REPRESENTATION OF LINEAR FUNCTIONALS ON KÖTHE SPACES

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1. Introduction. Köthe spaces, in the terminology of Diendonné [**2**], are certain spaces X of real valued integrable functions. In this paper we consider the problem of representation of continuous linear functionals on vector valued Köthe spaces. The elements of a Köthe space $X(B)$ are functions $\mathbf{f}(t)$, $0 \leqslant t \leqslant 1$, with values in a Banach space B (see §2). In the case $X = L^p$, this problem was solved by Dieudonné [**3**]. In May 1952, the second of the present authors found that Dieudonné's methods also apply to spaces[1] $\Lambda(\phi, p)$ for $p > 1$. However, difficulties arise even for spaces $\Lambda(\phi, 1)$, since Dieudonne's methods depend heavily on the reflexivity of the space X. This motivates an entirely new approach to the problem proposed here which is applicable for more general spaces X. Our main idea is the use of linear operators of class (b, o) of Kantorovitch [**6**; **8**], which seem to provide the most natural way of handling the problem. In fact, this method is applicable also to cross-spaces $B \otimes_\delta X$ (see Schatten [**12**]) with a certain cross-norm δ, not symmetric with respect to B and X. Köthe spaces $X(B)$ are special cases of these $B \otimes_\delta X$. In order not to complicate the exposition, we confine our attention to the simpler case of Köthe spaces; functionals on cross-spaces will be discussed in a separate paper.

In §2 we give the definition of a Köthe space X and consider its properties as an abstract Banach lattice. Care must be taken not to exclude spaces such as $X = L^1$ for which the conjugate space X^* does not satisfy condition (f) of §2. Therefore these properties (which must also hold for X^*) turn out to be partly weaker than those given in [**8**, p. 215]. In §3 we give our main results concerning linear functionals on spaces $X(B)$.

2. Köthe spaces and Banach lattices. Let C be a non-empty class of positive integrable functions $c(t)$. The Köthe space $X = X_C$ consists of all measurable functions $f(t)$ for which

$$(1) \qquad ||f||_X = \sup_{c \in C} \int_0^1 |f(t)|\, c(t)\, dt < +\infty.$$

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[1]Spaces $\Lambda(\phi, p)$ for $\phi = x^{\alpha-1}$ were defined by Lorentz [**10**]; in his talks in Tübingen (1948) and Kingston (1950) he indicated the generalization to an arbitrary decreasing ϕ [**11**]. Statements made in this connection in [**13**, p. 273] are misleading: the manuscipt mentioned there was written after these talks; in the 1950 Report of the Summer Research Institute, Halperin gives its content as obtained jointly with Lorentz.

Without changing X_C or the value of $||f||$ we may assume that

(a) C is *normal*: if $c \in C$ and $0 \leqslant c_1(t) \leqslant c(t)$, then $c_1 \in C$;

(b) C is *convex*: if $c_i \in C$, $0 \leqslant \alpha_i \leqslant 1$, $\Sigma_1^n \alpha_i = 1$, then $\Sigma \alpha_i c_i \in C$.

We shall also assume that

(c) if $c_n \in C$ and $c_n(t) \uparrow c(t)$, then $c \in C$;

(d) $\mathbf{1} \in C$, where $\mathbf{1}(t) = 1$ a.e. on $(0, 1)$;

(e) $$\int_0^1 c(t)\, dt \leqslant 1, c \in C.$$

Conditions (a), (d), (e) imply that all measurable functions $c(t)$ with $0 \leqslant c(t) \leqslant 1$ belong to C, condition (e) that all bounded functions belong to X_C. From the definition of $||f||$ it follows that X_C is normal: $f \in X$ and $|g(t)| \leqslant |f(t)|$ imply $g \in X$. Condition (d) and (1) also imply that all functions $f \in X_C$ are integrable and that

(2) $$\int_0^1 |f(t)|\, dt \leqslant ||f||_X.$$

It is easy to see that under these assumptions X_C is a Banach space. General Köthe spaces of integrable functions were considered by Dieudonné [**2**]. Köthe and Köthe and Toeplitz in a series of papers beginning in 1934 were dealing with spaces of sequences of similar type. Dieudonné defines the Köthe space X_C to consist of all $f \in L^1$ with $\int |f|\, c\, dt < +\infty$ for each $c \in C$; under the assumptions (a), (b), and (c) this is equivalent to our definition (given also in [**11**]). In fact, if $\int |f|\, c\, dt$, $c \in C$ is not bounded for some f, there is a $c \in C$ with $\int |f|\, c\, dt = +\infty$. For the proof, we choose $c_n \in C$ with $\int |f|\, c_n\, dt > n^3$ ($n = 1, 2, \ldots$); then $c = \Sigma n^{-2} c_n$ belongs to C by (a), (b), and (c), and $\int |f|\, c\, dt > n$ for each n. On the other hand, if the conditions (a), (b), (c) are not assumed, Dieudonné's definition is more general than that given above. In what follows a Köthe space X_C is always a space with C satisfying (a)–(e).

In X we introduce a partial ordering, writing $f \leqslant g$ if $f(t) \leqslant g(t)$ a.e. With this order X becomes a Banach lattice [**1**; **5**; **6**; **8**] with the following properties:

(I) $f \geqslant 0$, $g \geqslant 0$ imply $f + g \geqslant 0$;

(II) $f \geqslant 0$, $a \geqslant 0$ imply $af \geqslant 0$;

(III) each finite set $E \subset X$ is bounded from above;

(IV) each set $E \subset X$ bounded from above has a supremum $f_0 = \bigcup f = \sup f$ in X.

Property (IV) follows from the fact that the space S of all measurable functions has this property. The order induces the *order-convergence* (*o-convergence*) $f_n \to f$ (o), which for the space X_C is equivalent to $f_n(t) \to f(t)$ a.e. and to the existence of a $g \in X$ with $|f_n| \leqslant g$.

From (1) we deduce also:

(V) $|f| \leqslant |g|$ implies $||f|| \leqslant ||g||$;

(VI) if $0 \leqslant f_n \in X$, $f_n \uparrow f$ (where $f \in X$ or f is the element $+\infty$), then $||f_n|| \to ||f||$. (In particular, $f_n \uparrow +\infty$ implies $||f_n|| \to \infty$.)

A space X_C has a weak form of regularity:

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(VII) each set $E \subset X$ has a denumerable subset $E' \subset E$ with $\sup E' = \sup E$. (This also follows from the corresponding property of S.)

We finally note that a Köthe space X_C has a (weak) unit:

(VIII) there is an element $\mathbf{1} \in X$ such that $f \cap n\mathbf{1} \uparrow f$ for each $0 \leqslant f \in X$.

The following lemma is implicitly contained in [**8**, pp. 201–203]:

LEMMA. *For a Banach lattice X satisfying* I–VIII, *the o-boundedness of a set $E \subset X$ and the boundedness of the set*

$$H = \left\{ \| \bigcup_{i=1}^{k} |f_i| \| \right\},$$

$f_i \in E$ are equivalent. Moreover, with $g = \sup |f|$, where the supremum is taken for all $f \in E$, and $A = \sup H$ we have $A = \|g\|$.

First let $A < +\infty$. Each of the sets $E_n = \{|f| \cap n\mathbf{1}\}$ is bounded and for the $g_n = \sup E_n$ we have $g_n \uparrow g = \sup |f|$ (with $g \in X$ or $g = +\infty$). By VII,

$$g_n = \bigcup_{k=1}^{\infty} (|f_{kn}| \cap n\mathbf{1})$$

for properly chosen $f_{kn} \in E$. By VI,

$$\| g_n \| = \lim_{k \to \infty} \| \bigcup_{i=1}^{k} |f_{in}| \cap n\mathbf{1} \| \leqslant \lim \| \bigcup_{i=1}^{k} |f_{in}| \| \leqslant A.$$

By VI, this shows that $g \in X$ and that E is bounded. We also have $\|g\| \leqslant A$. The converse is obvious.

The *dual space* X' of a Köthe space X_C is the set of all measurable g such that

$$\| g \|_{X'} = \sup_{\|f\| \leqslant 1} \int_0^1 |g| \, |f| \, dt < +\infty.$$

In other words, $X' = X_{C'}$, where C' consists of all elements $f \in X$ with $0 \leqslant f$, $\|f\| \leqslant 1$. It follows from (a)–(e) and VI that C' also satisfies (a)–(e). Clearly, X' is a subspace of the conjugate space X^*. A Köthe space X is *perfect* if $(X')' = X$; a reflexive space X is perfect, but not conversely (example: $X = L^1$).

THEOREM 1. *If X is a separable perfect Köthe space, then X' is identical with X^*.*

Proof. In virtue of Lorentz [**11**, Theorem 3], X' is identical with X^* if $X = X_C$ satisfies the condition

(f) if $f \in X$ and χ_e is the characteristic function of the set e, then $\|f\chi_e\| \to 0$ with $me \to 0$.

Hence we have to show that a separable perfect space X_C has the property (f). The following proof uses an argument due to Köthe [**9**, pp. 105–106].

We have $Y' = X$, and X is separable. It follows (by the usual method of a diagonal subsequence) that the unit sphere U in Y is sequentially weakly compact (in the weak topology generated by X). If (f) is not fulfilled for X, there

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is an $f \in X$ and a sequence of sets e_n with $me_n \to 0$ such that $\|f\chi_{e_n}\| \geqslant 1$. Then for each n there is a function $c_n \in U$ such that $\|c_n\|_Y \leqslant 1$ and

$$\int_0^1 |f| \chi_{e_n} c_n \, dt \geqslant \tfrac{1}{2}.$$

The functions $\bar{c}_n = c_n \chi_{e_n}$ have the properties $\|\bar{c}_n\|_Y \leqslant 1$,

$$\int_0^1 |f| \bar{c}_n \, dt \geqslant \tfrac{1}{2}$$

and $\bar{c}_n(t) = 0$ outside of the set e_n. By passing to a subsequence of the $\bar{c}_n$, if necessary, we may assume that the $\bar{c}_n$ are weakly convergent to $\bar{c}$, say, and that $\Sigma me_n < +\infty$. Then the set lim sup e_n is of measure zero. Let e be disjoint with $\bigcup_{n \geqslant N} e_n$; then

$$\int_e \bar{c}(t) \, dt = \int_0^1 \chi_e \bar{c} \, dt = \lim_{n\to\infty} \int_0^1 \chi_e c_n \, dt = 0,$$

so that $\bar{c}$ vanishes outside of $\bigcup_{n \geqslant N} e_n$. It follows that $\bar{c}$ vanishes a.e. But this is a contradiction:

$$0 = \int_0^1 |f| \bar{c} \, dt = \lim_{n\to\infty} \int_0^1 |f| \bar{c}_n \, dt \geqslant \tfrac{1}{2}.$$

This completes the proof of Theorem 1.

It is easy to check that the spaces $\Lambda(\phi, p)$ satisfy (a)–(f) (compare [**10**; **11**]). The spaces $\Lambda(\phi, 1)$ are perfect [**10**], the spaces $\Lambda(\phi, p)$, $p > 1$ are reflexive [**11**].

Let B be a Banach space and $X = X_C$ a Köthe space of real-valued functions. The space $X(B)$ by definition consists of all functions $\mathbf{f}(t)$ from $(0, 1)$ to B which are weakly measurable and such that $\|\mathbf{f}(t)\|$ belongs to X. For separable spaces X, $\|\mathbf{f}(t)\|$ is measurable whenever $\mathbf{f}(t)$ is weakly measurable [**3**]. We put

$$\|\mathbf{f}\|_{X(B)} = \| \|\mathbf{f}(t)\|_B \|_X = \sup_{c \in C} \int_0^1 \|\mathbf{f}(t)\| c(t) \, dt. \tag{3}$$

It is easy to show that $X(B)$ is a Banach space. If B is separable and X_C satisfies condition (f), $X(B)$ is also separable, since elements of the type

$$\sum_{i=1}^n x_i \chi_{e_i}(t)$$

with $x_i \in B$ and measurable and disjoint sets e_i are everywhere dense in $X(B)$.

3. Linear operators of class (b, o) and continuous linear functionals on $X(B)$. Let B be a Banach space and X a Banach lattice. A linear mapping $f = U(x)$ from B to X is called an operator of class (b, o) if the set of the $|U(x)|$ for all $\|x\| \leqslant 1$ is o-bounded in X. Then

$$|U| = \bigcup_{\|x\| \leqslant 1} |U(x)| \tag{4}$$

is the "abstract norm" of U. (In [**8**], U is defined to be of class (b, o) if $x_n \to x$

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always implies $U(x_n) \to U(x)$ (o). From the lemma of §2 and [**8**, pp. 202, 187] it follows that a Banach lattice X satisfying I–VIII is a K^+ space, and then both definitions are equivalent [**8**, p 258]). We shall use sometimes the notation $\langle x, g\rangle$ for the value of the functional $g \in B^*$ at $x \in B$.

The Lemma of §2 shows that in case when X satisfies I–VIII, a linear operator U is of class (b, o) if and only if

$$(5) \qquad A = \sup_{\|x_i\| \leqslant 1} \| U(x_1) \cup U(x_2) \cup \ldots \cup U(x_n) \| < +\infty .$$

THEOREM 2. *The general form of an operator $f = U(x)$ of class (b, o) from a separable Banach space to a Köthe space X_C is given by*

$$(6) \qquad f(t) = \langle x, \mathfrak{g}(t)\rangle$$

where $\mathfrak{g}(t)$ belongs to $X_C(B^*)$. Moreover, $|U|$ is the function $\|\mathfrak{g}(t)\|_{B^*}$.

Proof. For $X = L^p$, $p \geqslant 1$, this was given by Kantorovitch and Vulich [**7**, Theorem 14; or **8**, p. 330], except for the last statement.

If U is an operator of desired class, then it is also an operator of class (b, o) from B to L^1. By the above theorem with $X = L^1$, $f(t) = \langle x, \mathfrak{g}(t)\rangle$ for each $x \in B$ and almost all t, where $\mathfrak{g}(t)$ is weakly measurable and $\|\mathfrak{g}(t)\| \in L^1$. We have

$$|\langle x, \mathfrak{g}(t)\rangle| = |f(t)| \leqslant |U|(t)\|x\| \qquad \text{a.e.};$$

this relation holds for each x and all t except for a set E_x of measure zero, which may depend on x. Since B is separable, it is easy to prove that there is a set E of measure zero such that

$$|\langle x, \mathfrak{g}(t)\rangle| \leqslant |U|(t)\|x\|, \qquad x \in B, t \notin E.$$

Then for $t \notin E$, $\|\mathfrak{g}(t)\| \leqslant |U|(t)$. This shows that $\mathfrak{g}(t) \in X(B^*)$.

On the other hand, from (6) we derive

$$|f(t)| = |\langle x, \mathfrak{g}(t)\rangle| \leqslant \|\mathfrak{g}(t)\| \cdot \|x\|,$$

so that the conditions are sufficient and $|U|(t) \leqslant \|\mathfrak{g}(t)\|$. This completes the proof.

If X is not separable, a representation formula for an operator of class (b, o) can still be given under stronger assumptions on X_C. However, this will not be used for our main theorems, and we do not give full proofs. Here also, the case $X_C = L^p$ has been discussed by Kantorovitch and Vulich [**7**].

We shall formulate the following properties of a class $C = \{c\}$:

(g) C is *average-invariant:* if $c \in C$, $e \subset (0, 1)$ and if $\tilde{c}$ is obtained from c by replacing its values on e by the average $(me)^{-1}\int_e c\,dt$, then $\tilde{c} \in C$.

(h) There is a function $A(\epsilon) > 0$, $A(\epsilon) \to \infty$ for $\epsilon \to 0$ such that for each set e of measure $me < \epsilon$ there is a $c \in C$ with $c(t) \geqslant A(\epsilon)$ on e.

This allows us to characterize the integrals

$$F(t) = \int_0^t f(u)\,du$$

of functions $f \in X_C$. We say that a real function $F(t)$ on $(0, 1)$ is of bounded X_C-variation if, for all subdivisions $0 = t_0 < t_1 < \ldots < t_n = 1$ of $(0, 1)$, the functions $\tilde{F}(t)$ defined by

$$\tilde{F}(t) = \frac{1}{t_{i+1} - t_i}[F(t_{i+1}) - F(t_i)], \qquad t \in (t_i, t_{i+1}) \ (i = 0, 1, \ldots, n-1)$$

have uniformly bounded norms $||\tilde{F}||_X$. Let X_C be a Köthe space. Then a function $F(t)$ with $F(0) = 0$ is an integral of a function $f \in X_C$ if F is of bounded X_C-variation and if (h) holds. Conversely, any such integral is of bounded X_C-variation if (g) holds. The proof is similar to the proof of F. Riesz's theorem on integrals of functions $f \in L^p$. By means of this result we obtain

THEOREM 3.[2] *If the Köthe space X_C satisfies* (g) *and* (h), *the general form of an operator $f = U(x)$ of class (b, o) from a Banach space B to X_C is given by*

$$f(t) = \frac{d}{dt}\langle x, \mathbf{G}(t)\rangle$$

where $\mathbf{G}(t)$ is a mapping from $(0, 1)$ *to B^* such that the functions*

$$\tilde{G}(t) = \frac{1}{t_{i+1} - t_i}||\mathbf{G}(t_{i+1}) - \mathbf{G}(t_i)||_{B^*}, \qquad t \in (t_i, t_{i+1}) \ (i = 0, 1, \ldots, n-1)$$

have uniformly bounded norms in X.

The proof consists in a direct construction of the function $|U|(t)$.

We now come to our main subject: to linear continuous functionals on spaces $X_C(B)$. We assume throughout the rest of this section that B is separable, that X satisfies condition (f), and that X' and X^* are identical. Theorem 1 provides examples of such X—all perfect separable Köthe spaces belong to this class. More particular examples of such X are spaces $\Lambda(\phi, p)$, which reduce to L^p for $\phi(x) = 1$.

THEOREM 4. *There is a natural isomorphism between the spaces of* (i) *all continuous linear functionals $L(f)$ on $X(B)$;* (ii) *all functions $\mathfrak{g}(t)$ belonging to $X^*(B^*)$*; (iii) *all (b, o)-operators $U(x)$ mapping B into X^*. The general form of a continuous linear functional $L(\mathbf{f})$ on $X(B)$ is given by*

$$L(\mathbf{f}) = \int_0^1 \langle \mathbf{f}(t), \mathfrak{g}(t)\rangle \, dt, \tag{7}$$

where $\mathfrak{g}(t)$ belongs to $X^*(B^*)$ and $||L|| = ||\mathfrak{g}||$.

Proof. To each $\mathfrak{g}$ the relation (7) lets correspond an $L(\mathbf{f})$ which is clearly a continuous linear functional on $X(B)$ with norm $||L|| \leqslant ||\mathfrak{g}||$. Theorem 2 establishes a (1, 1) correspondence between the U and the $\mathfrak{g}$. We shall show that to each L there corresponds a U. For a fixed $x \in B$ and variable $f \in X$, $L(xf)$ is a continuous linear functional on X, which is characterized by a function

[2]Added January 28, 1953.

$g(t)$ from X'. The mapping $x \to g$ defines a linear operator $U(x) = g$. The correspondence between x and g is given by

$$L(xf) = \int_0^1 f(t)\, g(t)\, dt. \tag{8}$$

To show that U is of class (b, o), we shall prove (5) with $A \leqslant ||L||$. We have to show that for each finite set of elements $x_i \in B$ $(i = 1, \ldots, n)$ with $||x_i|| \leqslant 1$ we have $||g|| \leqslant ||L||$ where $g = \bigcup_{i=1}^n |g_i|$, $g_i = U(x_i)$. There are disjoint measurable sets e_i with $\bigcup_1^n e_i = (0, 1)$ and $g(t) = |g_i(t)|$ on the set e_i. Put $\epsilon_i(t) = \text{sign}\, g_i(t)$, so that $\epsilon_i(t) = 0$ outside of e_i. Let $f \in X_C$ be arbitrary and

$$\mathbf{f}(t) = \sum_{i=1}^n x_i f(t)\, \epsilon_i(t).$$

Then by (8)

$$\int_0^1 fg\, dt = \sum_i \int_{e_i} f\, |\, g_i\, |\, dt = \sum_i L(x_i f \epsilon_i) = L(\mathbf{f}). \tag{9}$$

On the other hand,

$$\begin{aligned} |\, L(\mathbf{f})\, | &\leqslant ||\, L\, || \,.\, ||\, \mathbf{f}\, || = ||\, L\, || \Big\| \, ||\sum_i x_i f \epsilon_i\, ||_B \Big\|_X \\ &= ||\, L\, || \,.\, \Big\| \sum_i ||\, x_i\, ||\, |f|\, |\, \epsilon_i\, | \Big\| \\ &\leqslant ||\, L\, || \,.\, \Big\| \sum_i ||\, x_i\, ||\, |f|\, |\, \chi_{e_i}\, | \Big\| = ||\, L\, || \,.\, ||\, \mathbf{f}\, ||_X. \end{aligned}$$

Comparing this with (8) we obtain $||g|| \leqslant ||L||$.

Now let

$$(U(x))(t) = \langle x, \mathfrak{g}(t) \rangle, \qquad \mathfrak{g} \in X(B^*) \tag{10}$$

be the representation of U given by Theorem 2. For elements $\mathbf{f}$ of $X(B)$ of the form

$$\mathbf{f} = \sum_{i=1}^n x_i f_i, \qquad x_i \in B, f_i \in X$$

we have by (8) and (9)

$$\begin{aligned} L(\mathbf{f}) &= \sum_i L(x_i f_i) = \sum \int_0^1 f_i\, U(x_i)\, dt \\ &= \int_0^1 \sum_i f_i(t) \langle x, \mathfrak{g}(t) \rangle\, dt = \int_0^1 \langle \mathbf{f}(t), \mathfrak{g}(t) \rangle\, dt. \end{aligned} \tag{11}$$

Since both L and the last integral in (11) are continuous functionals on $X(B)$ and the set of the $\mathbf{f}$ of the above kind is everywhere dense in $X(B)$, we obtain (7) for an arbitrary $\mathbf{f} \in X(B)$.

We also have $||\, |U|\, || = ||g|| \geqslant ||L|| \geqslant A = ||\, |U|\, ||$ which proves that $||L|| = ||g||$. This completes the proof. Theorem 4 may also be stated in the form $(X(B))^* = X^*(B^*)$. As a corollary we have:

THEOREM 5. *If the spaces B, X are separable and reflexive, then so is $X(B)$.*

In particular, the vector-valued spaces $\Lambda(\phi, p, B)$, $p > 1$ are reflexive. In the case when B is a finite-dimensional Euclidean space, this was also proved by Ellis and Halperin [**4**].

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