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THE 2-SYLOW-SUBGROUP OF THE TAME KERNEL OF NUMBER FIELDS

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0. **Introduction.** For a number field F with ring of integers O_F the tame symbols yield a surjective homomorphism $\lambda : K_2(F) \longrightarrow \amalg_{\mathfrak{p}} K_1(O_F/\,\mathfrak{p})$ with a finite kernel, which is called the tame kernel, isomorphic to $K_2(O_F)$. For the relative quadratic extension E/F, where $E = F(\sqrt{-1})$ and $E \neq F$, let $C^S(E/F)(2)$ denote the 2-Sylow-subgroup of the relative S-class-group of E over F, where S consists of all infinite and dyadic primes of F, and let m be the number of dyadic primes of F, which decompose in E. Then the following formula holds (cf. [9],[15]):

$$4\text{-}\operatorname{rank}\big(K_2(O_F)\big) = m + 2\text{-}\operatorname{rank}\Big(C^S(E/F)/\operatorname{im}\big({}_2C^S(F)\big)\Big),$$

where ${}_2C^S(F)$ consists of all elements of order ≤ 2 in the S-class-group of F. For an odd prime p Coastes (cf. [6], [7]) has shown that the Main Conjecture in Iwasawa-theory implies the p-primary part of the Birch-Tate Conjecture (cf. [1], [20]). For the prime 2 Federer (cf. [10]) has given an analogous Main Conjecture and Kolster (cf. [17]) has shown that this conjecture implies the 2-primary part of the Birch-Tate Conjecture using his results about the 2-Sylow-subgroup of $K_2(O_F)$(cf. [16]).

By the work of Wiles (cf. [21]) the Main Conjecture for odd primes has been proved but Federer's analog is still open. The major aim of this paper is to determine all real quadratic number fields for which the 2-Sylow-subgroup of $K_2(O_F)$ is elementary abelian. According to Kolster (cf. [15]) the Birch-Tate Conjecture is valid for all these fields. In the course of this paper we also obtain similar results about imaginary quadratic and real biquadratic number fields (Theorem 2.8 and 2.11) and construct a family of number fields for which the Birch-Tate Conjecture is valid but the 2-Sylow-subgroup of the tame kernel is not elementary abelian (Theorem 3.1).

Throughout we fix the following notations:

For a (finite) commutative group A

$A(p)$	p-Sylow-subgroup of A
${}_pA(A^p)$	kernel (resp. image) of multiplication by p
p^k- rk(A)	p^k-rank of A, i.e., the number of cyclic components of A, whose order is divisible by p^k

For a number field F

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$\gamma(F)$	the set of all primes of F
S_F or S	a finite subset of $\gamma(F)$ containing the infinite primes
$F_{\mathfrak{p}}$	completion of F at $\mathfrak{p} \in \gamma(F)$
O_F (resp. O_F^S)	the ring of integers (resp. S-integers) of F
U_F (resp. U_F^S)	units (resp. S-units) of F
J_F (resp. J_F^S)	idelgroup (resp. S-idelgroup) of F
$C(F)$ (resp. $C^S(F)$)	class-group (resp. S-class-group) of F
$h(F)$ (resp. $h^S(F)$)	order of $C(F)$ (resp. $C^S(F)$)
$\mu(F)$	group of roots of unity in F
For a natural number n	
$n(p)$	p-primary part of n, i.e., the highest power of p dividing n
ζ_n	primitive n-th root of unity

1. **S-class numbers.** Let E/F be a cyclic extension of number fields, S_F be a finite set of primes containing the infinite primes and $S_E := \{\mathfrak{P} \in \gamma(E) : \mathfrak{P} \mid \mathfrak{p} \text{ and } \mathfrak{p} \in S_F\}$. If there is no confusion we write S instead of S_F resp. S_E. For an arbitrary number field K, let H_K^S be the Hilbert S-class field, i.e., the maximal abelian, unramified extension of K, in which all finite primes of S are totally decomposed. Furthermore we define $\mathfrak{N}_{E/F}^S := \{y \in F^x : y \text{ is a local norm at } E/F \text{ for all } \mathfrak{p} \notin S\}$.

LEMMA 1.1. *Let S contain all ramified primes in E/F and $\langle\sigma\rangle = \mathrm{Gal}(E/F)$. Then*

$$1 \longrightarrow \operatorname{Ker} N_{E/F} \longrightarrow J_E/J_E^S E^x \xrightarrow{N_{E/F}} J_F/J_F^S F^x \longrightarrow \mathrm{Gal}(H_F^S \cap E/F) \longrightarrow 1$$

and

$$1 \longrightarrow J_E^{1-\sigma} J_E^S E^x / J_E^S E^x \longrightarrow \operatorname{Ker} N_{E/F} \xrightarrow{\psi} \mathfrak{N}_{E/F}^S / N_{E/F}(E^x) U_F^S \longrightarrow 1$$

are exact sequences, where $\psi_{E/F}(\alpha J_E^S E^x) := y N_{E/F}(E^x) U_F^S$ with $N_{E/F}(\alpha) = \beta \cdot y$, $\beta \in J_F^S$.

PROOF. All we have to show is that $\psi_{E/F}$ is surjective.
For $y \in \mathfrak{N}_{E/F}^S$ we define $\beta = (\beta_{\mathfrak{p}}) \in J_F^S$ as follows:

$$\text{If } \mathfrak{p} \in S \text{ then } \beta_{\mathfrak{p}} = y^{-1}.$$
$$\text{If } \mathfrak{p} \notin S \text{ then } \beta_{\mathfrak{p}} = 1.$$

The idel $\beta \cdot y$ is locally everywhere a norm, hence there exists an idel $\alpha \in J_E$ such that $N_{E/F}(\alpha) = \beta \cdot y$. ■

Next we assume that E/F is a cyclic extension of prime degree p. For this, let $k_{E/F}^S$ be the sum of the number of primes of F that ramify in E/F and the number of primes in S that are inert in E/F and define $r_{E/F}^S$ as follows:

$$r_{E/F}^S := p\text{-}\mathrm{rk}\left(\ker\left(U_F^S/(U_F^S)^p \longrightarrow F^x/N_{E/F}(E^x)\right)\right)$$

LEMMA 1.2. *For S as above, let $S' := \{\mathfrak{p} \in S : \mathfrak{p}$ is ramified or inert in $E/F\}$.*

(i) If $k^S_{E/F} = 0$ then $\mathfrak{R}^S_{E/F} = N_{E/F}(E^x)$ and if $k^S_{E/F} > 0$,

$$1 \longrightarrow \mathfrak{R}^S_{E/F}/N_{E/F}(E^x) \xrightarrow{\chi} \prod_{\mathfrak{p}\in S'} \mathrm{Gal}(E_{\mathfrak{P}}/F_{\mathfrak{p}}) \xrightarrow{\varphi} \mathrm{Gal}(E/F) \longrightarrow 1$$

is an exact sequence, where χ is the global norm residue symbol and $\varphi\big((\sigma_{\mathfrak{p}})_{\mathfrak{p}\in S'}\big) := \prod \sigma_{\mathfrak{p}}$.

(ii) Now we assume that S contains all ramified primes in E/F and $k^S_{E/F} > 0$. Then the following holds:

$$p\text{-}\mathrm{rk}\left(\mathfrak{R}^S_{E/F}/N_{E/F}(E^x)U^S_F\right) = \begin{cases} r^S_{E/F} + |S'| - |S_F| - 1 & \text{if } \zeta_p \in \mu(F) \\ r^S_{E/F} + |S'| - |S_F| & \text{if } \zeta_p \notin \mu(F) \end{cases}$$

PROOF. For (i) use global class field theory, (ii) is then a trivial consequence of (i) and the unit theorem. ∎

Now let M be an arbitrary number field and $L = M(\sqrt{-1})$ be a quadratic extension; choose $d \not\equiv -1 \bmod M^{x^2}$ such that $F = M(\sqrt{d})$ and $K = M(\sqrt{-d})$ are quadratic extensions. Let $E = M(\sqrt{-1}, \sqrt{d})$ and S be the set of infinite and dyadic primes, $\langle\sigma\rangle = \mathrm{Gal}(E/F)$ and $\langle\tau\rangle = \mathrm{Gal}(E/K)$. We assume that

$$(a)\ h^S(M) \text{ is odd and } (b)\ h^S(L) \text{ is odd}.$$

Then the commutatives diagrams

$$\begin{array}{ccc} J_E & \xrightarrow{N_{E/F}} & J_F \\ \uparrow & & \uparrow \\ J_K & \xrightarrow{N_{K/M}} & J_M \end{array} \quad \text{resp.} \quad \begin{array}{ccc} J_E & \xrightarrow{N_{E/K}} & J_K \\ \uparrow & & \uparrow \\ J_F & \xrightarrow{N_{F/M}} & J_M \end{array}$$

where the horizontal arrows are the canonical injections, and the assumption (a) implies

$$J_K J^S_E E^x / J^S_E E^x(2) \subset \mathrm{Ker}\, N_{E/F}(2)$$

resp.

$$J_F J^S_E E^x / J^S_E E^x(2) \subset \mathrm{Ker}\, N_{E/K}(2)$$

Now let $\alpha \in J_E$ then we have the following formulas

$$\alpha \cdot \sigma(\alpha)^{-1} = \big(\alpha \cdot \tau(\alpha)\big) \cdot \Big(\tau(\alpha)^{-1} \cdot \tau\sigma\big(\tau(\alpha)^{-1}\big)\Big)$$

resp.

$$\alpha \cdot \tau(\alpha)^{-1} = \big(\alpha \cdot \sigma(\alpha)\big) \cdot \Big(\sigma(\alpha)^{-1} \cdot \sigma\tau\big(\sigma(\alpha)^{-1}\big)\Big)$$

So the assumption (b) implies

$$J_E^{1-\sigma} J^S_E E^x / J^S_E E^x(2) \subset J_K J^S_E E^x / J^S_E E^x(2)$$

resp.

$$J_E^{1-\tau} J^S_E E^x / J^S_E E^x(2) \subset J_F J^S_E E^x / J^S_E E^x(2)$$

In particular, if $N_{E/K}$(resp. $N_{E/F}$) is surjective we see that the following identity holds:

$$J_E^{1-\sigma} J_E^S E/ J_E^S E^x(2) = J_K J_E^S E/ J_E^S E^x(2)$$

resp.
$$J_E^{1-\tau} J_E^S E^x/ J_E^S E^x(2) = J_F J_E^S E^x/ J_E^S E^x(2)$$

COROLLARY 1.3. *For $M = \mathbb{Q}$ and $d \in \mathbb{N}$ square-free we have:*

(i) $h^S(E) = \left| \operatorname{Ker}\left(C^S(F) \longrightarrow C^S(E)\right)\right|^{-1} \cdot \left| \operatorname{Gal}(H_K^S \cap E/K)\right|^{-1} \cdot h^S(F) \cdot h^S(K)$

(ii) $h^S(E) = \left| \operatorname{Ker}\left(C^S(K) \longrightarrow C^S(E)\right)\right|^{-1} \cdot 2^{r_{E/F}^S + |S_F'| - 1 - |S_F|} \cdot h^S(F) \cdot h^S(K)$, *if* $d \not\equiv 1 \bmod 8$.

(iii) $$\left|\operatorname{Ker}\left(C^S(K) \longrightarrow C^S(E)\right)\right| = \begin{cases} \left|\operatorname{Ker}\left(C^S(F) \longrightarrow C^S(E)\right)\right| \cdot 2^{r_{E/F}^S - 1} & \text{for } d \not\equiv \pm 1 \ (8) \\ \left|\operatorname{Ker}\left(C^S(F) \longrightarrow C^S(E)\right)\right| & \text{for } d \equiv -1 \ (8) \end{cases}$$

PROOF.

(i),(ii) Using Lemma 1.1, 1.2 and the remark above we get the claim. Note that for $d \not\equiv 1 \bmod 8$ the norm map $N_{E/K}$ is not surjective.

(iii) This is only a trivial consequence of (i) and (ii). ■

COROLLARY 1.4. *Let $a, d \in \mathbb{N}$ be square-free and relatively prime, $M = \mathbb{Q}(\sqrt{a})$. With the notations as above the formula*

$$h^S(E)(2) = \left| \operatorname{Gal}(H_K^S \cap E/K)\right|^{-1} \cdot h^S(F)(2) \cdot h^S(K)(2)$$

holds in the cases:

(i) $(a = 2$ *or* $a = 2p$, $p \equiv \pm 3 \ (8))$ *and* $d \not\equiv 7 \ (8)$

(ii) $a = p \equiv 3 \ (8)$ *and* $d \not\equiv 5, 7 \ (8)$

(iii) $a = p \equiv 5 \ (8)$ *and* $d \not\equiv 3, 7 \ (8)$

PROOF. See Corollary 1.3. ■

Now let E/F be a quadratic extension and assume that the S-class number $h^S(F)$ of F is odd, which implies $k_{E/F}^S > 0$. Then the exact hexagon discussed by Conner and Hurrelbrink (cf. [8]) enables us to obtain a 2-rank formula for the S-class-group $C^S(E)$ of E.

LEMMA 1.5. *Notations as above.*

$$2\text{-}\operatorname{rk}\left(C^S(E)\right) = r_{E/F}^S + k_{E/F}^S - |S_F| - 1.$$

PROOF. By [8, Lemma 9.1] we have the following exact sequence

$$1 \longrightarrow \ker \pi \longrightarrow U_F^S/(U_F^S)^2 \xrightarrow{\pi} R_0^S(E/F) \longrightarrow C^S(E)/\left(C^S(E)\right)^2 \longrightarrow 1,$$

where $R_0^S(E/F)$ is an elementary abelian 2-group with

$$2\text{-}\operatorname{rk}\left(R_0^S(E/F)\right) = k_{E/F}^S - 1 \text{ and } 2\text{-}\operatorname{rk}(\ker \pi) = r_{E/F}^S$$

[cf. 8, Lemma 1.4.]. ■

At this point we would like to remark that it is also possible to obtain a 4-rank formula for $C^S(E)$, and that this formula is analogous to the one discussed by Redei and Reichardt for the class-group in the narrow sense [cf. 2], [cf. 19].

COROLLARY 1.6. *Let $E = \mathbb{Q}(\sqrt{d})$ with $d \neq 1$ square-free, S be the set of infinite and dyadic primes and let t denote the number of odd prime factors of d and $r := r^S_{E/\mathbb{Q}}$.*

$$(i) \qquad 2\text{-}\operatorname{rk}\big(C^S(E)\big) = \begin{cases} r+t+1-|S_E| & \text{if } d > 0 \\ r+t-1 & \text{if } d < 0, d \not\equiv 1 \ (8) \\ r+t-2 & \text{if } d < 0, d \equiv 1 \ (8) \end{cases}$$

$$(ii) \qquad 2\text{-}\operatorname{rk}\big(K_2(O_E)\big) = \begin{cases} r+t & \text{if } d > 0 \\ r+t-1 & \text{if } d < 0 \end{cases}$$

PROOF. (i) is only a special case of Lemma 1.5 and (ii) is then a consequence of [15, Lemma 1.4], see also [4, Theorem 1]. ∎

2. The 2-Sylow-subgroup of the tame kernel.

2.1. Quadratic number fields. Let $E = \mathbb{Q}(\sqrt{d}, \sqrt{-1})$, $F = \mathbb{Q}(\sqrt{d})$ and $K = \mathbb{Q}(\sqrt{-d})$ with $d > 1$ square-free. By S we denote the set of infinite and dyadic primes and by t the number of odd prime factors of d.

THEOREM 2.1. *The following assertions are equivalent:*

(i) $K_2(O_F)(2)$ is elementary abelian.

(ii) $C^S(K)(2)$ is elementary abelian and $r^S_{F/\mathbb{Q}} - 1 = r^S_{K/\mathbb{Q}}$.

(iii) $C^S(K)(2)$ is elementary abelian and for all $d' | d : d' \not\equiv 7 \ (8)$.

PROOF. (i)⇔(ii): By [15, Theorem 3.1] and Corollary 1.3 the 2-Sylow-subgroup of the tame kernel is elementary abelian if and only if $d \not\equiv 7 \ (8)$ and the following formula is valid: $h^S(K)(2) = |\operatorname{Gal}(H^S_K \cap E/K| \cdot 2^{2\text{-}\operatorname{rk}(C^2(F))}$. Using Corollary 1.6 we get the claim.

(ii)⇔(iii): trivial. ∎

COROLLARY 2.2. *If $p \equiv \pm 3 \ (8)$ is a prime factor of d, the $K_2(O_F)(2)$ is elementary abelian if and only if $C(K)(2)$ is elementary abelian and for all $d' \mid d : d' \not\equiv 7 \ (8)$.*

COROLLARY 2.3. *Assume that all odd prime factors of d are congruent to* 1 mod 8.

(i) $d = p : K_2(O_F)(2)$ is elementary abelian iff $p \neq x^2 + 32y^2$ for all $x, y \in \mathbb{Z}$.

(ii) $d = 2p : K_2(O_F)(2)$ is elementary abelian iff either $p \neq x^2 + 32y^2$ and $p \equiv 1 \ (16)$ or $p = x^2 + 32y^2$ and $p \equiv 9 \ (16)$.

(iii) $d = p \cdot q$, $p \cdot q = 2v^2 - u^2$ with $u, v \in \mathbb{N}$

$$K_2(O_F)(2) \text{ is elementary abelian iff } \left(\frac{p}{q}\right) = -1 \text{ and } \left(\frac{v}{p}\right)\left(\frac{v}{q}\right) = -1.$$

(iv) $d = 2p \cdot q$, $p \cdot q = 2v^2 - u^2$ with $u, v \in \mathbb{N}$:

$$K_2(O_F)(2) \text{ is elementary abelian iff } \left(\frac{p}{q}\right) = -1$$

$$\text{and } \left(\frac{v}{p}\right)\left(\frac{v}{q}\right) = \begin{cases} 1 & \text{if } pq \equiv 1 \ (16) \\ -1 & \text{if } pq \equiv 9 \ (16) \end{cases}$$

Here $(\frac{-}{-})$ denotes the Legendre-symbol.

PROOF. By Theorem 2.1 $K_2(O_F)(2)$ is elementary abelian iff $h(K) = 2^{t+1}$. So the corollary is a trivial consequence of [5], [12], [13], [18]. ∎

COROLLARY 2.4. *Each of the following two assertions implies, that $K_2(O_F)(2)$ is not elementary abelian:*

(i) *Let $d = 2d'$ and let the number of prime factors $p \equiv 3 \ (8)$ of d' be even and greater than zero.*

(ii) *Let $d \equiv 1 \ (8)$ and $p \equiv 5 \ (8)$ a prime factor of d.*

PROOF. Assume that $K_2(O_F)(2)$ is elementary abelian. We then get by Corollary 2.2:

$$C(K)(2) \text{ is elementary abelian and for all } m \mid d : m \not\equiv 7 \ (8).$$

But this is impossible, since $\alpha = (-8, d')$ resp. $\alpha = (-4, d)$ are of the second kind (cf. [19]). ∎

COROLLARY 2.5. *Let $d < 100$, then $K_2(O_F)(2)$ is elementary abelian if and only if*

$$d \in \{2, 3, 5, 6, 10, 11, 13, 17, 19, 22, 26, 29, 33, 34, 37, 38, 43, \\ 51, 53, 58, 59, 61, 67, 73, 74, 82, 83, 85, 86, 89, 97\}.$$

COROLLARY 2.6. *The 2-Sylow-subgroup of the tame kernel of real quadratic number fields can be elementary abelian of arbitrary rank.*

PROOF. Let $d_1 := 2p_1$ with a prime $p_1 \equiv 5 \ (8)$. If d_n is defined, then we choose a prime p_{n+1} with $p_{n+1} \equiv 5 \ (8)$ and $p_{n+1} \equiv -1 \ (p_i)$ for $1 \le i \le n$ and define $d_{n+1} := d_n \cdot p_{n+1}$. Let $K_n = \mathbb{Q}(\sqrt{-d_n})$ and $F_n = \mathbb{Q}(\sqrt{d_n})$, by induction the 2-Sylow-subgroup of the class-group of K_n is elementary abelian (cf. [11]) and we deduce from Corollary 2.2 that $K_2(O_{F_n})(2)$ is elementary abelian of rank $n + 1$. ∎

REMARK 2.7.

(i) For $F = \mathbb{Q}(\sqrt{d})$ the tame kernel is generated by Steinberg-symbols if and only if $d = 2$, $d = 5$ or $d = 13$.

(ii) For a suitable choice of d the tame kernel of $F = \mathbb{Q}(\sqrt{d})$ contains an element of order 2^k, where k is an arbitrary natural number.

PROOF. cf. [3], [4]. ∎

THEOREM 2.8. *The following assertions are equivalent:*

(i) *$K_2(O_K)(2)$ is elementary abelian.*

(ii) *$d \not\equiv 1 \ (8)$, $C^S(F)$ is elementary abelian and $r^S_{E/F} + r^S_{F/\mathbb{Q}} = r^S_{K/\mathbb{Q}} + 2$ for $d \not\equiv \pm 1 \ (8)$.*

PROOF. We conclude from [9]:

$K_2(O_K)(2)$ is elementary abelian iff $d \not\equiv 1 \ (8)$ and $\operatorname{Ker}\big(C^S(E)(2) \longrightarrow C^S(F)(2)\big)$ is equal to $\operatorname{im}\big({}_2C^S(K) \longrightarrow C^S(E)\big)$.

CASE (A). $d \not\equiv \pm 1 \ (8)$ It is well known that for $d \not\equiv -1 \ (8)$ the homomorphism $C^S(F) \longrightarrow C^S(E)$ is injective (cf. [15]), hence we get by Corollary 1.3 and Lemma 1.5:

$K_2(O_K)(2)$ is elementary abelian iff $C^S(F)(2)$ is elementary abelian and $r^S_{E/F} + r^S_{F/\mathbb{Q}} = r^S_{K/\mathbb{Q}} + 2$.

CASE (B). $d \equiv -1 \pmod 8$. Clearly $r^S_{F/\mathbb{Q}} = r^S_{K/\mathbb{Q}}$. By Corollary 1.3 we have:

$$\left|\mathrm{Ker}\big(C^S(K) \to C^S(E)\big)\right| = \left|\mathrm{Ker}\big(C^S(F) \to C^S(E)\big)\right|,$$

hence $K_2(O_K)(2)$ is elementary abelian iff $h^S(F)(2) = 2^{r+t-2}$ with $r := r^S_{F/\mathbb{Q}}$, i.e., $C^S(F)(2)$ is elementary abelian. ∎

COROLLARY 2.9. *Let $d \equiv \pm 3 \pmod 8$, then $K_2(O_K)(2)$ is elementary abelian if and only if $C^+(F)(2)$ is elementary abelian, where $C^+(F)$ denotes the class-group of F in the narrow sense.*

PROOF. All we have to show is:

$$(*)\qquad \begin{array}{c} C^S(F)(2) \text{ is elementary abelian and} \\ r^S_{E/F} + r^S_{F/\mathbb{Q}} = r^S_{K/\mathbb{Q}} + 2 \text{ iff } C^+(F)(2) \text{ is elementary abelian.} \end{array}$$

CASE (A). $d \equiv 5 \pmod 8$, then $C^S(F) = C(F)$. Let e_0 be the fundamental unit of F. If $N_{F/\mathbb{Q}}(e_0) = -1$, then $C^+(F) = C(F)$ and $r^S_{E/F} = r^S_{F/\mathbb{Q}} = 1$, $r^S_{K/\mathbb{Q}} = 0$, hence (*) is valid.

Now let $N_{F/\mathbb{Q}}(e_0) = +1$. Hence we get $r^S_{E/F} = 2$. If $C^+(F)(2)$ is elementary abelian, so is $C(F)(2)$ and $-1 \notin N_{F/\mathbb{Q}}(F^\times)$ (cf. [11]), which implies $r^S_{F/\mathbb{Q}} = r^S_{K/\mathbb{Q}}$. Conversely, if $C(F)(2)$ is elementary abelian and $r^S_{F/\mathbb{Q}} = r^S_{K/\mathbb{Q}}$, then we get $r^S_{F/\mathbb{Q}} = r^S_{K/\mathbb{Q}} = 0$, which implies that $C^+(F)(2)$ is elementary abelian.

CASE (B). $d \equiv 3 \pmod 8$, then $4\text{-}\mathrm{rk}\big(C^+(F)\big) = 4\text{-}\mathrm{rk}\big(C(f)\big)$ (cf. [11]). Let $(2)\cdot O_F = \mathfrak{p}^2$ and $U^S_F = \langle -1, \alpha, e_0\rangle$, where again e_0 is the fundamental unit of F. If $\mathfrak{p}$ is a principal ideal, i.e., $\mathfrak{p} = (\alpha)$ and $N_{F/\mathbb{Q}}(\alpha) = -2$, then we get $r^S_{E/F} = r^S_{F/\mathbb{Q}} = 1$, $r^S_{K/\mathbb{Q}} = 0$ and $C^S(F) = C(F)$, hence (*) is valid. If $\mathfrak{p}$ is not a principal ideal, then clearly $U^S_F = \langle -1, 2, e_0\rangle$ and $r^S_{E/F} = 2$. Furthermore, $r^S_{F/\mathbb{Q}} = r^S_{K/\mathbb{Q}}$ iff $r^S_{F/\mathbb{Q}} = r^S_{K/\mathbb{Q}} = 0$, so again (*) is valid.

COROLLARY 2.10. *Let $d \not\equiv 1 \pmod 8$ and $d < 200$; $K_2(O_K)(2)$ is elementary abelian if and only if $d \neq 34$ or $d \neq 194$.*

2.2. Real biquadratic number fields. Let $E = \mathbb{Q}(\sqrt{d}, \sqrt{a}, \sqrt{-1})$, $F = \mathbb{Q}(\sqrt{d}, \sqrt{a})$, $K = \mathbb{Q}(\sqrt{-d}, \sqrt{a})$ and $M = \mathbb{Q}(\sqrt{a})$ with $a, d \in \mathbb{N}$ such that the assumptions of Corollary 1.4 are valid; furthermore let S be the set of infinite and dyadic primes.

THEOREM 2.11. *The following assertions are equivalent:*

(i) $K_2(O_F)(2)$ is elementary abelian.

(ii) $C^S(K)(2)$ is elementary abelian and $r^S_{K/M} = r^S_{F/M} - 2$.

PROOF. For all choices of $a, d \in \mathbb{N}$ we have $1 \geq r^S_{K/M} \geq r^S_{F/M} - 2$. By [15, Theorem 3.1], Corollary 1.4 and Lemma 1.5 we get:

$$K_2(O_F)(2) \text{ is elementary abelian iff } h^S(K)(2) = 2^{k_{K/M} + r^S_{F/M} - 6}.$$

Using Lemma 1.5 again we get the claim. ∎

COROLLARY 2.12. *Let $d' \equiv 7 \ (8)$ and $d' \mid d$, then $K_2(O_F)(2)$ is not elementary abelian.*

PROOF. We assume that $K_2(O_F)(2)$ is elementary abelian. In particular, this implies:

$$(*) \qquad r^S_{K/M} = r^S_{F/M} - 2$$

CASE (A). $a = p$ or $a = 2 \cdot p$ with a prime $p \equiv 5 \ (8)$. In this case we have $U^S_M = \langle -1, 2, e_0 \rangle$, where e_0 is the fundamental unit of M and $N_{M/\mathbb{Q}}(e_0) = -1$. If e_0 or $-e_0$ is a norm at F/M, the -1 is a norm at $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$, which is impossible since $d' \equiv 7 \ (8)$. If 2 is a norm at F/M, then 2 is a norm at K/M. This implies together with (*), that $\pm e_0$ is a norm at F/M. Therefore we get $\pm 2 \cdot e_0$ is a norm at F/M, but then again -1 is a norm at $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$, which gives the desired contradiction.

CASE (B). $a = p$ or $a = 2 \cdot p$ or $a = 2$ with a prime $p \equiv 3 \ (8)$. Analogous to Case (A). ∎

EXAMPLES 2.13. *The 2-Sylow-subgroup of $K_2(O_F)$ is elementary abelian for:*

(i) $a = p \equiv 3 \ (8)$ *and* $d = q \equiv 3 \ (8)$.

(ii) $a = p \equiv 3 \ (8)$ *and* $d = 2 \cdot q$ *with* $q \equiv 3 \ (8)$ *and* $\left(\frac{q}{p}\right) = -1$.

(iii) $a = p \equiv 5 \ (8)$ *and* $d = 2 \cdot q$ *with* $q \equiv 5 \ (8)$ *and* $\left(\frac{q}{p}\right) = -1$.

Here p and q are, as always, primes and $(\frac{\cdot}{\cdot})$ denotes the Legendre-symbol.

PROOF.

(i) Since $h^S(E)$ is odd the claim is trivial.

(ii) According to Theorem 2.11 we have to show

(a) $C^S(K)(2)$ is elementary abelian

(b) $r^S_{K/M} = r^S_{F/M} - 2$.

Using the class-number formula for biquadratic number fields (cf. [14]) we get:

$$h^S(K)(2) = 1/2 \cdot h(K)(2) = 1/2 \cdot 1/2 \cdot h\big(\mathbb{Q}(\sqrt{-2q})\big)(2) \cdot h\big(\mathbb{Q}(\sqrt{-2pq})\big)(2) = 2.$$

So (A) is valid and by Lemma 1.5 it follows that $r^S_{K/M}$ is equal to 1. This yields $e_0 \in N_{F/M}(F^x)$, where e_0 is the fundamental unit of M. Now let $U^S_M = \langle -1, \alpha, e_0 \rangle$ with $(2) \cdot O_M = (\alpha)^2$ and $N_{M/\mathbb{Q}}(\alpha) = -2$. By reciprocity $\pm\alpha$ is a norm at F/M if and only if $\pm\alpha$ is a local norm at the dyadic prime, but this is trival since $N_{M_2/\mathbb{Q}_2}(\pm\alpha) \in N_{\mathbb{Q}_2(\sqrt{2q})/\mathbb{Q}_2}\big(\mathbb{Q}_2(\sqrt{2q})^x\big)$. So we get $r^S_{F/M} = 3$, hence (b) is valid.

(iii) Analogous to (ii). ∎

3. **The 2-primary part of the Birch-Tate Conjecture.** According to Kolster (cf. [15]) the 2-primary part of the Birch-Tate Conjecture for the totally real number field F is valid if the 2-Sylow-subgroup of the tame kernel is elementary abelian. For quadratic and several biquadratic number fields F, we can decide whether the 2-Sylow-subgroup of $K_2(O_F)$ is elementary abelian or not, cf. Theorem 2.1. and 2.11. Now we construct a

family of number fields, for which $K_2(O_F)(2)$ is not elementary abelian but the 2-primary part of the Birch-Tate Conjecture is valid.

For this let $M = \mathbb{Q}(\sqrt{2})$, $F = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ and $K = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ with a prime $p \equiv 1 \ (8)$. Theorem 2.11 implies that $K_2(O_F)(2)$ is not elementary abelian and by [15, Theorem 3.4] the Birch-Tate Conjecture is valid if $h^S(K)(2) = 2^{r^S_{F/M}}$. By Lemma 1.5 we get $r^S_{F/M} \geq 2$ and if we choose p such that $h\big(\mathbb{Q}(\sqrt{-p})\big) \equiv 4 \ (8)$, then $1 + \sqrt{2} \notin N_{F/M}(F^{x})$(cf. [8]), i.e., $r^S_{F/M} = 2$. Using the class-number formula for bi-quadratic number fields we get:

$$h^S(K)(2) = h\big(\mathbb{Q}(\sqrt{-2p})\big)(2).$$

If furthermore $p \equiv 1 \ \ (16)$ then $h^S(K)(2)$ is equal to 2^2; so we have proved the following.

THEOREM 3.1. *Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{p})$ and p be a prime satisfying the following condition:*

$$p \equiv 1 \ \ (16) \ \textit{and} \ p \neq x^2 + 32y^2 \ \textit{for all} \ x, y \in \mathbb{Z}$$

Then the Birch-Tate Conjecture is valid and the 2-Sylow-subgroup of $K_2(O_F)$ is not elementary abelian.

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