ON PENDENT DROPS IN A CAPILLARY TUBE

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The energy functional for a pendent drop in a capillary tube is
neither convex nor bounded from below. We obtain local minima of
the energy by making the physically reasonable assumption that
the gravitation or the prescribed volume of the drop is small.

0. Introduction

We consider a drop of prescribed volume $\dot{V}$ hanging from inside the
top of an upside down capillary tube. Let the bounded domain $\Omega \subset \mathbf{R}^n$,
$n \geq 2$, denote the cross-section of the tube and assume that the top of the
tube and the surface of the drop can be represented as graphs of functions
ψ_1 and u on Ω . For convenience of notation it is preferable to make a
change of the coordinate system, that is, we reverse the cylinder and
assume the gravitation to be directed upward. Then ψ_1 becomes the bottom
of the tube and we have

$$(0.1) \qquad u \geq \psi_1 \ , \quad V = \int_\Omega u - \psi_1 \, dx \ .$$

Now the physical principle of virtual work leads to the variational problem

$$(0.2) \qquad E(v) = \int_\Omega \left(1+|Dv|^2\right)^{\frac{1}{2}} dx - \frac{\kappa}{2} \int_\Omega v^2 dx + \int_{\partial\Omega} \beta v \, dH_{n-1}$$

$$\rightarrow \min \text{ in } K$$

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where K is some suitable convex function space. Here $\kappa \geq 0$ is the
capillarity constant and $\beta \in L^{\infty}(\partial\Omega)$ is the cosine of the contact angle
between u and the cylinder walls. E represents the energy of our system
and we have a minus sign in front of κ since the gravitation is upward
directed. We assumed that there is no contribution to the energy from the
bottom of the tube, that is that the liquid will 'wet' the obstacle ψ_1 .

     A natural choice of K would be

$$(0.3) \qquad K_1 = H^{1,\infty}(\Omega) \cap \{v \geq \psi_1\} \cap \left\{\int_\Omega v-\psi_1 dx = V\right\}$$

and in the case of downward directed gravitation the corresponding
variational problem was solved by Gerhardt in [2], [4]. In our case it is
not possible to minimize E in K_1 since the energy is not bounded from
below on this set. However, it was shown in [6], that the corresponding
variational inequality to (0.2) admits a solution under the physically
reasonable assumption that the capillarity constant κ or the volume V
is small enough (see [6] or Chapter 2 below). If we consider these
solutions as critical points of the energy functional, the question arises
whether there are critical points which actually minimize the energy at
least compared with functions which are not too far away.

     To make the terminology 'not too far away' precise, we introduce the
sets

$$(0.4) \qquad K_2 = BV(\Omega) \cap \{\psi_1 \leq v \leq \psi_2\} \cap \left\{\int_\Omega v-\psi_1 dx = V\right\}$$

and

$$(0.5) \quad K_3 = H^{1,\infty}(\Omega) \cap \{\psi_1 \leq v \leq \psi_2\} \cap \left\{\int_\Omega v-\psi_1 dx = V\right\} \cap \{|Dv|_\infty \leq N\}$$

where $\psi_2 > \psi_1$ is a second obstacle and N is some large constant. To
ensure that these sets are not empty, we have to assume

$$(0.6) \qquad \int_\Omega \psi_2 - \psi_1 dx \geq V .$$

Assuming suitable smoothness of the data we can show that the minimizing
problem (0.2) can be solved in K_2 and K_3 for any κ and V .

Moreover, if N and ψ_2 are large enough we can ensure that for suitable small κ or V the constraints ψ_2 and N are not reached by the solution. These solutions can then be considered as hanging drops, which minimize the energy in K_2 respectively K_3. Furthermore, a minimum of E in K_3, which is strictly away from the constraints ψ_2 and N, satisfies a variational inequality as in [6] and has therefore all the regularity properties established in [6] and [7].

Giusti [5] used a second obstacle to show the existence of pendent drops in a different setting.

Notation. We denote by $H^{m,p}(\Omega)$ the Sobolev spaces and by $BV(\Omega)$ the space of functions of bounded variation. H_n is the n-dimensional Hausdorff measure and $|\cdot|_\infty$ the supremum norm on Ω. We shall denote by $c = c(\ldots)$ various constants, whereas indices will be used if a constant recurs at another place.

1. Minima in $BV(\Omega)$

From now on we consider the more general energy functional

$$(1.1) \quad E(v) = \int_\Omega \left(1+|Dv|^2\right)^{\frac{1}{2}}dx + \int_\Omega \int_0^v H(x,\ t)dt dx - \frac{\kappa}{2}\int_\Omega v^2 dx + \int_{\partial\Omega}\beta v dH_{n-1}$$

where

$$(1.2) \qquad\qquad \beta \in L^\infty(\partial\Omega)\ ,\quad H \in C^{0,1}\left(\mathbf{R}^n \times \mathbf{R}\right)$$

satisfy

$$(1.3) \qquad\qquad |\beta| \le 1 - a\ ,\quad a > 0\ ,$$

and

$$(1.4) \qquad \frac{\partial H}{\partial t} \ge 0\ ,\quad \sup_\Omega H(x,\ t) \le \alpha(1+t)\ ,\quad \alpha > 0\ ,\ \text{for all}\ t > 0\ .$$

We assume that $\Omega \subset \mathbf{R}^n$, $n \ge 2$, is a bounded domain with Lipschitz boundary $\partial\Omega$ satisfying an interior sphere condition of radius R.

Here Ω is said to satisfy an ISC of radius R, if for any boundary

point $x_0 \in \partial\Omega$ there exists a ball B of radius R such that $B \subset \Omega$ and $x_0 \in \overline{B}$.

This interior sphere condition is imposed in view of the following result (see [3], Remark 2).

LEMMA 1. *Let* Ω *be as above; then*

$$(1.5) \qquad \int_{\partial\Omega} |v| \, dH_{n-1} \leq \int_{\Omega_\varepsilon} |Dv| \, dx + c_\varepsilon \cdot \int_\Omega |v| \, dx \quad \text{for all} \quad v \in BV(\Omega)$$

where $\Omega_\varepsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \varepsilon\}$, c_ε *depends on* ε , R, $\partial\Omega$ *and* $0 < \varepsilon \leq R/2$ *is arbitrary.*

Then, assuming all these conditions to hold, we have

THEOREM 1. *Let* $\psi_i \in C^{0,1}(\overline{\Omega})$, $i = 1, 2$. *Then the minimizing problem*

$$(1.6) \qquad\qquad\qquad E(v) \to \min \quad in \quad K_2$$

has a solution u *for any* κ *and* V , *if* (0.6) *holds.*

Proof. It is obvious that

$$(1.7) \qquad\qquad\qquad E(v) \geq -C \quad \text{for all} \quad v \in K_2$$

for some constant C , since v is between the two bounded functions ψ_1 and ψ_2 . Now let v_l be a minimizing sequence of the variational problem (1.6). If we define

$$(1.8) \qquad\qquad w = \psi_1 + V \cdot (\psi_2 - \psi_1) \Big/ \left(\int_\Omega \psi_2 - \psi_1 \, dx \right)$$

then $w \in K_2$ in view of (0.6) and eventually we have

$$(1.9) \qquad\qquad\qquad E\big(v_l\big) \leq E(w) \ .$$

Again using $\psi_1 \leq v \leq \psi_2$ we conclude

$$(1.10) \qquad\qquad \int_\Omega |Dv_l| \, dx \leq \int_\Omega \left(1 + |Dv_l|^2\right)^{\frac{1}{2}} dx \leq E(w) + C$$

where the constant depends on $|\psi_2|_\infty$, κ, Ω, $\partial\Omega$ and $|H(\cdot, \psi_i)|_\infty$,

$i = 1, 2$. Thus, the sequence

$$(1.11) \qquad \int_\Omega |Dv_l|\,dx + \int_\Omega |v_l|\,dx$$

is uniformly bounded. Then, as in ([2], Theorem 2.1) we can find a sub-sequence of the v_l converging in $L^1(\Omega)$ to some function $u \in BV(\Omega)$.

It was also shown in [2], that the expression

$$(1.12) \qquad \int_\Omega \left(1+|Dv|^2\right)^{\frac{1}{2}}dx + \int_\Omega \int_0^v H(x, t)\,dt\,dx + \int_{\partial\Omega} \beta v\,dH_{n-1}$$

is lower semicontinuous with respect to this convergence. To handle the term

$$(1.13) \qquad -\frac{\kappa}{2} \int_\Omega v_l^2\,dx$$

we observe that in view of $\psi_1 \leq v_l \leq \psi_2$ the Lebesgue dominated convergence theorem can be applied. Therefore

$$(1.14) \qquad E(u) \leq \lim\inf E(v_l)$$

and u is the desired solution.

THEOREM 2. *If we choose*

$$(1.15) \qquad \psi_2 = \psi_1 + M , \quad M > 0 ,$$

then

    (i) *for arbitrary but fixed κ and M we can find $V_0 > 0$, such that for any $0 < V < V_0$ the solution u of* (1.6) *lies strictly below ψ_2 ;*

    (ii) *for M large enough and V or κ small enough the solution u of* (1.6) *will be strictly below ψ_2 .*

Proof. The conclusion of the theorem follows from the estimate

$$(1.16) \qquad u - \psi_1 \leq c_1 \cdot \left(c_2 + \kappa\cdot|\psi_2|_\infty\right) \cdot V^{1/(n+1)}$$

where the constants depend on H, n, a, Ω, $\partial\Omega$, $|\psi_1|_\infty$ and $|D\psi_1|_\infty$. To establish the estimate, we choose the test function

$$(1.17) \qquad v = \psi_1 + \min(u-\psi_1,\, k) + \frac{1}{|\Omega|} \cdot \int_\Omega \max(u-\psi_1-k,\, 0)$$

for any $0 \leq k \leq M$. Since

$$(1.18) \qquad\qquad \min(u-\psi_1,\, k) + \max(u-\psi_1-k,\, 0) = u - \psi_1$$

we have $\displaystyle\int_\Omega v - \psi_1\, dx = V$. The estimate

$$(1.19) \qquad\qquad\qquad \psi_1 \leq v \leq \psi_2$$

follows from $0 \leq k \leq M$ and the fact that $u \in K_2$. Therefore $v \in K_2$ and

$$(1.20) \qquad\qquad\qquad E(u) \leq E(v)\ .$$

We obtain

$$(1.21) \quad \int_\Omega \left(1+|Du|^2\right)^{\frac{1}{2}}dx \leq \int_\Omega \left(1+|Dv|^2\right)^{\frac{1}{2}}dx + \int_{\partial\Omega} \beta(v-u)\,dH_{n-1}$$

$$+ \frac{\kappa}{2} \cdot \int_\Omega (u-v)(u+v)\,dx + \int_\Omega \int_u^v H(x,\,t)\,dt\,dx\ .$$

The first term on the right hand side can be estimated by

$$\int_{A(k)} \left(1+|D\psi_1|^2\right)^{\frac{1}{2}}dx + \int_{\Omega\backslash A(k)} \left(1+|Du|^2\right)^{\frac{1}{2}}dx$$

$$\leq 2 \cdot |D\psi_1|_\infty \cdot |A(k)| + \int_{\Omega\backslash A(k)} \left(1+|Du|^2\right)^{\frac{1}{2}}dx$$

where we used the notation $A(k) = \{x \in \Omega \mid u-\psi_1 > k\}$ and $|A(k)| = H_n(A(k))$. Furthermore we get from (1.3) and (1.5),

$$(1.23) \quad \int_{\partial\Omega} \beta(v-u)\,dH_{n-1} \leq (1-a) \int_\Omega |D(v-u)|\,dx + c_{R/2} \cdot \int_\Omega |v-u|\,dx$$

$$\leq (1-a) \int_{A(k)} |Du|\,dx + |D\psi_1|_\infty \cdot |A(k)|$$

$$+ 2 \cdot c_{R/2} \cdot \int_{A(k)} u - \psi_1 - k\,dx\ .$$

Moreover

$$(1.24) \quad \frac{\kappa}{2} \cdot \int_\Omega (u-v)(u+v)dx \leq 2 \cdot \kappa \cdot |\psi_2|_\infty \cdot \int_{A(k)} u - \psi_1 - kdx .$$

The last term in (1.21) can be estimated by

$$(1.25) \quad \left(|H(\cdot, 0)|_\infty + |H(\cdot, \psi_1)|_\infty + \alpha \left(1 + \frac{V}{|\Omega|} + |\psi_1|_\infty \right) \right) \cdot \int_{A(k)} u - \psi_1 - kdx$$

where we used the assumptions (1.4) in a similar manner as in ([6],
Estimate 2.14). So we finally conclude, from (1.21),

$$(1.26) \quad a \cdot \int_{A(k)} |Du|dx \leq 3 \cdot |D\psi_1|_\infty \cdot |A(k)| + c \cdot \int_{A(k)} u - \psi_1 - kdx$$

where

$$c = \left(2 \cdot \kappa \cdot |\psi_2|_\infty + 2 \cdot C_{R/2} + |H(\cdot, 0)|_\infty + |H(\cdot, \psi_1)|_\infty + \alpha \left(1 + \frac{V}{|\Omega|} + |\psi_1|_\infty \right) \right) .$$

Up to a numerical constant this is exactly inequality (2.16) in [6], and
the estimate (1.16) now follows in the same way as in [6].

REMARK. We can not expect the function u to be regular in general,
since there might be sections with vertical gradient. To avoid this
possibility, we impose an additional gradient constraint in the next
section.

2. Regular minima

In [6] we assumed that

$$(2.1) \qquad\qquad \beta \in C^{1,\alpha}(\partial\Omega) , \quad H \in C^{1,\alpha}(\mathbf{R}^n \times \mathbf{R})$$

satisfy the conditions (1.3) and (1.4). We denoted by A the minimal
surface operator

$$(2.2) \qquad Au = \sum_{i=1}^n - D_i \left(a^i(Du) \right) , \quad a^i = D_i u / \left(1 + |Du|^2 \right)^{\frac{1}{2}}$$

and used the notation

$$(2.3) \qquad \langle Au, \eta \rangle = \int_\Omega \sum_i a^i D_i \eta dx + \int_{\partial\Omega} \beta \eta dH_{n-1} .$$

Then we had the following result on the existence of critical points of the

energy functional.

THEOREM 3. *Let* Ω *be a bounded domain of* $\mathbb{R}^n$, $n \geq 2$, *with boundary of class* $C^{2,\alpha}$, *and let the functions* β *and* H *satisfy the conditions above. Then the variational inequality*

(2.4) $\langle Au + H(x, u) - \kappa u, v-u \rangle \geq 0$ *for all* $v \in K_1$

has a solution $u \in H^{1,\infty}(\Omega) \cap H^{2,2}(\Omega) \cap H^{2,\infty}_{\text{loc}}(\Omega)$ *if we assume* $\psi_1 \in C^2(\overline{\Omega})$ *and if* κ *or* V *is small enough. The solution has continuous tangential derivatives at the boundary and in the case* $n = 2$ *we have* $u \in C^1(\overline{\Omega})$. *If we impose on* ψ_1 *the further condition*

(2.5) $\sum_i - a^i (D\psi_1) \cdot \gamma_i \geq \beta$ *on* $\partial\Omega$

where γ *is the outer unit normal to* $\partial\Omega$ *and if* $\partial\Omega \in C^{3,\alpha}$, $\beta \in C^{1,1}(\partial\Omega)$, *then* $u \in H^{2,\infty}(\Omega)$.

Now assume that Ω, β and H are as in Theorem 3 and that ψ_2 has the form

(2.6) $\psi_2 = \psi_1 + M$, $\psi_1 \in C^2(\overline{\Omega})$, $M > 0$.

Then we have

THEOREM 4. *The minimizing problem*

(2.7) $E(v) \to \min$ *in* K_3

has a solution u_0 *for any* κ, V, M, N , *provided these quantities are such that* K_3 *is not empty.*

If N and M are large enough and κ or V is small enough, u_0 satisfies

(2.8) $|Du_0|_\infty \leq N - 1$, $u_0 \leq \psi_2 - 1$.

Then u_0 is also a solution to the variational inequality (2.4) and the same regularity results as in Theorem 3 apply.

REMARKS. (i) A solution u_0 satisfying (2.8) can be considered as a hanging drop which minimizes the energy in the set K_3 .

(ii) κ and V depend on N and M in the sense that if N, M are enlarged, κ or V has to be taken correspondingly smaller.

(iii) The proof of Theorem 4 yields an existence result for the variational inequality (2.4) without using an abstract fixed point theorem as in [6].

Proof of Theorem 4. *(i)* K_3 is a bounded, closed and convex set in $H_{\cdot}^{1,\infty}(\Omega)$ and E is bounded from below on K_3 . It is easy to see that then some minimizing sequence tends uniformly to a solution u_0 of (2.7).

(ii) We look at the new problem

(2.9) $E_1(v) \to \min$ in K_1

where

$$(2.10) \quad E_1(v) = \int_{\Omega} \left(1+|Dv|^2\right)^{\frac{1}{2}} dx + \int_{\Omega} \int_0^v H(x,\ t) dt dx$$
$$- \kappa \cdot \int_{\Omega} u_0 \cdot v dx + \int_{\partial\Omega} \beta v dH_{n-1} .$$

E_1 is a convex functional of the type Gerhardt considered in [2]. He minimized those functionals in the set

$$(2.11) \qquad BV(\Omega) \cap \{v \geq \psi_1\} \cap \left\{\int_{\Omega} v-\psi_1 dx = V\right\}$$

but the regularity results in [4] showed, that the solution is in $H^{1,\infty}(\Omega)$ and therefore a solution of (2.9).

Now, for the (unique) solution u_1 of (2.9) we can establish exactly the same estimate as in Theorem 2. This follows from the fact that the test function

$$(2.12) \qquad v = \psi_1 + \min\left(u_1-\psi_1,\ r\right) + \frac{1}{|\Omega|} \int_{\Omega} \max\left(u_1-\psi_1-r\right) dx$$

is in K_1 and from the inequality

(2.13) $u_0 \leq \psi_2$.

Thus we can choose M as large and κ or V as small that

(2.14) $|u_1|_\infty \leq \psi_2 - 1$.

In order to get a similar estimate for the gradient of u_1 , we observe
that u_1 also solves the variational inequality

(2.15) $\langle Au_1 + H(x, u_1) - \kappa u_0, v-u_1 \rangle \geq 0$ for all $v \in K_1$.

Moreover, it was shown in [2], that there is a Lagrange multiplier $\lambda \in \mathbb{R}$
such that u_1 also satisfies

(2.16) $\langle Au_1 + H(x, u_1) - \kappa u_0 + \lambda, v-u_1 \rangle \geq 0$

$$\text{for all } v \in K_1' = H^{1,\infty}(\Omega) \cap \{v \geq \psi_1\} \ .$$

To this variational inequality we want to apply the gradient estimate in
([6], Theorem 1.2):

        "We can choose N as large and κ or V as small that

(2.17) $|Du_1|_\infty \leq N - 1$, if $|Du_0|_\infty \leq N$."

Actually, the gradient estimate in [6] was established for solutions of the
related boundary value problem. But u_1 can be approximated by solutions
of boundary value problems and it was shown in [6], that the gradient
estimate for the approximating solutions is independent of the
approximation parameters.

        In view of (2.14), (2.15) and (2.17) the conclusion of the theorem now
follows from

        LEMMA 2. *If N, M, κ and V are such that $u_1 \in K_3$, then we have*
$u_0 \equiv u_1$.

        Proof of Lemma 2. We can compute

$$E_1(u_0) = E(u_0) - \frac{\kappa}{2} \cdot \int_\Omega u_0^2 dx \ ,$$

(2.18)

$$E_1(u_1) = E(u_1) - \kappa \cdot \int_\Omega u_0 u_1 dx + \frac{\kappa}{2} \cdot \int_\Omega u_1^2 dx \ ,$$

and therefore

$$(2.19) \qquad E_1(u_0) = E_1(u_1) + (E(u_0) - E(u_1)) - \frac{\kappa}{2} \int_\Omega |u_0 - u_1|^2 dx .$$

In view of the assumptions of the lemma we have

$$(2.20) \qquad\qquad\qquad E(u_0) \leq E(u_1) .$$

Thus, if $u_0 \not\equiv u_1$ we have

$$(2.21) \qquad\qquad\qquad E_1(u_0) < E_1(u_1) ,$$

a contradiction to the minimizing property of u_1 .

Finally, from Lemma 4.1 in [6] we have

COROLLARY. *The solution u_0 of (2.7) is unique if κ is small enough.*

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