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A NOTE ON THE SUM OF RECIPROCALS

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Abstract

We prove that, given a positive integer m, there is a sequence $\{n_i\}_{i=1}^{k}$ of positive integers such that

$$m = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

with the property that partial sums of the series $\{1/n_i\}_{i=1}^{k}$ do not represent other integers.

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1. Introduction

Let $a + n\mathbb{Z}$ denote the arithmetic progression $\{x \in \mathbb{Z} : x \equiv a \pmod{n}\}$. For a finite system $\mathcal{A} = \{a_i + n_i\mathbb{Z}\}_{i=1}^{k}$, we define the covering function $w_{\mathcal{A}}$ over $\mathbb{Z}$ by

$$w_{\mathcal{A}}(x) := |\{1 \le i \le k : x \in a_i + n_i\mathbb{Z}\}|.$$

The system $\mathcal{A}$ is called an m-cover of $\mathbb{Z}$ if $w_{\mathcal{A}}(x) \ge m$ for all $x \in \mathbb{Z}$ and an exact m-cover if $w_{\mathcal{A}}(x) = m$ for all $x \in \mathbb{Z}$.

The concept of covering in $\mathbb{Z}$ was first mentioned by Erdős [3] and has been investigated in many papers (see, for example, [1, 2, 4–6, 11]). In [7], Porubský first studied exact m-covers and showed that

$$\sum_{i=1}^{k} \frac{1}{n_i} = m$$

is a necessary condition for a set of integers $\{n_i\}_{i=1}^{k}$ to be the set of moduli of an exact m-cover. Clearly, for any m-cover, $\sum_{i=1}^{k} 1/n_i \ge m$. In [12], Zhang discovered a connection between covering systems and Egyptian fractions. He showed that if $\mathcal{A}$ is a 1-cover of $\mathbb{Z}$, then

$$\sum_{s\in I} \frac{1}{n_s} \in \mathbb{Z}^{+}$$

for some $I \subset \{1, \ldots, k\}$. Sun [8–11] investigated connections between sums of reciprocals of the residue class moduli with covering systems. For example, Sun [8] showed that for each $n = 1, \ldots, m$, there exist (at least) $\binom{k}{2}$ subsets I of $\{1, \ldots, k\}$ with

$$\sum_{s \in I} \frac{1}{n_i} = n.$$

Suppose that $\mathcal{A}_1, \mathcal{A}_2$ are an m_1-cover and an m_2-cover over $\mathbb{Z}$, respectively. Obviously, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ constitutes an $(m_1 + m_2)$-cover. Conversely, Porubský [7] asked whether for each $m \geq 2$ there exists an exact m-cover of $\mathbb{Z}$ which cannot be split into an exact n-cover and an exact $(m - n)$-cover with $1 \leq n < m$. Later, Zhang [13] answered Porubský's question affirmatively.

Motivated by these results, we consider whether for each $m \geq 1$ there exists a series $\{n_i\}_{i=1}^k$ such that $m = \sum_{i=1}^k 1/n_i$ is an integer, but no partial sum of $\sum_{i=1}^k 1/n_i$ belongs to $\mathbb{Z}$.

THEOREM 1.1. *For every given positive integer m, there exists a sequence* $\{n_i\}_{i=1}^k$ *of positive integers such that*

$$m = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

and $\sum_{i \in I} 1/n_i \notin \mathbb{Z}$, *where I is any nonempty proper subset of* $\{1, 2, \ldots, k\}$.

Let

$$\Omega\{n_1, n_2, \ldots, n_k\} = \left\{ \sum_{i \in I} \frac{1}{n_i} : I \subset \{1, 2, \ldots, k\} \right\} \cap \mathbb{Z}.$$

As usual, the sum over the empty set is taken to be zero. In fact, we can prove the following stronger result.

THEOREM 1.2. *Let m be a given positive integer. For any partition*

$$m = m_1 + m_2 + \cdots + m_e, \quad 1 \leq m_1 \leq m_2 \leq \cdots \leq m_e \leq m,$$

there exists a sequence $\{n_i\}_{i=1}^k$ *of positive integers such that*

$$m = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

and

$$\Omega\{n_1, n_2, \ldots, n_k\} = \left\{ \sum_{c \in C} m_c : C \subset \{1, \ldots, e\} \right\}.$$

One can also require that

$$\left| \left\{ I : I \subset \{1, 2, \ldots, k\}, \sum_{i \in I} \frac{1}{n_i} \in \mathbb{Z} \right\} \right| = 2^e.$$

Let us give a simple explanation about why Theorem 1.2 implies Theorem 1.1. We choose $e = 1$ and exclude the case that the subset I of $\{1, \ldots, k\}$ is empty. Then we obtain the desired result.

Corollary 1.3. *For every given positive integer m, there exists a sequence $\{n_i\}_{i=1}^k$ of positive integers such that*

$$m = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$$

and only two integers n and $m - n$ can be written in the form $\sum_{i\in I} 1/n_i$, where $1 \le n < m$ and I is a nonempty proper subset of $\{1, 2, \ldots, k\}$.

Remark 1.4. This corollary is immediate from Theorem 1.2 with $e = 2$. This result seems to be related to Zhang's result [13] mentioned above about splitting an exact m-cover of $\mathbb{Z}$ into an exact n-cover and an exact $(m - n)$-cover with $1 \le n < m$.

2. Proof of Theorem 1.2

We use p_j to denote the jth prime. Since the sum of the reciprocals of the primes diverges, there exist $e + 1$ integers $l_1 < l_2 < \cdots < l_e < l_{e+1}$ with $l_1 = 0$ such that

$$\sum_{j=l_c+1}^{l_{c+1}} \frac{1}{p_j} < m_c < \sum_{j=l_c+1}^{l_{c+1}+1} \frac{1}{p_j}$$

for $c = 1, 2, \ldots, e$. For $1 \le c \le e$, write

$$m_c = \sum_{j=l_c+1}^{l_{c+1}} \frac{1}{p_j} + \frac{g_c}{\prod_{j=l_c+1}^{l_{c+1}} p_j}.$$

Since $g_c < \prod_{j=l_c+1}^{l_{c+1}} p_j$ and $(g_c, \prod_{j=l_c+1}^{l_{c+1}} p_j) = 1$, we can choose $s_c, t_c \in \mathbb{Z}^+$ such that

$$g_c s_c - t_c \prod_{j=l_c+1}^{l_{c+1}} p_j = 1.$$

This equation remains true when we replace s_c and t_c by $\tilde{s}_c = s_c + a \prod_{j=l_c+1}^{l_{c+1}} p_j$ and $\tilde{t}_c = t_c + a g_c$ for any $a \in \mathbb{Z}^+$. By Dirichlet's theorem on primes in arithmetic progressions, we can choose $\tilde{s}_c$ to be e different primes with $p_{l_{e+1}} < \tilde{s}_1 < \cdots < \tilde{s}_e$. From the construction, $\tilde{t}_c < \tilde{s}_c$ and

$$m_c = \sum_{j=l_c+1}^{l_{c+1}} \frac{1}{p_j} + \frac{\tilde{t}_c}{\tilde{s}_c} + \frac{1}{\tilde{s}_c \prod_{j=l_c+1}^{l_{c+1}} p_j}.$$

Since $m = m_1 + \cdots + m_e$, we can write

$$m = \sum_{c=1}^{e} \left\{ \sum_{j=l_c+1}^{l_{c+1}} \frac{1}{p_j} + \frac{\tilde{t}_c}{\tilde{s}_c} + \frac{1}{\tilde{s}_c \prod_{j=l_c+1}^{l_{c+1}} p_j} \right\} := \sum_{i=1}^{k} \frac{1}{n_i}. \tag{2.1}$$

It remains to show that

$$\Omega\{n_1, n_2, \ldots, n_k\} = \left\{ \sum_{c\in C} m_c : C \subset \{1, \ldots, e\} \right\}$$

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and

$$\left|\left\{I : I \subset \{1, 2, \ldots, k\}, \sum_{i \in I} \frac{1}{n_i} \in \mathbb{Z}\right\}\right| = 2^e.$$

By the construction of m,

$$\left\{\sum_{c \in C} m_c : C \subset \{1, \ldots, e\}\right\} \subset \Omega\{n_1, n_2, \ldots, n_k\}$$

and

$$\left|\left\{C : C \subset \{1, \ldots, e\}\right\}\right| = 2^e.$$

Consequently, we only need to show that

$$\left|\left\{I : I \subset \{1, 2, \ldots, k\}, \sum_{i \in I} \frac{1}{n_i} \in \mathbb{Z}\right\}\right| = 2^e.$$

We shall do this by induction on e. We first consider the case $e = 1$ and write

$$m = \sum_{j=1}^{l_2} \frac{1}{p_j} + \frac{\tilde{t}_1}{\tilde{s}_1} + \frac{1}{\tilde{s}_1 \prod_{j=1}^{l_2} p_j}$$

corresponding with the form of (2.1). In this case, Theorem 1.2 is equivalent to showing that

$$\sum_{j \in J} \frac{1}{p_j} + \frac{q}{\tilde{s}_1} \notin \mathbb{Z} \tag{2.2}$$

for any $J \subset \{1, 2, \ldots, l_2\}$, where q is an integer no more than $\tilde{t}_1$. The case $J = \emptyset$ is trivial. Provided that $J \neq \emptyset$, since $(\tilde{s}_1, \prod_{j \in J} p_j)$=1, we can easily obtain (2.2) because

$$\tilde{s}_1\left(\sum_{j \in J} \frac{1}{p_j} + \frac{q}{\tilde{s}_1}\right) = \tilde{s}_1\left(\sum_{j \in J} \frac{1}{p_j}\right) + q \not\equiv 0 \pmod{\tilde{s}_1}.$$

Suppose that Theorem 1.2 holds for $e = 1, \ldots, b-1$, where b is an integer and $b \geq 2$, and consider the case $e = b$. According to (2.1),

$$m = \sum_{c=1}^{b}\left\{\sum_{j=l_c+1}^{l_{c+1}} \frac{1}{p_j} + \frac{\tilde{t}_c}{\tilde{s}_c} + \frac{1}{\tilde{s}_c \prod_{j=l_c+1}^{l_{c+1}} p_j}\right\}.$$

Choose some partial sums of the form

$$\sum_{c=1}^{b}\left\{\sum_{j=l_c+1}^{l_{c+1}} \frac{\delta_{c,j}}{p_j} + \frac{\tilde{q}_c}{\tilde{s}_c} + \frac{\delta_c}{\tilde{s}_c \prod_{j=l_c+1}^{l_{c+1}} p_j}\right\}$$

which lie in $\mathbb{Z}$, where the $\delta_{c,j}$ and δ_c are 0 or 1 and $\tilde{q}_c \leq \tilde{t}_c$ for $1 \leq c \leq b$. Let

$$A = \sum_{c=1}^{b-1}\left\{\sum_{j=l_c+1}^{l_{c+1}} \frac{\delta_{c,j}}{p_j} + \frac{\tilde{q}_c}{\tilde{s}_c} + \frac{\delta_c}{\tilde{s}_c \prod_{j=l_c+1}^{l_{c+1}} p_j}\right\}, \quad B = \sum_{j=l_b+1}^{l_{b+1}} \frac{\delta_{b,j}}{p_j} + \frac{\tilde{q}_b}{\tilde{s}_b} + \frac{\delta_b}{\tilde{s}_b \prod_{j=l_b+1}^{l_{b+1}} p_j},$$

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so that $A + B \in \mathbb{Z}$. Since both $(\prod_{c=1}^{b-1} \tilde{s}_c \prod_{j=l_c+1}^{l_{c+1}} p_j) \cdot A$ and $(\prod_{c=1}^{b-1} \tilde{s}_c \prod_{j=l_c+1}^{l_{c+1}} p_j) \cdot (A + B)$ lie in $\mathbb{Z}$, so does $(\prod_{c=1}^{b-1} \tilde{s}_c \prod_{j=l_c+1}^{l_{c+1}} p_j) \cdot B$. But all denominators of B are relatively prime to $\prod_{c=1}^{b-1} \tilde{s}_c \prod_{j=l_c+1}^{l_{c+1}} p_j$, so B must be an integer. Since $A + B$ is an integer, so is A.

By the inductive assumption, there are 2^{b-1} choices to construct an integer from partial sums of A and two independent choices to construct an integer from partial sums of B. Hence, for any $m \in \mathbb{Z}^+$, we conclude that

$$\left|\left\{I : I \subset \{1, 2, \dots, k\}, \sum_{i\in I} \frac{1}{n_i} \in \mathbb{Z}\right\}\right| = 2^b.$$

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