p(\xi)). \tag{16}$$

For each $\xi \in \Upsilon$ we let $T\Omega(L/\mathbb{Q})_{p,\xi}$ denote the component of $T\Omega(L/\mathbb{Q})_p$ in $K_0(\mathbb{Z}_p(\xi)[G_p], \mathbb{Q}_p(\xi))$ under the decomposition (16). Then one has $T\Omega(L/\mathbb{Q})_p = 0$ if and only if $T\Omega(L/\mathbb{Q})_{p,\xi} = 0$ for each $\xi \in \Upsilon$. We first show that the results of Section 4.1 imply $T\Omega(L/\mathbb{Q})_{p,\xi} = 0$ for certain pairs (p, ξ).

We assume until further notice that $p \nmid l_1 l_2$. For any field F we set $F' = F(\zeta_p)$ and write $\omega\colon \mathrm{Gal}(\mathbb{Q}'/\mathbb{Q}) \simeq \mathrm{Gal}(L'/L) \to \mathbb{Z}_p^\times$ for the p-adic Teichmüller character which is characterised by $\sigma(\zeta_p) = \zeta_p^{\omega(\sigma)}$. We set $\Delta := \mathrm{Gal}(\mathbb{Q}'/\mathbb{Q})$ and $H := G' \times \Delta$ and note that $\xi^{-1}\omega$ is an odd character of H for any character ξ of G'.

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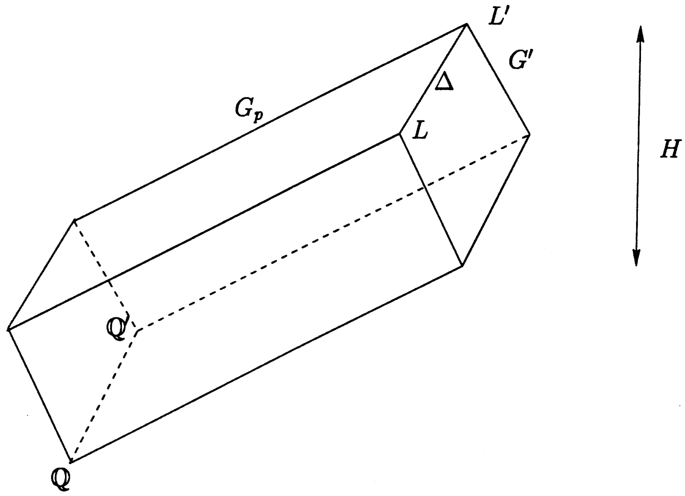
The following diagram of fields clarifies the situation.



We set $K_1 := \mathbb{Q}(l_1^a)^+$ and $K_2 := \mathbb{Q}(l_2^b)^+$ so that $L = K_1K_2$ and $K_1 \cap K_2 = \mathbb{Q}$, and we write G_1 and G_2 for $\mathrm{Gal}(L/K_1)$ and $\mathrm{Gal}(L/K_2)$ respectively.

PROPOSITION 3.4. *Let p be an odd prime which does not divide $l_1 l_2$ and let $l \in \{l_1, l_2\}$. Let D be the decomposition group of l in* $\mathrm{Gal}(L'/\mathbb{Q})$ *and suppose that the character $\xi^{-1}\omega$ is trivial on $D \cap H$. Then $T\Omega(L/\mathbb{Q})_{p,\xi} = 0$.*

Proof. It suffices to show that under the stated conditions ξ is trivial or factors through a character of either $\mathrm{Gal}(K_1/\mathbb{Q})$ or $\mathrm{Gal}(K_2/\mathbb{Q})$. Indeed $L^{G'}/\mathbb{Q}$ is an extension of the form considered in Proposition 3.2, and so for all such characters ξ Propositions 2.1(ii), 3.2 and 3.3 combine to imply that $T\Omega(L/\mathbb{Q})_{p,\xi} = 0$.

Without loss of generality we assume $l = l_1$. We suppose first that $p \mid l_1 - 1$. Then $D = G$ because l_1 splits completely in $\mathbb{Q}'/\mathbb{Q}$. Therefore $D \cap H = G'$ and so $\xi = \xi\omega^{-1} \mid_{G'}$ is trivial.

If on the other hand $p \nmid l_1 - 1$, then $G_2 \subseteq G' \subset H$. Moreover $G_2 \subseteq D$ because all primes of K_2' above l_1 are totally ramified in L'/K_2'. Therefore $\xi \mid_{G_2} = \xi\omega^{-1} \mid_{G_2}$ is trivial and so ξ factors through a character of $\mathrm{Gal}(K_2/\mathbb{Q})$. □

To compute $T\Omega(L/\mathbb{Q})_{p,\xi}$ for pairs (p, ξ) which do not satisfy the condition of Proposition 3.4 we shall use the approach described in Section 2. To do this we first recall some results from [2, §6] and [7, §8].

We let g_1, resp. g_2, denote the generator of G_1, resp. G_2, which restricts to give the Frobenius of l_1 on K_2, resp. of l_2 on K_1. In addition, for $i \in \{1, 2\}$ we set $\Sigma_i := \sum_{g \in G_i} g \in \mathbb{Z}[G_i]$. We let $S' = \{p_1, \ldots, p_{s'}\}$ be a set of rational primes which are fully split in $L/\mathbb{Q}$ and such that $S = \{\infty, l_1, l_2, p_1, \ldots, p_{s'}\}$ is admissible for $L/\mathbb{Q}$. Let $\mathfrak{l}_1$ and $\mathfrak{l}_2$ denote the unique primes of L above l_1 and l_2 and choose primes

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$\mathfrak{p}_1, \ldots, \mathfrak{p}_{s'}$ above the rational primes $p_1, \ldots, p_{s'}$ respectively. It is shown in [2, §6] that there is an exact sequence

$$0 \longrightarrow X_S(-2) \xrightarrow{\theta^{-1}} F \xrightarrow{\partial} F \xrightarrow{\theta^1} X_S \longrightarrow 0 \tag{17}$$

with

$$\begin{aligned}
X_S &= \mathbb{Z}(\mathfrak{l}_2 - \mathfrak{l}_1) \oplus \mathbb{Z}[G](\mathfrak{p}_\infty - \mathfrak{l}_1) \oplus \bigoplus_{i=1}^{s'} \mathbb{Z}[G](\mathfrak{p}_i - \mathfrak{l}_1), \\
F &= \mathbb{Z}[G]^{s'+2}, \\
X_S(-2) &= < (g_2 - 1, g_1 - 1, \mathbf{0}), (\Sigma_1, 0, \mathbf{0}), (0, \Sigma_2, \mathbf{0}), w_1, \ldots, w_{s'} >_{\mathbb{Z}[G]} \\
&\quad \text{where } w_i := (0, 0, \mathbf{e}_i) \text{ with } \mathbf{e}_i \text{ equal} \\
&\quad \text{to the } i\text{th unit vector of } \mathbb{Z}[G]^{s'}
\end{aligned}$$

$$\begin{aligned}
\theta^1(x, y, t_1, \ldots, t_{s'}) &= \epsilon(x)(\mathfrak{l}_2 - \mathfrak{l}_1) + y(\mathfrak{p}_\infty - \mathfrak{l}_1) + \sum_{i=1}^{s'} t_i(\mathfrak{p}_i - \mathfrak{l}_1), \\
&\quad (\epsilon = \text{augmentation map}), \\
\partial(x, y, t_1, \ldots, t_{s'}) &= ((g_1 - 1)x - (g_2 - 1)y, 0, \mathbf{0}), \\
\theta^{-1} &= \text{ inclusion.}
\end{aligned}$$

Following [7, §8] we fix integers i and j such that

$$il_2^{b-1} \equiv 1 (\bmod\ l_1^a), \qquad jl_1^{a-1} \equiv 1 (\bmod\ l_2^b) \tag{18}$$

and define

$$\begin{aligned}
\eta_1 &:= \mathrm{N}_{\mathbb{Q}(l_1^a)/K_1}(1 - \zeta_{l_1^a}) \in K_1, \\
\eta_2 &:= \mathrm{N}_{\mathbb{Q}(l_2^b)/K_2}(1 - \zeta_{l_2^b}) \in K_2, \\
\eta_3 &:= \mathrm{N}_{\mathbb{Q}(l_1^a l_2^b)/L}(1 - \zeta_{l_1^a}^i \zeta_{l_2^b}^j) \in L.
\end{aligned}$$

Then (18) implies that $\Sigma_1\eta_3 = (g_2 - 1)\eta_1$, $\Sigma_2\eta_3 = (g_1 - 1)\eta_2$, and so there exists an injective G-morphism $\varphi_S\colon X_S(-2) \to U_S$ which satisfies the conditions

$$\begin{aligned}
&\varphi_S((g_2 - 1, g_1 - 1, \mathbf{0})) = \eta_3, \\
&\varphi_S((\Sigma_1, 0, \mathbf{0})) = \eta_1, \\
&\varphi_S((0, \Sigma_2, \mathbf{0})) = \eta_2, \\
&\varphi_S(w_i) = x_i \text{ for each } i \in \{1, \ldots, s'\}
\end{aligned} \tag{19}$$

where here x_i is any choice of generator of the principal $\mathcal{O}_L$-ideal $\mathfrak{p}_i^h$ with h equal to the class number of L.

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LEMMA 3.5. *If* $\mathrm{Ext}^2_G(X_S, U_S)$ *is computed via the resolution (17), then* φ_S *represents the canonical class* $c_S(L/\mathbb{Q})$.

Proof. For each prime p we set $\mathcal{E}_p := \mathrm{Ext}^2_{\mathbb{Z}_p[G]}(X_S \otimes_{\mathbb{Z}} \mathbb{Z}_p, U_S \otimes_{\mathbb{Z}} \mathbb{Z}_p) \simeq \mathrm{Ext}^2_G(X_S, U_S) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, and we write ε_p for the element of $\mathcal{E}_p$ represented by $\varphi_S \otimes_{\mathbb{Z}} \mathbb{Z}_p$ when $\mathcal{E}_p$ is computed via the resolution $R^\bullet_p$ of $X_S \otimes_{\mathbb{Z}} \mathbb{Z}_p$ obtained by applying $- \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to (17). It suffices to show that ε_p is equal to the image $c(L)_p$ of $c_S(L/\mathbb{Q})$ in $\mathcal{E}_p$ for each prime divisor p of $|G|$, and we prove this by using results from [2, §6].

Regarding p as fixed, we write N_1 and N_2 for the maximal subfields of K_1 and K_2 which are of p-power degree over $\mathbb{Q}$. We let N denote the compositum of N_1 and N_2, set $P := \mathrm{Gal}(N/\mathbb{Q})$ and $H := \mathrm{Gal}(L/N)$, write $U_{S,N,p}, X_{S,N,p}$ and $c(N)_p$ for the analogues of $U_S \otimes_{\mathbb{Z}} \mathbb{Z}_p, X_S \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $c(L)_p$ for the field N, and set $\mathcal{E}_{N,p} := \mathrm{Ext}^2_{\mathbb{Z}_p[P]}(X_{S,N,p}, U_{S,N,p})$. Since $p \nmid |H|$, the Hochshild–Serre spectral sequence induces a canonical isomorphism $\delta_N : \mathcal{E}_p \simeq \mathcal{E}_{N,p}$. It is well known that $\delta_N(c(L)_p) = c(N)_p$ and so it suffices to prove $\delta_N(\varepsilon_p) = c(N)_p$.

To compute $\delta_N(\varepsilon_p)$ explicitly, one can proceed in the following manner. Let $\psi_S \in \mathrm{Hom}_G(X_S(-2), U_S)$ be the homomorphism which differs from φ_S only in that $\psi_S(w_i) = 0$ for each $i \in \{1, \dots, s'\}$, and set $\psi_{S,p} := \psi_S \otimes_{\mathbb{Z}} \mathbb{Z}_p$. By taking H fixed points of $R^\bullet_p$ one obtains a resolution $R^\bullet_{p,F}$ of $X_{S,N,p}$, and $\delta_N(\varepsilon_p)$ is represented by the restriction $\psi^H_{S,p}$ of $\psi_{S,p}$ to $(X_S(-2) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^H$ when $\mathcal{E}_{N,p}$ is computed via $R^\bullet_{p,F}$. Now $N/\mathbb{Q}$ is an extension of the form considered in [2, §6] (with the group G and primes l, p and q of loc. cit. now replaced by P, p, l_1 and l_2 respectively), $\psi^H_{S,p}$ can be naturally identified with the p-completion of the morphism ϕ described in [*loc. cit.*, Lem. 6.2], and the resolution $R^\bullet_{p,F}$ can be obtained from the resolution of [*loc. cit.*, (6.4)] by adding a free $\mathbb{Z}[P]$-module of rank s' to each term of the latter (with an obvious change of differentials), and then applying $- \otimes_{\mathbb{Z}} \mathbb{Z}_p$. The argument which begins at the top of [*loc. cit.*, p. 899] thus proves that $c(N)_p$ is represented by $\psi^H_{S,p}$ when $\mathcal{E}_{N,p}$ is computed via $R^\bullet_{p,F}$, as was required. □

The last result confirms that φ_S can be used in the context of Corollary 2.8. In addition, by using a simple adaptation of the computations which prove [7, Lem. 8 and 9] one obtains the following result.

LEMMA 3.6. $\vartheta(\varphi_S)\big(\mathrm{Det}_{\mathbb{Z}[G]} F^\bullet[1]\big) = (2h^{s'})$. □

Taking into account Proposition 3.4 and Lemmas 3.5 and 3.6, the proof that $T\Omega(L/\mathbb{Q})_p = 0$ for all primes $p \nmid 2l_1 l_2$ is completed by combining Corollary 2.8 with the following result.

THEOREM 3.7. *Let p be any prime which does not divide $2l_1l_2$, and let D_1 and D_2 denote the decomposition groups in $\mathrm{Gal}(L'/\mathbb{Q})$ of l_1 and l_2 respectively. Let ξ be a non-trivial character of G' such that $\xi^{-1}\omega$ is non-trivial on $D_i \cap H$ for both*

$i \in \{1, 2\}$. *Then*

$$\mathrm{Fitt}_{\mathbb{Z}_p(\xi)[G_p]}((\mathrm{cok}(\varphi_S) \otimes_{\mathbb{Z}} \mathbb{Z}_p)_\xi) = (h^{s'}). \tag{20}$$

Theorem 3.7 is proved in Section 3.4 after some preparations in Iwasawa theory which are given in Section 3.3.

The proof of Theorem 1.1 is then completed in Section 3.5 where we shall prove that $T\Omega(L/\mathbb{Q})_p = 0$ for both $p = l_1$ and $p = l_2$.

3.3. IWASAWA THEORETIC PRELIMINARIES

In this subsection we prove the results in Iwasawa theory which are essential for the proof of Theorem 3.7 given in Section 3.4. We keep the notation of Section 3.2 and continue to assume that p does not divide $2l_1l_2$.

For each non-negative integer n we let $\mathbb{Q}_n$ denote the n-th p-cyclotomic extension of $\mathbb{Q}$ and set $L_n := L\mathbb{Q}_n$. Since $p \nmid l_1l_2$ the groups $\mathrm{Gal}(L'_n/\mathbb{Q}_n)$ and $\mathrm{Gal}(L'/\mathbb{Q})$ are naturally isomorphic via restriction. Let M_n denote the maximal Abelian p-ramified pro-p-extension of L_n and write B_n for the Galois group of M_n/L_n. As usual we let $\Lambda = \mathbb{Z}_p[[T]]$ denote the Iwasawa algebra and for a $\mathbb{Q}_p^c$-valued character χ of G' we set $\Lambda(\chi) := \mathbb{Z}_p(\chi)[[T]]$. We define

$$\mathbb{Q}_\infty := \bigcup_{n=0}^{\infty} \mathbb{Q}_n, \quad L_\infty := L\mathbb{Q}_\infty \text{ and } M_\infty := \bigcup_{n=0}^{\infty} M_n.$$

Then L_∞ is the cyclotomic $\mathbb{Z}_p$-extension of L and we set $\Gamma := \mathrm{Gal}(L_\infty/L)$. We let Y denote the Galois group of M_∞/L_∞ and note that Y is a module over the group ring $\Lambda[G]$.

The key to applying Iwasawa theory in the context of Theorem 3.7 is the following observation.

THEOREM 3.8. *Let ξ be a non-trivial character of G' which satisfies the assumptions of Theorem 3.7. Then the projective dimension of Y_ξ over the ring $\Lambda(\xi)[G_p]$ is at most one.*

The proof of this theorem is based on the following result.

PROPOSITION 3.9. *Let ξ be a non-trivial character of G' which satisifies the assumptions of Theorem 3.7 and set $\chi := \xi^{-1}\omega$. Then $\mathrm{cl}(L'_n)_\chi$ is G_p-cohomologically trivial for all $n \geqslant 0$.*

We will give the proof of Proposition 3.9 at the end of this subsection.

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Proof of Theorem 3.8. Given the result of Proposition 3.9, the proof of Theorem 3.8 is completely analogous to the proof of [16, Prop. 5.1], but for the convenience of the reader we briefly recall the arguments.

By [16, Th. 2.2] it suffices to show that Y_ξ is $\mathbb{Z}_p$-torsion free and cohomologically trivial over G_p (note the assumption that G is cyclic in [16, Th. 2.2] is unnecessary). By Kummer duality (cf. for example the argument in [33, pp. 292–3]) Y_ξ is the Pontryagin dual of A_χ, where here A is the direct limit of $A_n = \mathrm{cl}(L'_n)_p$ and $\chi = \xi^{-1}\omega$. It follows that Y_ξ is $\mathbb{Z}_p$-torsion free since A_χ is divisible.

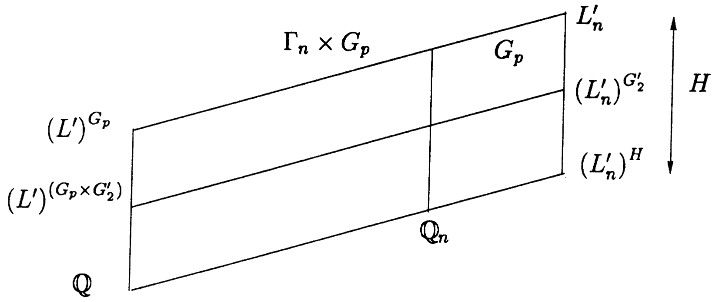
Proposition 3.9 implies that A_χ is cohomologically trivial over G_p. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is $\mathbb{Z}_p$-injective we have $\mathrm{Ext}_{\mathbb{Z}_p}(A_\chi, \mathbb{Q}_p/\mathbb{Z}_p) = 0$, and hence [29, Ch.IX, Th.9] implies that $Y_\xi \simeq \mathrm{Hom}_{\mathbb{Z}_p}(A_\chi, \mathbb{Q}_p/\mathbb{Z}_p)$ is also cohomologically trivial over G_p. □

Remark 3.10. In general the projective dimension of Y_ξ over $\Lambda(\xi)[G_p]$ is greater than one.

We prepare for the proof of Proposition 3.9 by first proving three lemmas. For these, we continue to use the notation and assumptions of Proposition 3.9. In addition, for any group Γ, any Γ-module M and any integer i we write $\hat{H}^i(\Gamma, M)$ for the Tate cohomology group in dimension i. In the case that $\Gamma = \mathrm{Gal}(E/F)$ for a Galois extension of fields E/F we also use the notation $\hat{H}^i(E/F, M)$ in place of $\hat{H}^i(\Gamma, M)$.

LEMMA 3.11. *Let n be a strictly positive integer, and let $\mathfrak{l}$ be any prime of $\mathbb{Q}_n$ above either l_1 or l_2. If D denotes the decomposition subgroup of $\mathfrak{l}$ in* $\mathrm{Gal}(L'_n/\mathbb{Q}_n)$, *then χ is non-trivial on $D \cap H$.*

Proof. Without loss of generality we assume that $\mathfrak{l} \mid l_1$. Since $p \nmid l_1 l_2$ the inertia group of l_1 in $\mathrm{Gal}(L'_n/\mathbb{Q})$ is equal to G_2 viewed as a subgroup of $\mathrm{Gal}(L'_n/\mathbb{Q})$. Setting $\Gamma_n := \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ and $G_2 := G'_2 \times G_{2,p}$ we have the following diagram of fields:



Let $\hat{\mathfrak{l}}$ be a prime of ${L'_n}^H$ above $\mathfrak{l}$. Then G'_2 is the inertia group of $\hat{\mathfrak{l}}$ in $\mathrm{Gal}(L'_n/{L'_n}^H)$ and so $D \cap H = < \mathrm{Fr}(\hat{\mathfrak{l}}, {L'_n}^{G'_2}/{L'_n}^H) > \times G'_2$. Now $[L'_n : {L'_n}^H]$ is coprime to $[{L'_n}^H : \mathbb{Q}]$ and hence restriction induces an isomorphism between G'_2 and the inertia group of l_1 in $\mathrm{Gal}(L'^{G_p}/\mathbb{Q})$, and also between the decomposition groups of $\hat{\mathfrak{l}}$ in

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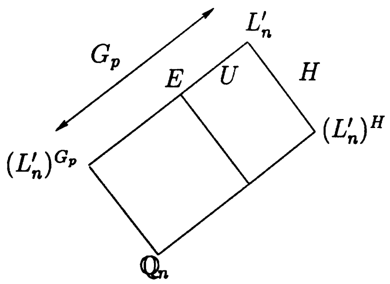
$\mathrm{Gal}({L'_n}^{G'_2}/{L'_n}^{H})$ and of l_1 in $\mathrm{Gal}(L'^{(G'_2\times G_p)}/\mathbb{Q})$. It follows that if χ is trivial on $D\cap H$, then it is trivial on the decomposition group of l_1 in $\mathrm{Gal}(L'^{G_p}/\mathbb{Q})$ and this cannot happen since ξ satisfies the assumptions of Theorem 3.7. $\square$

For any Abelian group Γ we let Γ_p denote its p-completion $\varprojlim \Gamma/p^n\Gamma$.

LEMMA 3.12. *Let U be a subgroup of G_p and set $E = \left(L'_n\right)^U$. Then*

$$\mathrm{cl}(L'_n)^U_{p,\chi} = \mathrm{cl}(E)_{p,\chi}.$$

Proof. The situation is clarified by the following diagram.



Note that there is a natural action of $\mathbb{Z}_p(\xi)[H]$ on objects coming from either L'_n or E.

For any number field N we write I_N for its group of fractional ideals and P_N for the subgroup of I_N consisting of those ideals which are principal. We consider the natural short exact sequence of $\mathrm{Gal}(L'_n/\mathbb{Q}_n)$-modules $0\longrightarrow P_{L'_n}\longrightarrow I_{L'_n}\longrightarrow \mathrm{cl}(L'_n)\longrightarrow 0$. Taking χ-eigenspaces (which is exact since $(|H|,p)=1$) and then U-invariants yields the exact sequence

$$0\longrightarrow P^U_{L'_n,\chi}\longrightarrow I^U_{L'_n,\chi}\longrightarrow \mathrm{cl}(L'_n)^U_\chi\longrightarrow \hat{H}^1(U,P_{L'_n,\chi}), \tag{21}$$

where we omit the index p which usually stands for p-completion. Let J be the subgroup of $I_{L'_n}$ which is generated by the primes above l_1 and l_2 (these are exactly the primes that ramify in L'_n/E).

Since χ is non-trivial on $D\cap H$ one has $J_\chi = 0$ and hence $I^U_{L'_n,\chi} = I_{E,\chi}$. As a consequence of (21), it is therefore sufficient to show that

$$P^U_{L'_n,\chi} = P_{E,\chi} \quad \text{and} \quad \hat{H}^1(U,P_{L'_n,\chi}) = 0. \tag{22}$$

To that end we consider the exact sequence

$$0\longrightarrow \mathcal{O}^\times_{L'_n}\longrightarrow {L'_n}^\times\longrightarrow P_{L'_n}\longrightarrow 0.$$

By taking χ-eigenspaces and U-invariants Hilbert's Theorem 90 yields the long exact

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cohomology sequence

$$0 \longrightarrow \mathcal{O}_{E,\chi}^{\times} \longrightarrow E_{\chi}^{\times} \longrightarrow P_{L'_n,\chi}^{U} \longrightarrow \hat{H}^1(U, \mathcal{O}_{L'_n,\chi}^{\times}) \longrightarrow 0 \longrightarrow \hat{H}^1(U, P_{L'_n,\chi})$$
$$\longrightarrow \hat{H}^2(U, \mathcal{O}_{L'_n,\chi}^{\times}).$$

From [33, Th. 4.12] we deduce that $\mathcal{O}_{L'_n,\chi}^{\times} = \mu(L'_n)_\chi \times \mathcal{O}_{(L'_n)^+,\chi}^{\times}$ (recall that we always work with p-completion for an odd prime p). Since χ is odd we have $\mathcal{O}_{L'_n,\chi}^{\times} = \mu(L'_n)_\chi$. But $\mu(L'_n)_\chi = \mu(L'_n)_{p,\chi} =< \zeta_{p^{n+1}} >_\chi = 1$, since G' acts trivially on $\zeta_{p^{n+1}} \in \mathbb{Q}'_n$ and ξ is non-trivial. Consequently $\hat{H}^i(U, \mathcal{O}_{L'_n,\chi}^{\times}) = 0$ for all integers i and (22) follows. □

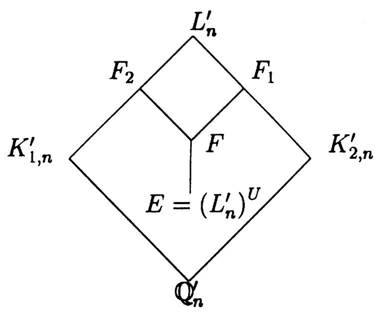
LEMMA 3.13. *Let E and F be fields with $L_n'^{G_p} \subseteq E \subseteq F \subseteq L'_n$ and such that there exists a chain of fields $E = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_s = F$ in which F_1/F_0 is unramified, and for each integer i with $2 \leqslant i \leqslant s$ the extension F_i/F_{i-1} contains no non-trivial unramified subextension. Then $\hat{H}^0(F/E, \mathrm{cl}(F)_\chi) = 0$.*

Proof. From Lemma 3.12 and [33, Th. 10.1] we derive

$$\hat{H}^0(F/E, \mathrm{cl}(F)_\chi) = \mathrm{cl}(E)_\chi / \mathrm{N}_{F_1/E}(\mathrm{cl}(F_1)_\chi),$$

and by global class field theory one has $\mathrm{cl}(E)/\mathrm{N}_{F_1/E}(\mathrm{cl}(F_1)) \simeq \mathrm{Gal}(F_1/E)$. Now since $\mathrm{Gal}(F_1/\mathbb{Q}_n)$ is Abelian, the group $H \subseteq \mathrm{Gal}(E/\mathbb{Q}_n)$ acts trivially on $\mathrm{Gal}(F_1/E)$ and hence $\mathrm{Gal}(F_1/E)_\chi = 0$. This therefore implies that $\mathrm{cl}(E)_\chi = \mathrm{N}_{F_1/E}(\mathrm{cl}(F_1))_\chi$ and hence that $\hat{H}^0(F/E, \mathrm{cl}(F)_\chi) = 0$. □

Proof of Proposition 3.9. By [8, Th. 9] it suffices to show that $\hat{H}^i(U, \mathrm{cl}(L'_n)_\chi) = 0$ for both $i \in \{0, 1\}$ and all subgroups U of G_p. We fix a subgroup U of G_p and set $E = L_n'^{U}$. For each $i \in \{1, 2\}$ let F_i denote the inertia field of primes above l_i in L'_n/E and set $F := F_1 \cap F_2$. With obvious notation we have the following diagram of fields.



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We observe that all primes above l_2 are totally ramified in L'_n/F_2, that all primes above l_1 are totally ramified in F_2/F and that F/E is unramified. Therefore Lemma 3.13 implies that $\hat{H}^0(U, \mathrm{cl}(L'_n)_\chi) = 0$.

So we are left to show that $\hat{H}^1(U, \mathrm{cl}(L'_n)_\chi) = 0$. The inflation-restriction exact sequence for the subgroup $U_1 = \mathrm{Gal}(L'_n/F)$ gives

$$0 \longrightarrow \hat{H}^1(U/U_1, \mathrm{cl}(L'_n)_\chi^{U_1}) \longrightarrow \hat{H}^1(U, \mathrm{cl}(L'_n)_\chi) \longrightarrow \hat{H}^1(U_1, \mathrm{cl}(L'_n)_\chi),$$

and so it is certainly sufficient to show that both

(i) $\hat{H}^1(U_1, \mathrm{cl}(L'_n)_\chi) = 0$

(ii) $\hat{H}^1(U/U_1, \mathrm{cl}(L'_n)_\chi^{U_1}) = 0$.

To prove (i) we set $U_2 := \mathrm{Gal}(L'_n/F_2)$ and consider the exact sequence

$$0 \longrightarrow \hat{H}^1(U_1/U_2, \mathrm{cl}(L'_n)_\chi^{U_2}) \longrightarrow \hat{H}^1(U_1, \mathrm{cl}(L'_n)_\chi) \longrightarrow \hat{H}^1(U_2, \mathrm{cl}(L'_n)_\chi). \tag{23}$$

By Lemma 3.12 we have $\mathrm{cl}(L'_n)_\chi^{U_2} = \mathrm{cl}(F_2)_\chi$. The groups $\hat{H}^0(U_1/U_2, \mathrm{cl}(F_2)_\chi)$ and $\hat{H}^0(U_2, \mathrm{cl}(L'_n)_\chi)$ are both trivial as a consequence of Lemma 3.13. Since U_1/U_2 and U_2 are cyclic a Herbrand quotient argument therefore implies that the second and fourth terms in (23) are trivial, and this proves (i).

We now choose a chain of subfields $E = E_0 \subseteq E_1 \subseteq \ldots \subseteq E_t = F$ such that each of the extensions E_{i+1}/E_i is cyclic. Then inflation-restriction together with Lemma 3.12 leads to exact sequences

$$0 \longrightarrow \hat{H}^1(E_1/E, \mathrm{cl}(E_1)_\chi) \longrightarrow \hat{H}^1(U/U_1, \mathrm{cl}(F)_\chi) \longrightarrow \hat{H}^1(F/E_1, \mathrm{cl}(F)_\chi).$$

By Lemma 3.13 we have $\hat{H}^0(E_1/E, \mathrm{cl}(E_1)_\chi) = 0$ and since E_1/E is cyclic this implies that $\hat{H}^1(E_1/E, \mathrm{cl}(E_1)_\chi) = 0$. To prove (ii) it therefore remains to show that $\hat{H}^1(F/E_1, \mathrm{cl}(F)_\chi) = 0$ and to prove this one can proceed by induction (over t). □

We conclude this subsection with the following observation. Set $B := \mathrm{Gal}(M_0/L_0)$.

LEMMA 3.14. *For any non-trivial character ξ of G' there is a canonical isomorphism $B_\xi \simeq Y_\xi/TY_\xi$.*

Proof. By Iwasawa theory one has $\mathrm{Gal}(M_0/L_\infty) \simeq Y/TY$ (cf. [33, p. 291]). In addition, G acts trivially on Γ since $L_\infty/\mathbb{Q}$ is Abelian and hence the lemma follows upon taking ξ-eigenspaces of the canonical extension

$$0 \longrightarrow \mathrm{Gal}(M_0/L_\infty) \longrightarrow B \longrightarrow \Gamma \longrightarrow 0.$$ □

3.4. PROOF OF THEOREM 3.7

In this section we prove Theorem 3.7. We continue to use the notations of Sections 3.2 and 3.3. In particular, p is a prime which divides $|G|$ and is coprime to $2l_1l_2$.

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We fix a character ξ of G' which satisfies the assumptions of Theorem 3.7. Our proof of this theorem closely follows [16, §7].

Let T be a set of finite places of L which contains all places above p and also $S(L) \setminus S_\infty(L)$. For a finite place v of L we write $L_v^\times$ for the multiplicative group of the completion L_v of L with respect to v. We let $L_{v,p}^\times$ denote the p-completion of $L_v^\times$ and define an idèle group

$$J_{p,T} := \prod_{v\in T, v\nmid p} \mathbb{Z}_p \times \prod_{v|p} L_{v,p}^\times.$$

By global class field theory one has

$$\mathrm{Gal}(F/L) \simeq J_L/\overline{L^\times \prod_{v\nmid p} U_v},$$

where J_L denotes the idèles of L and F the maximal Abelian extension of L unramified outside p (cf. [33, proof of Th. 13.4]). By taking p-completion one shows that

$$B \simeq \frac{\prod_{v\nmid p} \mathbb{Z}_p \times \prod_{v|p} L_{v,p}^\times}{\overline{L^\times}}.$$

Since S is admissible and $S(L) \setminus S_\infty(L) \subseteq T$ it easily follows that $B \simeq J_{p,T}/\overline{U_T}$, where (by abuse of notation) U_T denotes the group of $T \cup S_\infty(L)$-units of L and overbar stands for 'closure of image'. We define

$$J' := \begin{cases} J_{p,S(L)\setminus S_\infty(L)}, & \text{if } p \in S,\\ \prod_{v\in S(L)\setminus S_\infty(L)} \mathbb{Z}_p \times \prod_{v|p} \mathcal{O}_{v,p}^\times, & \text{otherwise,} \end{cases}$$

where $\mathcal{O}_{v,p}^\times$ is the p-completion of the multiplicative group of the valuation ring $\mathcal{O}_v$ of L_v. Since S is admissible one checks that in both cases $J_{p,T}/\overline{U_T} \simeq J'/\overline{U_S}$. Recall now the definition of φ_S in (19) and set

$$P := \mathrm{im}(\varphi_S) = \langle \eta_1, \eta_2, \eta_3, x_1, \ldots, x_{s'} \rangle_{\mathbb{Z}[G]},$$
$$C := \langle \eta_1, \eta_2, \eta_3 \rangle_{\mathbb{Z}[G]},$$

so that $\mathrm{cok}(\varphi_S) = U_S/P$. By using the fact that $(\eta_1) = \mathfrak{l}_1$ and $(\eta_2) = \mathfrak{l}_2$ one can show that there is a natural exact sequence

$$1 \longrightarrow \left((\prod_{v|p} \mathcal{O}_{v,p}^\times)/\bar{C} \right)_\xi \longrightarrow (J'/\bar{P})_\xi \longrightarrow (\mathbb{Z}_p[S'(L)]/h \cdot \mathbb{Z}_p[S'(L)])_\xi \longrightarrow 0. \tag{24}$$

(In this context we remark that the sequence (10) in [16] is only correct after multiplying by e_ξ where ξ is a non-trivial character G'.) Note that $\mathbb{Z}_p[S'(L)]$ is a free $\mathbb{Z}_p[G]$-module of rank s'.

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On the other hand one has an obvious exact sequence

$$1 \longrightarrow (\overline{U_S}/\bar{P})_\xi \longrightarrow (J'/\bar{P})_\xi \longrightarrow (J'/\overline{U_S})_\xi \simeq B_\xi \longrightarrow 1. \tag{25}$$

The precise choice of S ensures $X_{S,p,\xi}$ is cohomologically trivial and the exactness of diagram (8) therefore implies that the same is true for both $U_{S,p,\xi}$ and $X_S(-2)_{p,\xi}$. Since φ_S is injective it follows that $P_{p,\xi}$ is cohomologically trivial and since $P = C \oplus x_1^{\mathbb{Z}[G]} \oplus \ldots \oplus x_{s'}^{\mathbb{Z}[G]}$ this means that $C_{p,\xi}$ is also cohomologically trivial. Finally, the module $\prod_{v|p} \mathcal{O}_{v,p}^\times \simeq \mathrm{ind}_{\mathrm{G_v}}^{\mathrm{G}} \mathcal{O}_{\mathrm{v,p}}^\times$ is cohomologically trivial since p is unramified in L, and so (24) implies that $(J'/\bar{P})_\xi$ is cohomologically trivial. It follows that all terms in (24) and (25) are of projective dimension at most 1 over $\mathbb{Z}_p(\xi)[G_p]$, and so [16, Cor. 1.2] implies that

$$\begin{aligned}&\mathrm{Fitt}_{\mathbb{Z}_p(\xi)[G_p]}\Big((U_S/P)_{p,\xi}\Big)\\ &\quad = h^{s'} \cdot \mathrm{Fitt}_{\mathbb{Z}_p(\xi)[G_p]}\left(((\prod_{v|p} \mathcal{O}_{v,p}^\times)/\bar{C})_\xi\right) \cdot \big(\mathrm{Fitt}_{\mathbb{Z}_p(\xi)[G_p]}(B_\xi)\big)^{-1}.\end{aligned}$$

It therefore remains to prove that

$$\mathrm{Fitt}_{\mathbb{Z}_p(\xi)[G_p]}(B_\xi) = \mathrm{Fitt}_{\mathbb{Z}_p(\xi)[G_p]}\left(((\prod_{v|p} \mathcal{O}_{v,p}^\times)/\bar{C})_\xi\right). \tag{26}$$

Still following [16] we will prove (26) by explicitly computing the left and right hand sides in terms of Iwasawa power series. For a finite Abelian extension $M/\mathbb{Q}$ with conductor m and any integer a with $(a, m) = 1$ we write $\sigma(a)$ for the associated element of $\mathrm{Gal}(\mathbb{Q}(m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$ and also for its restriction to M. For a character ψ of $\mathrm{Gal}(M/\mathbb{Q})$ we will usually write $\psi(a)$ in place of $\psi(\sigma(a))$.

The Stickelberger element for $L'_n/\mathbb{Q}$ is defined by

$$\theta_n := \frac{1}{p^{n+1} l_1^a l_2^b} \sum_{\substack{0<s<p^{n+1} l_1^a l_2^b \\ (s, p l_1 l_2)=1}} s\sigma(s)^{-1} \in \mathbb{Q}[\mathrm{Gal}(L'_n/\mathbb{Q})].$$

(cf. [33, §6.2]). By using the argument of [33, Prop. 7.6] one shows that $\omega\xi^{-1}(\theta_n) \in \mathbb{Z}_p(\xi)[G_p \times \mathrm{Gal}(L_n/L)]$ and that $\omega\xi^{-1}(\theta_m)$ maps to $\omega\xi^{-1}(\theta_n)$ for each $m \geqslant n$ under the projection which is induced by the restriction morphism $\mathrm{Gal}(L_m/L) \to \mathrm{Gal}(L_n/L)$. Hence we can define an element

$$F := -\varprojlim_n \omega\xi^{-1}(\theta_n) \in \Lambda(\xi)[G_p].$$

Let ψ be any character of G which extends ξ and write $f_{\omega\psi^{-1}}$ for the (primitive) Iwasawa power series which is associated to the p-adic L-function $L_p(s, \psi)$ (note that in [33, Th. 7.10] $f_{\omega\psi^{-1}}(T)$ is denoted $f(T, \psi)$).

LEMMA 3.15. *Let ψ be any non-trivial Abelian character of G. Then*

$$\psi^{-1}(F) = -\varprojlim_n \omega\psi^{-1}(\theta_n)$$
$$= \begin{cases} (1-\omega^{-1}\psi(l_2))f_{\omega\psi^{-1}}(T), & \text{if } f_\psi = l_1^s, s > 0, \\ (1-\omega^{-1}\psi(l_1))f_{\omega\psi^{-1}}(T), & \text{if } f_\psi = l_2^t, t > 0, \\ f_{\omega\psi^{-1}}(T), & \text{if } f_\psi = l_1^s l_2^t, s > 0, t > 0. \end{cases}$$

Proof. By the construction of $f_{\omega\psi^{-1}}$ (cf. [33, §. 7.2]) it suffices to show

$$\psi^{-1}(\theta_n) = \begin{cases} (1-\omega^{-1}\psi(l_2))\psi^{-1}(\theta(p^{n+1}l_1^s)), & \text{if } f_\psi = l_1^s, s > 0, \\ (1-\omega^{-1}\psi(l_1))\psi^{-1}(\theta(p^{n+1}l_2^t)), & \text{if } f_\psi = l_2^t, t > 0, \\ \psi^{-1}(\theta(p^{n+1}l_1^s l_2^t)), & \text{if } f_\psi = l_1^s l_2^t, s > 0, t > 0, \end{cases}$$

where we set $\theta(q) = \frac{1}{q}\sum_{0<s<q,(s,q)=1} s\sigma(s)^{-1}$. This is a straightforward computation which we leave to the reader. □

The assumptions of Theorem 3.7 ensure that the Euler factors which occur in Lemma 3.15 are units in $\mathbb{Z}_p(\psi)$. Indeed, if for example $f_\psi = l_1^s$ with $s > 0$, then $\omega^{-1}\psi$ is a character of $\mathrm{Gal}(K_1'/\mathbb{Q})$. But the decomposition group of l_2 in $K_1'/\mathbb{Q}$ is generated by $\sigma(l_2)$, and so the assumption of Theorem 3.7 implies that $\omega^{-1}\psi(l_2)$ is a (non-trivial) root of unity whose order is not a power of p.

For any commutative $\mathbb{Z}_p$-algebra R we let * denote the involution of $R[[T]][G_p] = R[[\Gamma]][G_p]$ which is induced by $\gamma \mapsto \omega_\infty(\gamma)\gamma^{-1}$ for $\gamma \in \Gamma$ and $\sigma \mapsto \sigma^{-1}$ for $\sigma \in G_p$. Note that $\psi(F^*) = \psi^{-1}(F)^*$ for any non-trivial character ψ of G.

The required equality (26) is an immediate consequence of the following two results.

THEOREM 3.16. $\mathrm{Fitt}_{\mathbb{Z}_p(\xi)[G_p]}(B_\xi) = (F^*(0))$.

THEOREM 3.17.

$$\mathrm{Fitt}_{\mathbb{Z}_p(\xi)[G_p]}\left((\prod_{v|p} \mathcal{O}_{v,p}^\times/\bar{C})_\xi \right) = (F^*(0)).$$

Proof of Theorem 3.16. By Lemma 3.14 and [23, App. 4] it is enough to show that $\mathrm{Fitt}_{\Lambda(\xi)[G_p]}(Y_\xi) = (F^*)$. The key to proving this equality is a purely algebraic observation of Greither. Indeed, [16, Lem. 3.7] implies it is sufficient for us to prove that

(i) $\mathrm{Fitt}_{\Lambda(\psi)}(Y_\psi) = (\psi(F^*))$ for all characters ψ of G which extend ξ,
(ii) Y_ξ/pY_ξ is finite

For a character ψ of G and a torsion $\Lambda(\psi)$-module M we write $\mathrm{char}(M)$ for the characteristic ideal of M over the two-dimensional regular local ring $\Lambda(\psi)$. By [16, Lem. 3.5] and Kummer duality one has $\mathrm{Fitt}_{\Lambda(\psi)}(Y_\psi) = \mathrm{char}(Y_\psi) = \mathrm{char}(X_{\infty,\chi})^*$,

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where $X_\infty = \varprojlim \mathrm{cl}(L'_n)_p$ is the Galois group of the maximal Abelian unramified pro-p extension of L'_∞ and $\chi := \omega\psi^{-1}$. From the proof of the Iwasawa conjecture in this case (cf. [23], [34]) one has $\mathrm{char}(X_{\infty,\chi})^* = f_\chi(T)^*$, and so Lemma 3.15 implies that

$$\mathrm{Fitt}_{\Lambda(\psi)}(Y_\psi) = (f_\chi(T)^*) = \begin{cases} ((1-\omega^{-1}\psi(l_2))^{-1}\psi(F^*)), & \text{if } f_\psi = l_1^s,\ s > 0, \\ ((1-\omega^{-1}\psi(l_1))^{-1}\psi(F^*)), & \text{if } f_\psi = l_2^t,\ t > 0, \\ (\psi(F^*)), & \text{if } f_\psi = l_1^s l_2^t,\ s > 0,\ t > 0. \end{cases}$$

This implies (i) because the factors $(1-\omega^{-1}\psi(l_i))$ are units of $\mathbb{Z}_p(\psi)$ for both $i \in \{1, 2\}$.

Finally we observe that, just as in [16], the finiteness of Y_ξ/pY_ξ can be deduced by using Kummer duality and the Theorem of Ferrero-Washington. $\square$

Proof of Theorem 3.17. From [33, Th. 7.10] and Lemma 3.15 one has

$$\begin{aligned} \psi(F^*)(0) &= \psi^{-1}(F)^*(0) \\ &= \begin{cases} (1-\omega^{-1}\psi(l_2))L_p(1,\psi), & \text{if } f_\psi = l_1^s,\ s > 0, \\ ((1-\omega^{-1}\psi(l_1))L_p(1,\psi), & \text{if } f_\psi = l_2^t,\ t > 0, \\ L_p(1,\psi), & \text{if } f_\psi = l_1^s l_2^t,\ s > 0,\ t > 0. \end{cases} \end{aligned} \tag{27}$$

Let $\mathcal{E}_{L,p}$ denote the torsion subgroup of $(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L)^\times$. Since $L_v/\mathbb{Q}_p$ is unramified for places $v \mid p$ the module $\prod_{v|p} \mathcal{O}_{v,p}^\times$ is canonically isomorphic to the quotient $(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L)^\times/\mathcal{E}_{L,p}$. Now $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L \simeq \prod_{v|p} \mathcal{O}_{L_v}$ and so [24, Satz (5.5)] implies that the p-adic logarithm induces a G-equivariant isomorphism

$$(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L)^\times/\mathcal{E}_{L,p} \xrightarrow{\simeq} p(\mathbb{Z}_p \otimes \mathcal{O}_L).$$

In order to specify an explicit $\mathbb{Z}_p[G]$-generator of $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_L$ we recall the main result of [21] as it is stated in [9, pp. 124–5] (see also [22]). For the moment, let $N/\mathbb{Q}$ be any finite Abelian extension with group G. For each complex Abelian character χ of G we decompose the conductor f_χ of χ as a product $f_\chi = f_{\chi,t} \cdot f_{\chi,w}$ with $f_{\chi,t} := \prod_p p$, where here the product extends over the primes p such that $p \mid f_\chi$ and $p^2 \nmid f_\chi$, and we write e_χ for the idempotent $|G|^{-1}\sum_{g\in G}\chi(g^{-1})g$ of $\mathbb{C}[\mathbb{G}]$. Complex Abelian characters χ and ϕ are said to be equivalent if $f_{\chi,w} = f_{\phi,w}$. For each equivalence class Φ of this relation we set $e_\Phi := \sum_{\phi\in\Phi} e_\phi$, $f_\Phi := \mathrm{lcm}\{f_\phi : \phi \in \Phi\}$ and $\ker(\Phi) := \bigcap_{\phi\in\Phi}\ker(\phi)$. To each character $\phi \in \Phi$ one associates a Gauss sum $\tau_\Phi(\phi) := \sum_{x\in(\mathbb{Z}/f_\Phi)^\times} \phi(x)\zeta_{f_\Phi}^x$ and sets

$$T_\Phi := \frac{1}{[N^{\ker(\Phi)} : \mathbb{Q}]}\sum_{\phi\in\Phi}\tau_\Phi(\phi).$$

Then Leopoldt's famous 'Hauptsatz' is the equality

$$\mathcal{O}_N = \mathcal{A}(N/\mathbb{Q})\left(\sum_\Phi T_\Phi\right) \tag{28}$$

where here Φ runs over the equivalence classes of complex Abelian characters of G and $\mathcal{A}(N/\mathbb{Q})$ is the $\mathbb{Z}$-order $\sum_\Phi e_\Phi \mathbb{Z}[G]$ in $\mathbb{Q}[G]$.

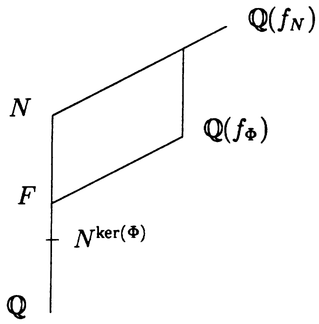
LEMMA 3.18. *Let $N/\mathbb{Q}$ be a finite Abelian extension and set*

$$y := \sum_\Phi \frac{1}{|\ker(\Phi)|} T_\Phi,$$

where here Φ runs over the equivalence classes of complex Abelian characters of $\mathrm{Gal}(N/\mathbb{Q})$. Then for each prime p which does not ramify in $N/\mathbb{Q}$ the element y is a $\mathbb{Z}_p[\mathrm{Gal}(N/\mathbb{Q})]$-generator of $\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_N$.

Proof. Set $G := \mathrm{Gal}(N/\mathbb{Q})$, and fix a prime p which does not ramify in $N/\mathbb{Q}$. One has $e_\Phi T_\Psi = \delta_{\Phi,\Psi} T_\Psi$ (Kronecker delta) and so (28) implies that it suffices to prove $p \nmid |\ker(\Phi)|$ and $\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{A}(N/\mathbb{Q}) = \mathbb{Z}_p[G]$.

Let $f_N = p_1^{e_1} \cdots p_s^{e_s}$ be the conductor of N and set $F := N \cap \mathbb{Q}(f_\Phi)$. Then we have the following diagram of fields:



The definition of f_Φ implies that $[\mathbb{Q}(f_N) : \mathbb{Q}(f_\Phi)] = p_1^{f_1} \cdots p_s^{f_s}$ with $0 \leqslant f_i < e_s$ for each i with $1 \leqslant i \leqslant s$, and hence p does not divide $[N : F]$. In addition [22, Lem. 1d] implies that $F = N^{\ker(\Phi)}$ and so $p \nmid |\ker(\Phi)|$.

Regarding e_Φ as an idempotent in $\mathbb{Q}_p[\mathrm{Gal}(F/\mathbb{Q})]$ it is therefore enough to prove that $e_\Phi \in \mathbb{Z}_p[\mathrm{Gal}(F/\mathbb{Q})]$. Let f'_Φ denote the maximal square-free divisor of f_Φ and set $F' := N \cap \mathbb{Q}(f'_\Phi)$. Then $p \nmid [\mathbb{Q}(f_\Phi) : \mathbb{Q}(f'_\Phi)]$ and so $p \nmid [F']$. It follows that $\mathbb{Z}_p[\mathrm{Gal}(F/F')]$ is the maximal $\mathbb{Z}_p$-order in $\mathbb{Q}_p[\mathrm{Gal}(F/F')]$ and so it suffices to prove that e_Φ belongs to $\mathbb{Q}_p[\mathrm{Gal}(F/F')] \subseteq \mathbb{Q}_p[\mathrm{Gal}(F/\mathbb{Q})]$. This is in turn an easy exercise which we leave to the reader. $\square$

We now return to the proof of Theorem 3.17 and use Lemma 3.18 to define an isomorphism $\alpha : (\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_L)^\times / \mathcal{E}_{L,p} \xrightarrow{\sim} \mathbb{Z}_p[G]$ by $\log_p(u) = p\alpha(u) \cdot y$ for each $u \in (\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_L)^\times$. This isomorphism implies that the first Fitting ideal of $\big(((\mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_L)^\times / \mathcal{E}_{L,p}) / \bar{C}\big)_\xi$ in $\mathbb{Z}_p(\xi)[G_p]$ is equal to

$$\xi(\alpha(\bar{C})) = \langle \xi(\alpha(\eta_1)), \xi(\alpha(\eta_2)), \xi(\alpha(\eta_3)) \rangle_{\mathbb{Z}_p(\xi)[G_p]}.$$

LEMMA 3.19. *Let ψ be any non-trivial complex Abelian character of G. Then one has:*

(a) $$\psi(\alpha(\eta_1)) = \begin{cases} [L\colon K_1]\frac{1}{p-\psi(p)}\psi(l_2)L_p(1,\psi), & \text{if } f_\psi = l_1^s, s>0,\\ 0, & \text{otherwise;}\end{cases}$$

(b) $$\psi(\alpha(\eta_2)) = \begin{cases} [L\colon K_2]\frac{1}{p-\psi(p)}\psi(l_1)L_p(1,\psi), & \text{if } f_\psi = l_2^t, t>0,\\ 0, & \text{otherwise;}\end{cases}$$

(c) $$\psi(\alpha(\eta_3)) = \begin{cases} -\psi(il_2^b + jl_1^a)\frac{1}{p-\psi(p)}L_p(1,\psi), & \text{if } f_\psi = l_1^s l_2^t, s>0, t>0,\\ \psi(il_2^b + jl_1^a)\frac{1}{p-\psi(p)}(\psi(l_2)-1)L_p(1,\psi), & \text{if } f_\psi = l_1^s, s>0,\\ \psi(il_2^b + jl_1^a)\frac{1}{p-\psi(p)}(\psi(l_1)-1)L_p(1,\psi), & \text{if } f_\psi = l_2^t, t>0.\end{cases}$$

Proof. For each $k \in \{1,2,3\}$ we set $\beta_k := p\alpha(\eta_k) \in \mathbb{Z}_p[G]$ so that $\log_p(\eta_k) = \beta_k y$. These definitions imply that

$$\sum_{\sigma\in G}\bar{\psi}(\sigma)\log_p(\sigma\eta_k) = \psi(\beta_k)\cdot\sum_{\sigma\in G}\bar{\psi}(\sigma)\sigma y. \tag{29}$$

By an entirely standard computation one finds that if $\psi \in \Psi$, then

$$\sum_{\sigma\in G}\bar{\psi}(\sigma)\sigma y = \sum_{x\in(\mathbb{Z}/f_\Psi)^\times}\bar{\psi}(x)\zeta_{f_\Psi}^x = \tau_\Psi(\bar{\psi}).$$

Note that the Gauss sums $\tau_\Psi(\bar{\psi})$ are usually not primitive. Indeed, according to the conductor of ψ, one has

$$\sum_{\sigma\in G}\bar{\psi}(\sigma)\sigma y = \begin{cases} -\bar{\psi}(l_2)\tau(\bar{\psi}), & \text{if } f_\psi = l_1^s, s>0,\\ -\bar{\psi}(l_1)\tau(\bar{\psi}), & \text{if } f_\psi = l_2^t, t>0,\\ \tau(\bar{\psi}), & \text{otherwise,}\end{cases} \tag{30}$$

where here $\tau(\bar{\psi})$ denotes the primitive Gauss sum as defined in [33, Th. 5.18]. Multiplying (29) by $\tau(\psi)/f_\psi$ we deduce from [33, Lem. 4.7 and 4.8] that

$$\begin{aligned} \tfrac{\tau(\psi)}{l_1^s}\textstyle\sum_{\sigma\in G}\bar{\psi}(\sigma)\log_p(\sigma\eta_k) &= -\bar{\psi}(l_2)\cdot\psi(\beta_k), \text{ if } f_\psi = l_1^s, s>0,\\ \tfrac{\tau(\psi)}{l_2^t}\textstyle\sum_{\sigma\in G}\bar{\psi}(\sigma)\log_p(\sigma\eta_k) &= -\bar{\psi}(l_1)\cdot\psi(\beta_k), \text{ if } f_\psi = l_2^t, t>0,\\ \tfrac{\tau(\psi)}{l_1^s l_2^t}\textstyle\sum_{\sigma\in G}\bar{\psi}(\sigma)\log_p(\sigma\eta_k) &= \psi(\beta_k), \text{ otherwise.}\end{aligned} \tag{31}$$

We must now compute more explicitly the left-hand side of (31). Recalling

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$\eta_1 = \mathrm{N}_{\mathbb{Q}(l_1^a)/K_1}(1 - \zeta_{l_1^a}) \in K_1$ one finds that

$$\sum_{\sigma \in G} \bar{\psi}(\sigma) \log_p(\sigma\eta_1) = \begin{cases} [L: K_1] \sum_{y \in (\mathbb{Z}/l_1^s)^\times} \bar{\psi}(y) \log_p(1 - \zeta_{l_1^s}^y), & \text{if } f_\psi = l_1^s, s > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

There is a completely analogous result for $k = 2$ in (31).

Finally we have to consider the above sum for $\eta_3 = \mathrm{N}_{\mathbb{Q}(l_1^a l_2^b)/L}(1 - \zeta_{l_1^a}^i \zeta_{l_2^b}^j)$. In this case, a standard computation (depending on the precise choice of i and j as in (18)) leads to equalities

$$\sum_{\sigma \in G} \bar{\psi}(\sigma) \log_p(\sigma\eta_3)$$
$$= \begin{cases} (\psi(l_2) - 1) \sum_{y \in (\mathbb{Z}/l_1^s)^\times} \bar{\psi}(y) \log_p(1 - \zeta_{l_1^s}^y), & \text{if } f_\psi = l_1^s, s > 0, \\ (\psi(l_1) - 1) \sum_{y \in (\mathbb{Z}/l_2^t)^\times} \bar{\psi}(y) \log_p(1 - \zeta_{l_2^t}^y), & \text{if } f_\psi = l_2^t, t > 0, \\ \psi(il_2^b + jl_1^a) \sum_{y \in (\mathbb{Z}/l_1^s l_2^t)^\times} \bar{\psi}(y) \log_p(1 - \zeta_{l_1^s l_2^t}^y), & \text{if } f_\psi = l_1^s l_2^t, s > 0, t > 0. \end{cases} \tag{33}$$

For $k = 1$ we derive from (31), (32) and [33, Th. 5.18] that

$$\psi(\beta_1) = \begin{cases} [L: K_1]\psi(l_2)\left(1 - \frac{\psi(p)}{p}\right)^{-1} L_p(1, \psi), & \text{if } f_\psi = l_1^s, s > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Analogously we get

$$\psi(\beta_2) = \begin{cases} [L: K_2]\psi(l_1)\left(1 - \frac{\psi(p)}{p}\right)^{-1} L_p(1, \psi), & \text{if } f_\psi = l_2^t, t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, for $k = 3$ we obtain from (31) and (33) together with [33, Th. 5.18]

$$\psi(\beta_3) = \begin{cases} \psi(l_2)\left(1 - \frac{\psi(p)}{bp}\right)^{-1} (\psi(l_2) - 1) L_p(1, \psi), & \text{if } f_\psi = l_1^s, s > 0, \\ \psi(l_1)\left(\frac{1 - \psi(p)}{p}\right)^{-1} (\psi(l_1) - 1) L_p(1, \psi), & \text{if } f_\psi = l_2^t, t > 0, \\ -\psi(il_2^b + jl_1^a)\left(1 - \frac{\psi(p)}{p}\right)^{-1} L_p(1, \psi), & \text{if } f_\psi = l_1^s l_2^t, s > 0, t > 0. \end{cases}$$

To conclude the proof we now observe that the congruences (18) imply that $\psi(il_2^b + jl_1^a) = \psi(l_2)$ if $f_\psi = l_1^s$ and that $\psi(il_2^b + jl_1^a) = \psi(l_1)$ if $f_\psi = l_2^t$. □

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Note that $\xi(e_\psi) = 0$ if ψ is not an extension of ξ and that $e_\xi = \sum_{\psi|\xi} e_\psi$. From (27) and Lemma 3.19 we conclude that

(a) $\xi(\alpha(\eta_1))$

$$= \xi\left([L\colon K_1]\frac{\sigma(l_2)}{p-\sigma(p)}(1-\omega^{-1}(l_2)\sigma(l_2))^{-1}F^*(0)\sum_{\substack{\psi,f_\psi=l_1^s\\ s>0}} e_\psi\right),$$

(b) $\xi(\alpha(\eta_2))$

$$= \xi\left([L\colon K_2]\frac{\sigma(l_1)}{p-\sigma(p)}(1-\omega^{-1}(l_1)\sigma(l_1))^{-1}F^*(0)\sum_{\substack{\psi,f_\psi=l_2^t\\ t>0}} e_\psi\right),$$

(c) $$\xi(\alpha(\eta_3)) = \xi\left(-\frac{\sigma(il_2^b+jl_1^a)}{p-\sigma(p)}F^*(0)\sum_{\substack{\psi,f_\psi=l_1^s l_2^t\\ s>0,t>0}} e_\psi - \frac{\sigma(il_2^b+jl_1^a)}{p-\sigma(p)}\frac{1-\sigma(l_2)}{1-\omega^{-1}(l_2)\sigma(l_2)}F^*(0)\sum_{\substack{\psi,f_\psi=l_1^s\\ s>0}} e_\psi - \frac{\sigma(il_2^b+jl_1^a)}{p-\sigma(p)}\frac{1-\sigma(l_1)}{1-\omega^{-1}(l_1)\sigma(l_1)}F^*(0)\sum_{\substack{\psi,f_\psi=l_2^t\\ t>0}} e_\psi\right).$$

We now claim that $\xi(\alpha(\bar{C}))$ is generated by $F^*(0)$. To prove this we first observe that the factor $\sigma(il_2^b+jl_1^a)/(p-\sigma(p))$ is a unit in $\mathbb{Z}_p[G]$. Indeed, it obviously belongs to $\mathbb{Z}_p[G]$ and is a unit in the maximal $\mathbb{Z}_p$-order $\mathcal{M}$ of $\mathbb{Q}_p[G]$, and this suffices since $\mathcal{M}^\times \cap \mathbb{Z}_p[G] = \mathbb{Z}_p[G]^\times$.

We next show that $(F^*(0))$ is generated by $\xi(\alpha(\eta_3))$. To that end we consider three separate cases

(i) $p \mid \frac{l_1-1}{2}$ and $p \mid \frac{l_2-1}{2}$,

(ii) $p \mid \frac{l_1-1}{2}$ and $p \nmid \frac{l_2-1}{2}$,

(iii) $p \nmid \frac{l_1-1}{2}$ and $p \mid \frac{l_2-1}{2}$.

In case (i) we have $\omega(l_1) = \omega(l_2) = 1$ and so it follows immediately that $(\xi(\alpha(\eta_3))) = (F^*(0))$. In case (ii) we note that $G_1 \subseteq G'$. Thus, if there exists a character ψ which extends ξ and which has conductor $f_\psi = l_1^s$ with $s > 0$, then $\xi\,|_{G_1} = 1$.

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Therefore the conductor of ψ is a non-trivial power of l_1 for every extension of ξ, and so (c) implies an equality of $\mathbb{Z}_p(\xi)[G_p]$-ideals

$$(\xi(\alpha(\eta_3))) = \left(\xi\left(\frac{1-\sigma(l_2)}{1-\omega^{-1}(l_2)\sigma(l_2)}\right)F^*(0)\right).$$

It therefore suffices to prove that both $\xi(1-\sigma(l_2))$ and $\xi(1-\omega^{-1}(l_2)\sigma(l_2))$ are units in $\mathbb{Z}_p(\xi)[G_p]$. But the arguments used immediately after Lemma 3.15 imply that these elements are units in the maximal $\mathbb{Z}_p(\xi)$-order of $\mathbb{Q}_p(\xi)[G_p]$, and this in turn implies that they are units in $\mathbb{Z}_p(\xi)[G_p]$.

Still considering the case (ii) we can now assume that no character ψ which extends ξ has conducter of the form l_1^s with $s > 0$. Then we derive from (c) and $\omega(l_1) = 1$ an equality of $\mathbb{Z}_p(\xi)[G_p]$-ideals $(\xi(\alpha(\eta_3))) = (F^*(0))$.

The case (iii) is completely analogous to (ii).

So far we have proved that $(\xi(\alpha(\eta_3))) = (F^*(0))$. Now

$$\sum_{\substack{\psi, f_\psi = l_1^s \\ s>0}} e_\psi = \frac{1}{|G_1|}\sum_{h\in G_1} h - \frac{1}{|G|}\sum_{g\in G} g,$$

and so

$$\xi\left(\sum_{\substack{\psi, f_\psi = l_1^s \\ s>0}} e_\psi\right) = \frac{1}{|G_1|}\xi\left(\sum_{h\in G_1} h\right) \in \mathbb{Q}_p(\xi)[G_{1,p}] \subseteq \mathbb{Q}_p(\xi)[G_p].$$

It follows from equality (a) that $\xi(\alpha(\eta_1)) \in (F^*(0))$ and analogously we obtain $\xi(\alpha(\eta_2)) \in (F^*(0))$.

This concludes the proof of Theorem 3.17 and hence also that of Theorem 3.7.

3.5. THE CASE $p \mid l_1 l_2$

In this subsection we complete the proof of Theorem 1.1 by proving that $T\Omega(L/\mathbb{Q})_p = 0$ if p is equal to either l_1 or l_2.

We set $l = l_1$ and assume that $p = l$ (the case $p = l_2$ being completely analogous). By Proposition 3.1 we may assume that $l \mid |G|$. We set $F := K_2(\zeta_l)^+$, so that $\mathrm{Gal}(F/\mathbb{Q}) \simeq G'$ and $[L : F] = l^{a-1}$. We let F_∞ be the cyclotomic $\mathbb{Z}_l$-extension of F, and we note that $L = F_{a-1}$.

The strategy of proof is once again the same as in [16]. We write $\mathfrak{l}$ for the unique prime of L above l and define an idèle group

$$J := L_{\mathfrak{l},l}^\times \times \prod_{v\in S'(L)} \mathbb{Z}_l.$$

Since the decomposition subgroup of l is G we may consider both $L_\mathfrak{l}^\times$ and $\mathcal{O}_\mathfrak{l}^\times$ as

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$\mathbb{Z}[G]$-modules. The $\mathfrak{l}$-adic valuation gives the short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow \mathcal{O}_{\mathfrak{l}}^{\times} \longrightarrow L_{\mathfrak{l}}^{\times} \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{34}$$

Taking l-completion and then ξ-eigenspaces for a non-trivial character ξ (both functors are exact) shows that $L_{\mathfrak{l},l,\xi}^{\times} = \mathcal{O}_{\mathfrak{l},l,\xi}^{\times}$. In addition, $\mathcal{O}_{\mathfrak{l},l,\xi}^{\times}$ is a free $\mathbb{Z}_l(\xi)[G_l]$-module. Indeed, since $\mathcal{O}_{\mathfrak{l},l,\xi}^{\times}$ is torsion-free it is free over $\mathbb{Z}_l(\xi)[G_l]$ if and only if it is cohomologically trivial over G_l. But for $U \leqslant G_l$ the cohomology sequence for (34) implies $\hat{H}^1(U, \mathcal{O}_{\mathfrak{l}}^{\times})_{\xi} = 0$ by Hilbert's Theorem 90 and $\hat{H}^0(U, \mathcal{O}_{\mathfrak{l}}^{\times})_{\xi} \simeq \hat{H}^0(U, L_{\mathfrak{l}}^{\times})_{\xi}$. Furthermore, $\hat{H}^0(U, L_{\mathfrak{l}}^{\times}) \simeq \mathrm{Gal}(L_{\mathfrak{l}}/L_{\mathfrak{l}}^{U})$. Since $L_{\mathfrak{l}}/\mathbb{Q}_l$ is Abelian and ξ is non-trivial we deduce that $\hat{H}^i(U, \mathcal{O}_{\mathfrak{l}}^{\times})_{\xi} = 0$ for both $i \in \{0, 1\}$ as required.

Recall the modules P and C defined just prior to (24). There is a natural analogue of the exact sequence (24)

$$0 \longrightarrow \left(\mathcal{O}_{\mathfrak{l},l}^{\times}/\bar{C}\right)_{\xi} \longrightarrow \left(J/\bar{P}\right)_{\xi} \longrightarrow (\mathbb{Z}_l[S'(L)]/h \cdot \mathbb{Z}_l[S'(L)])_{\xi} \longrightarrow 0$$

in which all modules are of projective dimension 1 over $\mathbb{Z}_l(\xi)[G_l]$. By combining this sequence with the obvious analogue of (25) we deduce that

$$\mathrm{Fitt}_{\mathbb{Z}_l(\xi)[G_l]}\big((U_S/P)_{l,\xi}\big) = h^{s'} \cdot \mathrm{Fitt}_{\mathbb{Z}_l(\xi)[G_l]}\big((\mathcal{O}_{\mathfrak{l},l}^{\times}/\bar{C})_{\xi}\big) \cdot \big(\mathrm{Fitt}_{\mathbb{Z}_l(\xi)[G_l]}(B_{\xi})\big)^{-1},$$

where B is now the Galois group of the maximal Abelian l-ramified pro-l extension of $L = F_{a-1}$. Taking into account Corollary 2.8, Lemma 3.6 and the fact that $U_S/P = \mathrm{cok}(\varphi_S)$ this equality means it suffices to show that

$$\mathrm{Fitt}_{\mathbb{Z}_l(\xi)[G_l]}(B_{\xi}) = \mathrm{Fitt}_{\mathbb{Z}_l(\xi)[G_l]}\big((\mathcal{O}_{\mathfrak{l},l}^{\times}/\bar{C})_{\xi}\big). \tag{35}$$

Just as in the proof of Lemma 3.14 we derive from [33, p. 291] that

$$B_{\xi} \simeq (Y/\omega_{a-1}(T)Y)_{\xi}$$

where Y is the Galois group of the maximal Abelian l-ramified pro-l extension of F_{∞} and $\omega_n(T) = (1+T)^{l^n} - 1$ for each $n \geqslant 0$. Since the Iwasawa algebra $\Lambda(\xi) = \mathbb{Z}_l(\xi)[[\Gamma]]$ with $\Gamma = \mathrm{Gal}(F_{\infty}/F)$ is regular the $\mathbb{Z}_l$-torsion free $\Lambda(\xi)$-torsion module Y_{ξ} is of projective dimension at most one. Therefore the Fitting ideal and characteristic ideal of Y_{ξ} over $\Lambda(\xi)$ coincide by [16, Lem. 3.5]. The known validity of the Iwasawa conjecture in this case therefore implies that $\mathrm{Fitt}_{\Lambda(\xi)}(Y_{\xi}) = (f_{\omega\xi^{-1}}(T))$, where $f_{\omega\xi^{-1}}(T)$ is the Iwasawa power series associated to $L_l(s, \xi)$. Now $\Lambda(\xi)/\omega_{a-1}(T)\Lambda(\xi) \simeq \mathbb{Z}_l(\xi)[G_l]$ and $B_{\xi} \simeq Y_{\xi}/\omega_{a-1}(T)\Lambda(\xi)$ and so [23, App. 4] implies that $\mathrm{Fitt}_{\mathbb{Z}_l(\xi)[G_l]}(B_{\xi})$ is generated by the image of $f_{\omega\xi^{-1}}(T)$ in $\Lambda(\xi)/\omega_{a-1}(T)\Lambda(\xi)$.

We now set $U_{\infty} := \varprojlim U_n$, where U_n denotes the group of principal units of the completion of F_n with respect to the unique prime above l. Note that $U_{a-1} = \mathcal{O}_{\mathfrak{l},l}^{\times}$. Moreover, [33, Th. 13.56] or [28, §8] implies that

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$\mathrm{Fitt}_{\Lambda(\xi)}(U_\infty/\bar{C}_\infty)_\xi = (f_{\omega\xi^{-1}}(T))$, where C_∞ is the projective limit of the group of cyclotomic units in F_n.

For each integer $n \geqslant 0$ we set $\Gamma_n := \mathrm{Gal}(F_\infty/F_n)$, and for any $\Lambda(\xi)$-module M we write M_{Γ_n} for its Γ_n-coinvariants $M/\omega_n(T)M$. Using [23, App. 4] we conclude that $\mathrm{Fitt}_{\mathbb{Z}_l(\xi)[G_l]}((U_\infty/\bar{C}_\infty)_{\xi,\Gamma_{a-1}})$ is generated by the image of $f_{\omega\xi^{-1}}(T)$ in $\Lambda(\xi)/\omega_{a-1}(T)\Lambda(\xi)$ and so to prove (35) it is now enough to show that

$$(U_\infty/\bar{C}_\infty)_{\xi,\Gamma_{a-1}} = (\mathcal{O}_{\mathfrak{l},l}^\times/\bar{C})_\xi.$$

To that end it suffices to note that by [28, Th. 6.1] one has

$$(U_\infty)_{\xi,\Gamma_{a-1}} = \mathcal{O}_{\mathfrak{l},l,\xi}^\times, \quad (\bar{C}_\infty)_{\xi,\Gamma_{a-1}} = \bar{C}_\xi.$$

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