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Quasi-adelic measures and equidistribution on $\mathbb{P}^1$

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Abstract. Baker and Rumely, Favre, Rivera, and Letelier, and Chambert-Loir proved an important arithmetic equidistribution theorem for points of small height associated to an adelic measure. To broaden the scope in which arithmetic equidistribution may be employed, we generalize the notion of an adelic measure to that of a quasi-adelic measure and show that arithmetic equidistribution holds for quasi-adelic measures as well. We exhibit examples of non-adelic, quasi-adelic measures arising from the dynamics of quadratic rational maps. In fact, we show that the measures that arise in applications of arithmetic equidistribution theorems are typically not adelic. Finally, we motivate our definition of a quasi-adelic measure by relating it to a seemingly different problem in arithmetic dynamics arising from results of Call, Tate, and Silverman in the study of abelian varieties.

Key words: quasi-adelic measures, equidistribution, Berkovich space, variation of heights, unlikely intersections
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Contents

1. *Introduction*

Baker and Rumely [6, 7] and Favre, Rivera and Letelier [29], employing a potential-theoretic approach, proved an important arithmetic equidistribution theorem for points of small height associated to *adelic* measures defined on the Berkovich compactifications of the projective line. Around the same time, Chambert-Loir, Thuillier, and Yuan [12, 54, 56], using an approach based on Arakelov theory, established generalizations of this arithmetic equidistribution theorem to projective varieties other than $\mathbb{P}^1$. The 'adelic' hypothesis appears in all the aforementioned results. It appeared first in Zhang's notion of an adelic metrized line bundle [58, 59].

We briefly describe the arithmetic equidistribution theorem of [6, 7, 12, 29] next. Let k be a product formula field, for example, a number field or the function field of a smooth and projective curve. Denote by $\mathcal{M}_k$ its set of places. Let $\mu = \{\mu_v\}_{v\in\mathcal{M}_k}$ be a collection of probability measures μ_v on the Berkovich projective line $\mathbb{P}^{1,an}_v$ for $v \in \mathcal{M}_k$. The measure μ is adelic if all measures μ_v have continuous potentials g_v and the potentials g_v are trivial at all but finitely many places v of k, meaning $g_v(\cdot) = \log^+ |\cdot|_v$. To each adelic measure, we associate a height function $\hat{h}_\mu : \mathbb{P}^1(k^{\mathrm{sep}}) \to \mathbb{R}$ given by the sum of potential functions of the measures μ_v. The main theorems of [7, 29] then assert that Galois orbits of points $t_n \in \mathbb{P}^1(k^{\mathrm{sep}})$ with $\hat{h}_\mu(t_n) \to 0$ must equidistribute with respect to μ_v at all places v of k.

In this article, we extend the notion of an adelic measure in [6, 7, 12, 29] to that of a quasi-adelic measure. Details appear in §2. Our results may be briefly summarized as follows.

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- We prove an equidistribution theorem for heights associated to quasi-adelic measures (Theorem 1.1).
- We show that non-adelic measures arise naturally in studying families of dynamical systems $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ (Theorem 1.3).
- We provide examples for which non-adelic measures are quasi-adelic (Theorem 1.4).
- We relate the notion of a quasi-adelic measure to the variation of canonical height as studied by Silverman, Tate, and Call and Silverman originally in the setting of abelian varieties (Theorem 1.6).

An early version of this article appeared in the preprint [**55**], containing a proof of Theorem 1.1. The theorem was used in [**22**] to obtain an 'unlikely-intersections' statement of dynamics on $\mathbb{P}^1$ in the setting where the measures were quasi-adelic but not adelic.

1.1. *Quasi-adelic arithmetic equidistribution theorem.* Our first result generalizes the arithmetic equidistribution theorem in [**6**, **7**, **12**, **29**]. We extend the notion of an adelic measure to that of a *quasi-adelic* measure $\mu = \{\mu_v\}_{v\in\mathcal{M}_k}$ allowing more flexibility for the potentials of μ_v. Instead of requiring them to be trivial at all but finitely many places, we impose a certain summability condition. Any adelic measure is quasi-adelic. There is a natural height function $\hat{h}_\mu : \mathbb{P}^1(k^{\mathrm{sep}}) \to \mathbb{R}$ associated to each quasi-adelic measure; see §2.2. In contrast to the adelic case, this height can have non-trivial contributions from infinitely many local heights. We establish that arithmetic equidistribution holds for quasi-adelic measures as well. Our approach is potential-theoretic and follows the proof strategy of [**6**, **7**, **29**].

THEOREM 1.1. *Let k be a product formula field and let $\mu = \{\mu_v\}_{v\in\mathcal{M}_k}$ be a quasi-adelic measure. Suppose that S_n is a sequence of* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$*-invariant subsets of* $\mathbb{P}^1(k^{\mathrm{sep}})$ *such that $|S_n| \to \infty$ and $\hat{h}_\mu(S_n) \to 0$ as $n \to \infty$. Then for each $v \in \mathcal{M}_k$, the sequence of probability measures $[S_n]_v$, weighted equally on the points in S_n, converges weakly to μ_v on $\mathbb{P}^{1,an}_v$ as $n \to \infty$.*

Theorem 1.1 can be motivated by conjectures in arithmetic geometry fitting in the theme of 'unlikely intersections'; see [**57**] for a great overview of related problems. We discuss applications of Theorem 1.1 later. Broadly speaking, people are interested in the study of families of rational maps $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ parametrized by t in a quasi-projective algebraic curve X. When X is the punctured Riemann sphere and f_t is defined over a product formula field k, Theorem 1.1 allows us to understand the distribution of parameters λ in X for which the value $c(\lambda)$ of a rational map $c \in k(t)$ is preperiodic for f_λ.

A family $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ of rational maps parametrized by t in $\mathbb{P}^1$ and defined over a field k yields a rational function $f \in k(t)(z)$. We call a pair $(f, c) \in k(t)(z) \times k(t)$ where $\deg_z f \geq 2$, a *dynamical pair*.

1.2. *Dynamical pairs are typically not adelic.* A dynamical pair induces a *canonical measure* $\mu_{f,c} = \{\mu_{f,c,v}\}_{v\in\mathcal{M}_k}$; see §4.3 for the precise definition. If $\mu_{f,c}$ is adelic (respectively quasi-adelic), then we call the dynamical pair (f, c) adelic (respectively quasi-adelic). Many instances of these measures have been previously studied and are well known to be adelic. For example, when $f \in k(z)$ and $c(t) = t$, the measure $\mu_{f,c}$ was

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introduced in [5, 6, 29] and is adelic. When f is a Lattès map induced by a multiplication by m on an elliptic surface, the measure $\mu_{f,c}$ is studied in [20, 23] and is adelic as well. It came as a surprise when the first non-adelic dynamical pair appeared in [22]. Our next main theorem demonstrates that, in fact, the measures $\mu_{f,c}$ are typically not adelic. To state our theorem, we introduce some terminology.

For $f \in k(t)(z)$, we write $f_t(z) = P_t(z)/Q_t(z)$ with coprime $P_t, Q_t \in k(t)[z]$. We say that $\alpha \in \mathbb{P}^1(k^{\mathrm{sep}})$ is a *degenerating parameter* if $\deg_z(f_\alpha) < \deg_z(f)$. The degenerating parameters of f form a finite set denoted by $\mathrm{Sing}(f)$. For each $\alpha \in \mathrm{Sing}(f)$, we denote by $\mathcal{H}_{f,\alpha}$ the common zero locus of P_α and Q_α viewed as homogeneous polynomials of degree $\deg_z(f)$. Equivalently, $\mathcal{H}_{f,\alpha}$ is the complement of the maximal affine open subset of $\mathbb{P}^1$ on which f_α is a morphism. We say that the elements of $\mathcal{H}_{f,\alpha}$ are the *α-holes* of f; see §4. The notion of holes has appeared in [15, 16] where it played a crucial role in studying compactifications of moduli spaces of rational maps.

Example 1.2. If $f_t(z) = ((z^3 + z + t)/z) \in \mathbb{Q}(t)(z)$, then $\mathrm{Sing}(f) = \{0, \infty\}$, as we have

$$f_0(z) = z^2 + 1 \quad \text{and} \quad f_\infty(z) = \infty.$$

Moreover,

$$\mathcal{H}_{f,0} = \{0\} \quad \text{and} \quad \mathcal{H}_{f,\infty} = \{\infty\}.$$

We say that $h \in \overline{k}$ is an *exceptional point* of $\varphi \in \overline{k}(z)$ if it is a totally ramified fixed point of φ^2, or equivalently if $(\varphi^2)^{-1}(\{h\}) = \{h\} = \varphi^2(\{h\})$. Recall that over a number field, φ has an exceptional point if and only if φ^2 can be conjugated to a polynomial in $\overline{k}[z]$. Also, any φ can have at most 2 exceptional points, with equality only if it is conjugate to $z^{\pm d}$ [50].

We can now state our second main result.

THEOREM 1.3. *Let k be a number field or the function field of a smooth projective curve defined over a field of any characteristic, and let $f \in k(t)(z)$ be non-isotrivial of degree* $\deg_z f \geq 2$. *Assume that there is an $N \in \mathbb{N}$ and an $\alpha \in \mathbb{P}^1(\overline{k})$ such that, for $g := f^N$,*

(G1) $\alpha \in \mathrm{Sing}(g)$ *and* $\deg_z(g_\alpha) \geq 2$ *and*

(G2) *there exists an element of $\mathcal{H}_{g,\alpha}$ that is not an exceptional point of g_α and*

(G3) *if k is a function field, assume further that g_α is not isotrivial.*

Then, for any $c \in \overline{k}(t)$ such that

(C1) $g_\alpha^n(c(\alpha)) \notin \mathcal{H}_{g,\alpha}$ *for all $n \in \mathbb{N}$ and*

(C2) $c(\alpha)$ *is not preperiodic under the action of g_α,*

the dynamical pair (f, c) is not adelic.

A few remarks regarding Theorem 1.3 are in order.

(1) The conditions that guarantee the failure of the adelic hypothesis are geometric in essence. They are related to degenerations of a dynamical pair. Remarkably though, the reason for the said failure is arithmetic. Our proof of Theorem 1.3 relies on results concerning integral points in orbits, established by Silverman [50, Theorem B] over number fields and in [11, 38] for function fields of characteristic zero and p respectively.

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These allow us to generate infinitely many primitive divisors of certain dynamical sequences, which in turn yield measures corresponding to non-archimedean places with non-trivial potentials.

(2) In fact, when $\deg_z(f) \geq 3$, the conditions in our theorem are typically satisfied, even if we only allow $N = 1$ in Theorem 1.3. Indeed, let $f_t(z) = (P_t(z)/Q_t(z)) \in k(t)(z)$ but not belonging to $k(z)$. Such f will degenerate at some α and we assume that $\alpha = 0$ for simplicity. Then condition (G1) fails only if the specializations $P_0(z)$ and $Q_0(z)$ have more than $d - 1 \geq 2$ common factors. This in turn imposes strong algebraic relations between the coefficients of P_t and Q_t as functions of t. Thus in general, condition (G1) holds. Condition (G2) is automatically satisfied when f_0 has no exceptional point, which is the case for 'most' maps. It is possible that f_0 has an exceptional point and condition (G2) still holds. For instance, note that ∞ is an exceptional point of f_0, as in Example 1.2. However, condition (G2) holds since ∞ is not a 0-hole. The only 'easy' way for condition (G2) to fail is if $f \in k(t)[z]$ is a polynomial. Then the only holes are at ∞ and all specializations are polynomials, so have ∞ as an exceptional point. Other examples in which (G2) fails come from Lattès maps, which are adelic.

(3) Finally, as long as conditions (G1) and (G2) are satisfied, one can easily construct $c(t) \in \overline{k}(t)$ satisfying (C1) and (C2) at will. In fact, our conditions for c are satisfied as soon as the logarithmic Weil height of $c(0)$ is large enough.

1.3. *Quasi-adelic dynamical pairs.* The first example of a non-adelic, quasi-adelic dynamical pair was studied by DeMarco, Wang, and Ye [22], who considered the family of rational maps $g_{\lambda,t}(z) = \lambda z/(z^2 + tz + 1)$ for $t \in \mathbb{P}^1(\overline{\mathbb{Q}})$ where $\lambda \in \overline{\mathbb{Q}}\backslash\{0\}$ is not a root of unity, or λ equals 1. They showed that the non-adelic dynamical pairs $(g_\lambda, 1)$ and $(g_\lambda, -1)$ are quasi-adelic. It is more delicate to show the quasi-adelicity when λ is a root of unity and the starting point $c(t)$ is generic, so new techniques are required. As our third main theorem, we provide more examples of quasi-adelic measures arising from dynamical pairs $(g_{\lambda,t}, c)$ where λ is a root of unity.

THEOREM 1.4. *Let $\lambda \neq 1$ be a primitive root of unity and k be a number field with $\lambda \in k$. Consider the rational function $g_{\lambda,t}(z) = \lambda z/(z^2 + tz + 1) \in k(t)(z)$ and let $c(t) \in k(t)$ be such that* 0 *and* ∞ *are not in the orbit of $c(\infty)$ iterated under the map $f_\infty(z) = z/(z^2 + 1)$. Then the measure $\mu_{g_\lambda,c} = \{\mu_{g_\lambda,c,v}\}_{v\in\mathcal{M}_k}$ is quasi-adelic. Furthermore, if $c(\infty)$ is not a preperiodic point of f_∞, then the measure $\mu_{g_\lambda,c}$ is not adelic.*

Notice that $g_{\lambda,t}$ degenerates only at $t = \infty$. The stated degeneration is the constant map equal to 0. However, the behavior of the degeneration differs depending on whether λ is a root of unity or not. If λ is not a root of unity, the degeneration of $g^n_{\lambda,t}$ at $t = \infty$ is *regular* in that it is the nth iterate of $g^n_{\lambda,\infty}$; the maps $g^n_{\lambda,t}$ degenerate at $t = \infty$ to the same constant map. A major difficulty in [22] was proving that the potentials of $\mu_{g_\lambda,\pm 1,v}$ are continuous along this singularity. To this end, DeMarco, Wang, and Ye computed certain homogeneous capacities explicitly. Their calculations were aided by the fact that the degeneration of $g^n_{\lambda,t}$ at $t = \infty$ is regular.

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However, when λ is a root of unity, the behavior of the degeneration at $t = \infty$ is more complicated. More precisely, if λ is of order $\ell \geq 2$, although the nth iterate $g^n_{\lambda,t}$ degenerates to a constant map when $n < \ell$, the ℓth iterate $f_t := g^{\ell}_{\lambda,t}$ degenerates to the degree 2 map $f_\infty(z) = z/(z^2+1)$; see Proposition 5.1. Clearly, $f_\infty(z)$ is not the ℓth iterate of $g_{\lambda,\infty}$. This phenomenon, studied extensively in [**16**, **25**, **40**], adds an extra layer of difficulty to controlling the potentials near ∞. We overcome the difficulties by working with the ℓth iterate of $g_{\lambda,t}$ instead.

More importantly, in contrast to [**22**], we allow the starting point c to be general—it only has to satisfy condition (C1) for $N = \ell$. In the absence of an explicit formula for c, we cannot compute the precise homogeneous capacities as in [**22**]. However, using our assumption for c, we are able to control the behavior of the potentials near the degenerating parameter.

As a special case, Theorem 1.4 holds when the starting point $c \in \{1, -1\}$ is a critical point of $g_{\lambda,t}$ and the corresponding dynamical pairs $(g_{\lambda,t}, \pm 1)$ are not adelic. Combining the techniques in [**22**] and the new ones developed in the proof of Theorem 1.4, one should be able to prove that many other dynamical pairs are quasi-adelic as well. It is now natural to ask the following.

Question 1.5. Are all dynamical pairs $(f, c) \in k(t)(z) \times k(t)$ quasi-adelic?

1.4. *Variation of canonical heights and applications of arithmetic equidistribution.* Recall that for each rational map $\varphi \in F(z)$ of degree at least 2, defined over a product formula field F, we have the *Call–Silverman canonical height* [**10**] defined as

$$\begin{aligned} \hat{h}_\varphi : \mathbb{P}^1(F^{\mathrm{sep}}) &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \lim_{n\to\infty} \frac{h(\varphi^n(x))}{\deg(\varphi)^n}. \end{aligned}$$

Here $h : \mathbb{P}^1(F^{\mathrm{sep}}) \to \mathbb{R}_{\geq 0}$ is the standard logarithmic *Weil height*. Thus, there are two natural heights associated with a dynamical pair $(f, c) \in k(t)(z) \times k(t)$; the Call–Silverman canonical height $\hat{h}_f(c)$ of $c \in k(t)$ associated with the map f defined over the function field $k(t)$, and for each specialization of $t \in \mathbb{P}^1(k^{\mathrm{sep}})$ such that $\deg_z(f_t) \geq 2$, the Call–Silverman canonical height $\hat{h}_{f_t}(c(t))$.

By definition, the function $t \mapsto \hat{h}_{\mu_{f,c}}(t)$ is equal to a multiple of $t \mapsto \hat{h}_{f_t}(c(t))$ plus a constant term. In Proposition 6.1, we see that the stated constant term equals zero. Thus, assuming that (f, c) is quasi-adelic, we infer that $t \mapsto \hat{h}_{f_t}(c(t))$ behaves like a Weil height and further Theorem 1.1 yields that Galois orbits of points with small Call–Silverman height equidistribution.

THEOREM 1.6. *Let k be a number field. Let $f \in k(t)(z)$ and $c \in k(t)$ be such that $\deg_z f \geq 2$ and the dynamical pair (f, c) is quasi-adelic. Then, the following hold.*

(1) *As $t \in \mathbb{P}^1(\bar{k})$ varies, we have $\hat{h}_{f_t}(c(t)) = \hat{h}_f(c)h(t) + O(1)$, where the implicit constant only depends on $f \in k(t)(z)$ and $c \in k(t)$.*

(2) *Let $\{t_n\}_{n\in\mathbb{N}} \subset \bar{k}$ be a non-repeating sequence of points with $\hat{h}_{f_{t_n}}(c(t_n)) \to 0$. The sequence of probability measures, weighted equally on the points in the*

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$\mathrm{Gal}(\bar{k}/k)$-orbit of t_n, converges weakly to $\mu_{f,c,v}$ on $\mathbb{P}^{1,an}_v$ as $n \to \infty$ for each $v \in \mathcal{M}_k$.

The first part of Theorem 1.6 relates our definition of a quasi-adelic measure with a theorem of Tate [53] from 1983 and its dynamical analogs. Let X be a smooth projective curve defined over a number field k. Tate considered an elliptic surface $E \to X$ and a section $P : X \to E$, defined over k, and proved that the map $t \to \hat{h}_{E_t}(P_t)$, associating to each $t \in X(\bar{k})$ the Néron–Tate height of P_t in the corresponding fiber E_t is actually a height function on the curve X corresponding to a divisor of degree equaling the geometric canonical height $\hat{h}_E(P)$ of $P \in E(k(X))$. More precisely, Tate showed that there exists a divisor $D = D(E, P) \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ of degree $\hat{h}_E(P)$ such that $\hat{h}_{E_t}(P_t) = h_D(t) + O(1)$, as $t \in X(\bar{k})$ varies. Tate thus strengthened an earlier result of Silverman [48] positing that $\hat{h}_{E_t}(P_t) = \hat{h}_E(P)h(t) + o(h(t))$. Silverman [49, 51, 52] strengthened Tate's result to show that not only is the error term $O(1)$, but the difference behaves quite regularly. Replacing the Nèron–Tate height by the Call–Silverman canonical height, it is natural to ask whether the analog of Tate's result in the dynamical setting holds.

Question 1.7. (Variation of Call–Silverman heights) Let k be a number field and $(f, c) \in k(t)(z) \times k(t)$. Do we have

$$\hat{h}_{f_t}(c(t)) = \hat{h}_f(c)h(t) + O(1)? \tag{1.1}$$

Theorem 1.6 establishes that Question 1.7 is closely related to Question 1.5. To prove that a dynamical pair is quasi-adelic, one has to understand whether it satisfies the variation of heights. Our summability condition in the definition of a quasi-adelic measure is manufactured so that (1.1) is satisfied. That said, it is not clear whether a pair satisfying the variation of heights is quasi-adelic. Given that the definition of a quasi-adelic pair is given by local conditions, more comparable to a local version of (1.1), it is *a priori* possible that a pair fails to be quasi-adelic while still satisfying (1.1). We know do not know of any such examples and expect that all pairs (f, c), as in Question 1.7, are quasi-adelic.

It seems that establishing the variation of Call–Silverman heights is a hard problem in general, especially as there are only a few known partial results. Tate's theorem [53] implies that the answer to Question 1.7 is affirmative if f is a Lattès map associated to an elliptic surface. Call and Silverman [10, Theorem 4.1] show that a weaker form of the variation of Call–Silverman heights holds in general: $\hat{h}_{f_t}(c(t)) = \hat{h}_f(c)h(t) + o(h(t))$ as $h(t) \to \infty$. Ingram [39] gave an affirmative answer to Question 1.7 when f is a polynomial in z. Ghioca, Hsia, and Tucker [33, Theorem 5.4] proved that the answer to Question 1.7 is still positive for dynamical pairs (f, c) upon imposing technical conditions on f, including that ∞ is a superattracting fixed point of f and that f does not degenerate at $t \neq \infty$. Finally, Ghioca and the first author [36] showed that the variation of heights still holds for (f, c) if $f(z) = (z^d + t)/z$ for $d \geq 2$. A common feature in all the aforementioned examples is that the corresponding dynamical pairs (f, c) are adelic, something that is not often the case by Theorem 4.5. (While this paper was under review, DeMarco and the first author [21] established that the variation of heights holds for 'globally Fatou' dynamical pairs.)

The second part of Theorem 1.6 has applications toward establishing a conjecture motivated by far-reaching conjectures in arithmetic geometry from the theme of 'unlikely intersections'; see [4, Conjecture 1.10], [17, Conjecture 6.1], [32, Question 1.3], and [33, Conjecture 2.3].

Conjecture 1.8. (Baker and DeMarco; Ghioca, Hsia and Tucker) Let k be a number field and $(f, c_i) \in k(t)(z) \times k(t)$ be non-preperiodic dynamical pairs for $i = 1, 2$, where $\deg_z(f) \geq 2$. Assume that there are infinitely many $t_n \in \mathbb{P}^1(\bar{k})$ such that $\hat{h}_{f_{t_n}}(c_1(t_n)) + \hat{h}_{f_{t_n}}(c_2(t_n)) \to 0$ as $n \to \infty$. Then c_1, c_2 are *coincident*, that is, there exists $j \in \{1, 2\}$ and a finite set $E \subset \mathbb{P}^1$ such that for $U = \mathbb{P}^1 \setminus E$, we have

$$\begin{aligned}\{t \in U(\bar{k}) : c_1(t) \text{ and } c_2(t) \text{ are preperiodic for } f_t\}\\ = \{t \in U(\bar{k}) : c_j(t) \text{ is preperiodic for } f_t\}.\end{aligned}$$

Recall that over a number field k, preperiodic points are points with canonical height equal to zero. Hence, a special case of Conjecture 1.8 asserts that if there are infinitely many parameters $t \in \mathbb{P}^1(\bar{k})$ such that both $c_1(t)$ and $c_2(t)$ are preperiodic under iteration by f_t, then either one of the c_i is identically preperiodic for f or for each $t \in U(\bar{k})$, $c_1(t)$ is preperiodic under iteration by f_t if and only if $c_2(t)$ is preperiodic under iteration by f_t.

A special case of this conjecture, namely when $f = z^d + t$ and c_i are constant, is due to Zannier who proposed a dynamical analog of his theorem with Masser [42–44] in the setting of elliptic surfaces. Baker and DeMarco [3] answered Zannier's question, establishing the first groundbreaking result leading to Conjecture 1.8. In fact, they also proved that c_1 and c_2 are coincident if and only if $c_1^d = c_2^d$. Their theorem was generalized in [20, 23, 31–33] to allow for more dynamical pairs and for replacing preperiodic points with points of small canonical height in the spirit of Zhang's dynamical Bogomolov conjecture. Notably, Favre and Gauthier [28] settled the Baker–DeMarco conjecture in the case of a polynomial map f. The common key ingredient in the aforementioned results is the adelic equidistribution theorem in [7, 12, 29, 54, 56]. The second part of Theorem 1.6 implies that Conjecture 1.8 holds under the weaker assumption that the dynamical pairs are quasi-adelic.

Other applications of adelic equidistribution theorems include the study of the distribution of postcritically finite maps in the moduli space of rational functions, see [4, 22, 26, 27, 34, 35, 37]. Part 2 of Theorem 1.6 has various implications in this setting as well. Moreover, a general strategy was introduced recently [18], using a quantitative (adelic) equidistribution theorem on $\mathbb{P}^1$ [29], to obtain a uniform Manin–Mumford result for a family of genus 2 curves defined over $\mathbb{C}$. See also [19] for a result concerning the uniform finiteness for the distribution of the 'special' points on $\mathbb{P}^1$. From Theorem 1.3, generic dynamical pairs are non-adelic, so we expect a quantitative (quasi-adelic) equidistribution theorem to be useful.

Finally, we point out that more general versions of Question 1.7 and Conjecture 1.8 have been proposed in which dynamical pairs are allowed to be parametrized by curves other

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than $\mathbb{P}^1$ or by higher dimensional varieties. It would be interesting to have a quasi-adelic equidistribution theorem in that setting as well.

Outline of the article. In §2, we introduce the notion of a quasi-adelic measure on $\mathbb{P}^1$ and study some properties of this measure. In §3, we prove the quasi-adelic equidistribution Theorem 1.1 and establish an important finiteness property for the height associated with a quasi-adelic measure (see Proposition 3.2). In §4, we prove Theorems 1.3 and 4.5. In §5, we prove Theorem 1.4, thus giving examples of dynamical pairs (f, c) that are quasi-adelic, but fail to be adelic. Finally, in §6, we prove Theorem 1.6.

2. *Quasi-adelic measure and some properties*

In this section, we introduce the notion of a quasi-adelic measure and a quasi-adelic set. Further, we define canonical heights associated with these measures and establish some of their properties.

2.1. *Preliminaries and basic notation.*

A *product formula field* is a field k together with a set $\mathcal{M}_k$ consisting of pairwise inequivalent non-trivial absolute values, and a unique positive integer N_v associated to each element of $\mathcal{M}_k$ such that the following holds.

- For each $\alpha \in k^*$, we have $|\alpha|_v = 1$ for all but finitely many places $v \in \mathcal{M}_k$, and the *product formula* holds,

$$\prod_{v\in\mathcal{M}_k} |\alpha|_v^{N_v} = 1. \tag{2.1}$$

In what follows, we often refer to the elements of $\mathcal{M}_k$ as places of k. Important examples of product formula fields include number fields and function fields of smooth projective curves. Let $\overline{k}$ and k^{sep} be the algebraic and respectively separable closure of k. If the characteristic of k is zero, then $\overline{k} = k^{\mathrm{sep}}$. For each $v \in \mathcal{M}_k$, let k_v be the completion of k with respect to $|\cdot|_v$, $\overline{k}_v$ be an algebraic closure of k_v, and $\mathbb{C}_v$ denote the completion of $\overline{k}_v$. We also let $\mathbb{P}_v^{1,an}$ be the Berkovich projective line over $\mathbb{C}_v$. This is a canonically defined path-connected compact Hausdorff space containing $\mathbb{P}^1(\mathbb{C}_v)$ as a dense subspace. For each $v \in \mathcal{M}_k$, we fix an embedding of $\overline{k}$ into $\mathbb{C}_v$. We remark here that if v is archimedean, we have $\mathbb{C}_v \simeq \mathbb{C}$ and $\mathbb{P}_v^{1,an} \simeq \mathbb{P}^1(\mathbb{C})$.

For each $v \in \mathcal{M}_k$, there is a distribution-valued Laplacian operator Δ on $\mathbb{P}_v^{1,an}$. For its definition and some examples, we refer the reader to [7, Ch. 5]. An important example is the Laplacian of $\log^+ |z|_v := \max\{\log |z|_v, 0\}$. Note that the function $\log^+ |z|_v$, which is originally defined on $\mathbb{P}^1(\mathbb{C}_v) \setminus \{\infty\}$, extends naturally to a continuous real valued function defined on $\mathbb{P}_v^{1,an}\setminus\{\infty\}$. The Laplacian of its extension, also denoted by $\log^+ |z|_v$ and taking images in $\mathbb{R} \cup \{+\infty\}$, is

$$\Delta \log^+ |z|_v = \delta_\infty - \lambda_v, \tag{2.2}$$

where λ_v is the uniform probability measure on the complex unit circle $\{|z|_v = 1\}$ when v is archimedean and a point mass at the Gauss point of $\mathbb{P}_v^{1,an}$ when v is non-archimedean.

A probability measure μ_v on $\mathbb{P}_v^{1,an}$ is said to have *continuous potentials* if $\mu_v - \lambda_v = \Delta g$ for some continuous function $g : \mathbb{P}_v^{1,an} \to \mathbb{R}$. We call the function g a *potential* of μ_v and note that any two potentials of μ_v differ by a constant.

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If μ_v has continuous potentials, then there is a function $g^0_{\mu_v} : \mathbb{P}^{1,an}_v \to \mathbb{R} \cup \{+\infty\}$ such that $g : \mathbb{P}^{1,an}_v \to \mathbb{R}$, defined by

$$g(z) = \log^+ |z|_v - g^0_{\mu_v}(z),$$

is a potential for μ_v and is continuous. We call the function $G^{\text{hom}}_{\mu_v}(x, y)$ uniquely determined by $g^0_{\mu_v}$ and defined as

$$G^{\text{hom}}_{\mu_v}(x, y) := \begin{cases} g^0_{\mu_v}(x/y) + \log |y|_v & \text{for } x, y \in \mathbb{C}_v \text{ and } y \neq 0, \\ \log |x|_v - g(\infty) & \text{for } x \in \mathbb{C}_v, x \neq 0, \text{ and } y = 0, \\ -\infty & \text{for } x = y = 0, \end{cases} \tag{2.3}$$

where $g(\infty) = \lim_{z\to\infty}(\log^+ |z|_v - g^0_{\mu_v}(z))$, a *homogeneous potential* of μ_v. If we further require that the set

$$\{(x, y) \in \mathbb{C}^2_v : G^{\text{hom}}_{\mu_v}(x, y) \leq 0\}$$

has homogeneous capacity equal to 1 (see [6, §3.3] and [14] for the definition of homogeneous capacity), then the homogeneous potential is uniquely determined. We denote it by G_{μ_v} and call it the *normalized homogeneous potential* of μ_v. Further, we write

$$M_{\mu_v} := \{(x, y) \in \mathbb{C}^2_v : G_{\mu_v}(x, y) \leq 0\}. \tag{2.4}$$

Note that any homogeneous potential of μ_v differs from G_{μ_v} by a constant. An important property of a homogeneous potential $G^{\text{hom}}_{\mu_v}$ is that it *scales logarithmically*,

$$G^{\text{hom}}_{\mu_v}(\alpha x, \alpha y) = G^{\text{hom}}_{\mu_v}(x, y) + \log |\alpha|_v. \tag{2.5}$$

For example, from (2.2), we get that the normalized homogeneous potential of λ_v is

$$G_{\lambda_v}(x, y) = \log \|(x, y)\|_v \text{ and } M_{\lambda_v} = \bar{D}^2(0, 1) \subset \mathbb{C}^2_v,$$

where $\|\cdot\|_v$ is the maximum norm defined as $\|(x, y)\|_v := \max\{|x|_v, |y|_v\}$.

Finally, we point out that for each probability measure μ_v on $\mathbb{P}^{1,an}_v$ with continuous potentials, we have a unique *normalized Arakelov–Green function* $g_{\mu_v} : \mathbb{P}^{1,an}_v \times \mathbb{P}^{1,an}_v \to \mathbb{R} \cup \{+\infty\}$. This is characterized by the differential equation $\Delta_x g_{\mu_v}(x, y) = \delta_y - \mu_v$ and the normalization

$$\iint g_{\mu_v}(x, y)\, d\mu_v(x)\, d\mu_v(y) = 0. \tag{2.6}$$

Note that for points $(x, y) \in \mathbb{C}^2_v$, the normalized Arakelov–Green function g_{μ_v} is given by

$$g_{\mu_v}(x, y) = -\log |\tilde{x} \wedge \tilde{y}|_v + G_{\mu_v}(\tilde{x}) + G_{\mu_v}(\tilde{y}), \tag{2.7}$$

where $\tilde{x} = (x_1, x_2) \in \mathbb{C}^2_v$ and $\tilde{y} = (y_1, y_2) \in \mathbb{C}^2_v$ are lifts of x and y respectively and $|\tilde{x} \wedge \tilde{y}|_v := |x_1 y_2 - y_1 x_2|_v$; see [6, 7] for more details.

2.2. *Quasi-adelic measure and canonical height function.* Let μ_v be a probability measure on $\mathbb{P}_v^{1,an}$ with continuous potentials. We define the *outer radius* and *inner radius* for μ_v as

$$r_{\text{out}}(\mu_v) := \inf\{r > 0 : M_{\mu_v} \subset \bar{D}(0, r) \times \bar{D}(0, r)\},$$
$$r_{\text{in}}(\mu_v) := \sup\{r > 0 : \bar{D}(0, r) \times \bar{D}(0, r) \subset M_{\mu_v}\}.$$

A *quasi-adelic measure* on $\mathbb{P}^1$ with respect to a product formula field k is a collection $\mu = \{\mu_v\}_{v\in\mathcal{M}_k}$ of probability measures on $\mathbb{P}_v^{1,an}$, one for each $v \in \mathcal{M}_k$, such that:

- μ_v has continuous potentials for each $v \in \mathcal{M}_k$, and
- $\prod_{v\in\mathcal{M}_k} r_{\text{in}}(\mu_v)^{N_v} > 0$ and $\prod_{v\in\mathcal{M}_k} r_{\text{out}}(\mu_v)^{N_v} < \infty$.

Remark 1. Since $\text{Cap}(M_{\mu_v}) = 1$ and $\text{Cap}(\bar{D}^2(0, r)) = r^2$, the radii satisfy $0 < r_{\text{in}}(\mu_v) \le 1 \le r_{\text{out}}(\mu_v)$. The measure $\mu = \{\mu_v\}_{v\in M_k}$ is *adelic* if we replace the second condition by $\mu_v = \lambda_v$ or equivalently $r_{\text{in}}(\mu_v) = r_{\text{out}}(\mu_v) = 1$ for all but finitely many $v \in \mathcal{M}_k$; see [7, 29]. In other words, adelic measures satisfy $M_{\mu_v} = \bar{D}^2(0, 1)$ for all but finitely many places $v \in \mathcal{M}_k$.

If ρ, ρ' are probability measures on $\mathbb{P}_v^{1,an}$, we define the μ_v*-energy* of ρ and ρ' as

$$(\rho, \rho')_{\mu_v} := \frac{1}{2} \iint_{\mathbb{P}_v^{1,an} \times \mathbb{P}_v^{1,an} \setminus \text{Diag}} g_{\mu_v}(x, y)\, d\rho(x)\, d\rho'(y).$$

The μ_v*-energy of* ρ is defined as $I_{\mu_v}(\rho) := (\rho, \rho)_{\mu_v}$.

We can now define the height associated with a quasi-adelic measure. Let $S \subset \mathbb{P}^1(k^{\text{sep}})$ be a finite $\text{Gal}(k^{\text{sep}}/k)$-invariant set with cardinality $|S| > 1$. Let $\tilde{S} \subset \mathbb{C}_v^2$ be a *lift of* S, that is, a $\text{Gal}(k^{\text{sep}}/k)$-invariant set consisting of lifts $\tilde{x} \in k^{\text{sep}} \times k^{\text{sep}} \setminus \{(0, 0)\}$ of elements $x \in S$; in particular, $|\tilde{S}| = |S|$. We denote by $[S]_v$ the discrete probability measure on $\mathbb{P}_v^{1,an}$ supported equally on all elements of S. The *canonical height* of S associated to a quasi-adelic measure $\mu = \{\mu_v\}_{v\in M_k}$ is a number given by

$$\begin{aligned}
\hat{h}_\mu(S) :&= \frac{|S|}{|S|-1} \sum_{v\in\mathcal{M}_k} N_v \cdot ([S]_v, [S]_v)_{\mu_v} \\
&= \frac{|S|}{|S|-1} \sum_{v\in\mathcal{M}_k} \frac{N_v}{2|S|^2} \sum_{x,y\in S, x\neq y} g_{\mu_v}(x, y) \\
&= \sum_{x,y\in S,x\neq y} \sum_{v\in\mathcal{M}_k} \frac{N_v \cdot (-\log|\tilde{x} \wedge \tilde{y}|_v + G_{\mu_v}(\tilde{x}) + G_{\mu_v}(\tilde{y}))}{2|S|(|S|-1)} \\
&= \frac{1}{2|S|(|S|-1)} \sum_{\tilde{x},\tilde{y}\in\tilde{S},x\neq y} \sum_{v\in\mathcal{M}_k} N_v \cdot (G_{\mu_v}(\tilde{x}) + G_{\mu_v}(\tilde{y})), \quad \text{by (2.1)} \\
&= \frac{1}{|S|} \cdot \sum_{\tilde{x}\in\tilde{S}} \sum_{v\in\mathcal{M}_k} N_v \cdot G_{\mu_v}(\tilde{x}).
\end{aligned} \tag{2.8}$$

Here, the constants N_v are the same as those appearing in the product formula. Therefore, we have

$$\hat{h}_\mu(S) = \frac{1}{|S|} \cdot \sum_{\tilde{x}\in\tilde{S}} \sum_{v\in\mathcal{M}_k} N_v \cdot G_{\mu_v}(\tilde{x}). \tag{2.9}$$

Equation (2.9) allows us to extend the definition of our height to the case $|S| = 1$. If $x \in k^{\mathrm{sep}}$, we may take S to be equal to $\mathrm{Gal}(k^{\mathrm{sep}}/k) \cdot x$ and use (2.9) to define the canonical height of x as

$$\hat{h}_\mu(x) := \hat{h}_\mu(\mathrm{Gal}(k^{\mathrm{sep}}/k) \cdot x).$$

In the next proposition, we show that our height is finite and well defined.

PROPOSITION 2.1. *Let k be a product formula field and $\mu = \{\mu_v\}_{v\in\mathcal{M}_k}$ a quasi-adelic measure. For each* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$*-invariant set S, the height $\hat{h}_\mu(S)$ from (2.9) is independent of the choice of lift for S and is a finite real number.*

Proof. That $\hat{h}_\mu(S)$ is independent of the choice of lift $\tilde{S}$ of S follows from the product formula of k once we recall that the homogeneous potential G_{μ_v} scales logarithmically; see (2.5). To see that $\hat{h}_\mu(S)$ is a finite number, notice that from the definition of inner and outer radii, we have

$$\log r_{\mathrm{in}}(\mu_v) \leq \frac{1}{|S|} \sum_{\tilde{x}\in\tilde{S}} (\log \|\tilde{x}\|_v - G_{\mu_v}(\tilde{x})) \leq \log r_{\mathrm{out}}(\mu_v).$$

Summing this inequality over all places $v \in \mathcal{M}_k$, we get

$$\sum_{v\in\mathcal{M}_k} N_v \log r_{\mathrm{in}}(\mu_v) \leq \sum_{v\in\mathcal{M}_k} \frac{N_v}{|S|} \sum_{\tilde{x}\in\tilde{S}} (\log \|\tilde{x}\|_v - G_{\mu_v}(\tilde{x})) \leq \sum_{v\in\mathcal{M}_k} N_v \log r_{\mathrm{out}}(\mu_v),$$

which, using the definition of our height, yields

$$\log \prod_{v\in\mathcal{M}_k} r_{\mathrm{in}}(\mu_v)^{N_v} \leq \frac{1}{|S|} \sum_{v\in\mathcal{M}_k} \sum_{\tilde{x}\in\tilde{S}} N_v \log \|\tilde{x}\|_v - \hat{h}_\mu(S) \leq \log \prod_{v\in\mathcal{M}_k} r_{\mathrm{out}}(\mu_v)^{N_v}. \tag{2.10}$$

Recall that μ is a quasi-adelic measure. Thus, $\log \prod_{v\in\mathcal{M}_k} r_{\mathrm{in}}(\mu_v)^{N_v}$ and $\log \prod_{v\in\mathcal{M}_k} r_{\mathrm{out}}(\mu_v)^{N_v}$ are real numbers. The proposition follows. □

Remark 2. Equation (2.10) gives a comparison between the naive Weil height and the height associated with a quasi-adelic measure μ; see Proposition 2.3 for a precise statement in the case of a number field k.

Remark 3. The canonical height we defined is slightly different from that appearing in [**7**, **29**], but agrees with that in [**22**, **23**]. The factor of $|S|/|S| - 1$ is included here to allow for a better comparison of this measure-theoretic height with the Call–Silverman height; see Proposition 6.1.

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2.3. *Quasi-adelic set.* In this section, we introduce the notion of a quasi-adelic set. This has a geometric interpretation; hence in many applications, it is easier to manipulate than a quasi-adelic measure. Analogous to the notion of the *homogeneous filled Julia set* in [6, §3.2], we define a *homogeneous set with continuous potential* as

$$M_v := \{(x, y) \in \mathbb{C}_v^2 : G_v(x, y) \leq 0\}, \tag{2.11}$$

where G_v is a homogeneous potential for a probability measure on $\mathbb{P}_v^{1,an}$ having continuous potentials. In what follows, we write G_{M_v} for G_v as in (2.11), thinking of G_{M_v} as the potential associated to the homogeneous set M_v with continuous potential. We also denote the measure corresponding to G_{M_v}, by μ_{M_v}. When v is an archimedean place, M_v having continuous potential is equivalent to saying that $M_v \subset \mathbb{C}_v^2 \simeq \mathbb{C}^2$ is a compact, circled, and pseudoconvex set, or that G_{M_v} is a continuous and plurisubharmonic function satisfying:

(1) $G_{M_v}(\alpha z) = G_{M_v}(z) + \log |\alpha|_v$ for all $\alpha \in \mathbb{C}_v$; and

(2) $G_{M_v}(z) = \log \|z\|_v + O(1)$.

See [14]. We point out here that there are many homogeneous sets with continuous potential. If $F_n : \mathbb{C}_v^2 \to \mathbb{C}_v^2$ is a sequence of homogeneous polynomials with $\deg(F_n) \geq 1$ such that the sequence of functions $\{\log \|F_n\|_v / \deg(F_n)\}_{n\geq 1}$ converges uniformly to G_v on $\mathbb{C}_v^2 \backslash \{(0, 0)\}$, then G_v is a homogeneous potential for some probability measure with continuous potentials on $\mathbb{P}_v^{1,an}$; see [6, §3]. Hence, $M_v = \{(x, y) \in \mathbb{C}_v^2 : G_v(x, y) \leq 0\}$ is a homogeneous set with continuous potential. As seen in [23, §2], its capacity can be computed by the following limit:

$$\mathrm{Cap}(M_v) = \lim_{n\to\infty} |\mathrm{Res}(F_n)|_v^{-1/\deg(F_n)^2}.$$

Analogous to the definition of the radii of μ_v, we define the *outer* and *inner* radii of M_v as

$$r_{\mathrm{out}}(M_v) := \inf\{r > 0 : M_v \subset \bar{D}(0, r) \times \bar{D}(0, r)\},$$

$$r_{\mathrm{in}}(M_v) := \sup\{r > 0 : \bar{D}(0, r) \times \bar{D}(0, r) \subset M_v\}.$$

A product $\prod_{v\in\mathcal{M}_k} r_v^{N_v}$, with $r_v > 0$ for each $v \in \mathcal{M}_k$, *converges strongly* if

$$\sum_{v\in\mathcal{M}_k} N_v \cdot |\log r_v| < \infty.$$

We define a *quasi-adelic set* (with respect to a product formula field k) to be a collection $\mathbb{M} = \{M_v\}_{v\in\mathcal{M}_k}$ of sets such that the following hold.

- For each $v \in \mathcal{M}_k$, the set M_v is a homogeneous set with continuous potential.
- The products $\prod_{v\in\mathcal{M}_k} r_{\mathrm{out}}(M_v)^{N_v}$ and $\prod_{v\in\mathcal{M}_k} r_{\mathrm{in}}(M_v)^{N_v}$ converge strongly.

Note that there is a unique probability measure μ_{M_v} with continuous potential associated to a homogeneous set M_v with continuous potential. Hence, a quasi-adelic set $\mathbb{M} = \{M_v\}_{v\in M_k}$ gives a measure $\mu_{\mathbb{M}} = \{\mu_{M_v}\}_{v\in M_k}$ on $\mathbb{P}^1$. In the next theorem, proved in §3, we will see that this measure is also quasi-adelic.

THEOREM 2.2. *Let k be a product field and $\mathbb{M} = \{M_v\}_{v\in\mathcal{M}_k}$ be a collection of homogeneous sets with continuous potential. Then we have the following.*

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- *If the set* $\mathbb{M} = \{M_v\}_{v\in\mathcal{M}_k}$ *is quasi-adelic, then the corresponding measure* $\mu_{\mathbb{M}} = \{\mu_{M_v}\}_{v\in M_k}$ *is quasi-adelic.*
- *Suppose that for each* $v \in \mathcal{M}_k$*, there are positive constants* r_v', r_v *such that* $\bar{D}^2(0, r_v') \subset M_v \subset \bar{D}^2(0, r_v)$ *and the products* $\prod_{v\in\mathcal{M}_k} {r_v'}^{N_v}$, $\prod_{v\in\mathcal{M}_k} {r_v}^{N_v}$ *converge strongly. Then the set* $\mathbb{M} = \{M_v\}_{v\in\mathcal{M}_k}$ *is quasi-adelic. Moreover, the product* $\prod_{v\in\mathcal{M}_k} \mathrm{Cap}(M_v)^{N_v}$ *converges strongly and for any* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$*-invariant* $S \subset \mathbb{P}^1(k^{\mathrm{sep}})$*, we have*

$$\hat{h}_{\mu_{\mathbb{M}}}(S) = \frac{1}{|S|} \cdot \sum_{\tilde{x}\in\tilde{S}} \sum_{v\in\mathcal{M}_k} N_v \cdot G_{M_v}(\tilde{x}) + \frac{1}{2} \log \prod_{v\in\mathcal{M}_k} \mathrm{Cap}(M_v)^{N_v}. \tag{2.12}$$

2.4. *Some properties.* A number field k is naturally equipped with a set of inequivalent absolute values $\mathcal{M}_k$ and positive integers $\{N_v\}_{v\in\mathcal{M}_k}$ making it a product formula field. For any $x \in k$, the *logarithmic Weil height* of x is defined as follows:

$$h(x) := \frac{1}{[k:\mathbb{Q}]} \sum_{v\in\mathcal{M}_k} N_v \cdot \log^+ |x|_v. \tag{2.13}$$

It is well defined and does not depend on the embedding of a number field $k \hookrightarrow \overline{\mathbb{Q}}$. We show that the canonical height associated to a quasi-adelic measure differs from a multiple of the Weil height by a bounded amount.

PROPOSITION 2.3. *Let* k *be a number field. Suppose* $\mu = \{\mu_v\}_{v\in\mathcal{M}_k}$ *is a quasi-adelic measure. Then the canonical height* $\hat{h}_\mu$ *is bounded by the logarithmic Weil height* h *on* $\mathbb{P}^1(\overline{k})$ *as*

$$\log \prod_{v\in\mathcal{M}_k} r_{\mathrm{in}}(\mu_v)^{N_v} \le [k:\mathbb{Q}]h(x) - \hat{h}_\mu(x) \le \log \prod_{v\in\mathcal{M}_k} r_{\mathrm{out}}(\mu_v)^{N_v},$$

for all $x \in \overline{k}$.

Proof. Recall that $\hat{h}_\mu(x) = \hat{h}_\mu(\mathrm{Gal}(\overline{k}/k) \cdot x)$. The proposition follows by applying (2.10) for $S = \mathrm{Gal}(\overline{k}/k) \cdot x$, once one notices that then

$$\frac{1}{|S|} \sum_{v\in\mathcal{M}_k} \sum_{\tilde{x}\in\tilde{S}} N_v \log \|\tilde{x}\|_v = [k:\mathbb{Q}]h(x). \qquad \square$$

Finally, we note that similar to the set of adelic metrized line bundles which is closed under taking tensor products, we have that the set of quasi-adelic metrics is closed under taking certain linear combinations. For example, the average of two quasi-adelic measures is a quasi-adelic measure.

3. *Equidistribution of small points*

In this section, we prove Theorems 1.1 and 2.2. The structure of the proof of Theorem 1.1 follows that of [**6**, Theorem 2.3] and [**29**, Theorem 6]. Moreover, we prove an important finiteness property for the height associated to a quasi-adelic measure.

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3.1. *Proof of quasi-adelic arithmetic equidistribution Theorem 1.1.* By assumption, $\{S_n\}_{n\geq 1}$ is a sequence of subsets of $\mathbb{P}^1(k^{\mathrm{sep}})$ which are $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-invariant and the cardinality $|S_n|$ tends to infinity. For each $v \in \mathcal{M}_k$, the μ_v-energy of the probability measure $[S_n]_v$ on $\mathbb{P}^{1,an}_v$ is given by

$$\begin{aligned}([S_n]_v, [S_n]_v)_{\mu_v} &= \frac{1}{2|S_n|^2} \sum_{x\neq y\in S_n} g_{\mu_v}(x, y)\\ &= \frac{1}{2|S_n|^2} \sum_{x\neq y\in S_n} (-\log|\tilde{x} \wedge \tilde{y}|_v + G_{\mu_v}(\tilde{x}) + G_{\mu_v}(\tilde{y})),\end{aligned}$$

where $\tilde{x} \in \mathbb{C}_v^2$ and $\tilde{y} \in \mathbb{C}_v^2$ are lifts of x and y respectively. We begin with a lemma, which follows the idea from [6, 29]. It asserts that the sequence $\{S_n\}_{n\geq 1}$ is pseudo-equidistributed with respect to g_{μ_v}; compare with [6, Definition 4.4], [6, Theorem 4.6], and [29, p. 349].

LEMMA 3.1. *Let k be a product formula field and let $\mu = \{\mu_v\}_{v\in\mathcal{M}_k}$ be a quasi-adelic measure. Suppose that S_n is a sequence of* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$*-invariant subsets of* $\mathbb{P}^1(k^{\mathrm{sep}})$ *such that $|S_n| \to \infty$ and* $\limsup_{n\to\infty} \hat{h}_\mu(S_n) \leq 0$. *Then for each $v \in \mathcal{M}_k$, we have*

$$\lim_{n\to\infty} ([S_n]_v, [S_n]_v)_{\mu_v} = 0.$$

Proof. First we will show that for each $v \in \mathcal{M}_k$, we have

$$\liminf_{n\to\infty} ([S_n]_v, [S_n]_v)_{\mu_v} \geq 0. \tag{3.1}$$

For this, we follow the proof of [6, Lemma 3.17]. From the definition of the homogeneous capacity, we have

$$\liminf_{n\to\infty} \inf_{\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n\in M_{\mu_v}} \frac{1}{n(n-1)} \sum_{i\neq j} -\log|\tilde{x}_i \wedge \tilde{x}_j|_v \geq -\log \mathrm{Cap}(M_{\mu_v}) = 0. \tag{3.2}$$

Let $\epsilon > 0$ be an arbitrary small number. Since $\{|\alpha|_v : \alpha \in \mathbb{C}_v\}$ is dense in $\mathbb{R}_{\geq 0}$, we can choose lifts of $x, y \in S_n$, denoted by $\tilde{x}, \tilde{y} \in M_{\mu_v}$, such that

$$-\epsilon < G_{\mu_v}(\tilde{x}) \leq 0 \quad \text{and} \quad -\epsilon < G_{\mu_v}(\tilde{y}) \leq 0.$$

Now, the definition of the energy function and (3.2) yield

$$\begin{aligned}\liminf_{n\to\infty} ([S_n]_v, [S_n]_v)_{\mu_v} &= \liminf_{n\to\infty} \frac{1}{2|S_n|^2} \sum_{x\neq y\in S_n} (-\log|\tilde{x} \wedge \tilde{y}|_v + G_{\mu_v}(\tilde{x}) + G_{\mu_v}(\tilde{y}))\\ &\geq -\epsilon + \liminf_{n\to\infty} \left(-\frac{1}{2|S_n|^2} \sum_{x\neq y\in S_n} \log|\tilde{x} \wedge \tilde{y}|_v\right) \geq -\epsilon.\end{aligned}$$

Shrinking ϵ to zero, (3.1) follows. It remains to prove that for each $v \in \mathcal{M}_k$, we have

$$\limsup_{n\to\infty} ([S_n]_v, [S_n]_v)_{\mu_v} \leq 0.$$

Let $v_0 \in \mathcal{M}_k$ be a fixed place. Let $\epsilon > 0$. We will show that

$$\limsup_{n\to\infty} ([S_n]_{v_0}, [S_n]_{v_0})_{\mu_{v_0}} \leq \epsilon. \tag{3.3}$$

The conclusion then follows letting $\epsilon \to 0$. For each $v \in \mathcal{M}_k$ and any $\delta > 0$, we lift $x, y \in S_n$ to $\tilde{x}, \tilde{y} \in M_{\mu_v}$ such that

$$G_{\mu_v}(\tilde{x}) \geq -\delta, G_{\mu_v}(\tilde{y}) \geq -\delta.$$

Since we know that M_{μ_v} is bounded by the polydisc with outer radius $r_{\text{out}}(\mu_v)$, we have $\|\tilde{x}\|_v, \|\tilde{y}\|_v \leq r_{\text{out}}(\mu_v)$. Hence, for a non-archimedean place $v \in \mathcal{M}_k$, we have

$$\log |\tilde{x} \wedge \tilde{y}|_v \leq \log r_{\text{out}}(\mu_v)^2,$$

and

$$\begin{aligned} ([S_n]_v, [S_n]_v)_{\mu_v} &= \frac{1}{2|S_n|^2} \sum_{x \neq y \in S_n} (-\log |\tilde{x} \wedge \tilde{y}|_v + G_{\mu_v}(\tilde{x}) + G_{\mu_v}(\tilde{y})) \\ &\geq \bigg(\frac{1 - |S_n|}{|S_n|}\bigg)\delta - \sum_{x \neq y \in S_n} \frac{\log r_{\text{out}}(\mu_v)}{|S_n|^2} \\ &= \bigg(\frac{1 - |S_n|}{|S_n|}\bigg)(\delta + \log r_{\text{out}}(\mu_v)). \end{aligned} \tag{3.4}$$

Shrinking δ to zero, we get that for all non-archimedean places $v \in \mathcal{M}_k$:

$$([S_n]_v, [S_n]_v)_{\mu_v} \geq \bigg(\frac{1 - |S_n|}{|S_n|}\bigg) \log r_{\text{out}}(\mu_v). \tag{3.5}$$

It is well known that for a product formula field k, there are only finitely many archimedean places; see [1, Ch. 12, Theorem 3]. Since $\mu = \{\mu_v\}_{v \in \mathcal{M}_k}$ is quasi-adelic, the product $\prod_{v \in \mathcal{M}_k} r_{\text{out}}(\mu_v)^{N_v}$ converges. Hence, we can choose a set $\mathcal{M}_k' \subset \mathcal{M}_k$ such that the following hold.

- $\mathcal{M}_k'' := \mathcal{M}_k \backslash (\mathcal{M}_k' \cup \{v_0\})$ has only finitely many places.
- All places in $\mathcal{M}_k'$ are non-archimedean and $v_0 \notin \mathcal{M}_k'$.
- $\sum_{v \in \mathcal{M}_k'} N_v \cdot \log r_{\text{out}}(\mu_v) \leq \epsilon$.

Now the definition of the canonical height $\hat{h}_\mu$ gives

$$\begin{aligned} N_{v_0} \cdot ([S_n]_{v_0}, [S_n]_{v_0})_{\mu_{v_0}} &= \bigg(\frac{|S_n| - 1}{|S_n|}\bigg)\hat{h}_\mu(S_n) - \sum_{v \in \mathcal{M}_k \backslash \{v_0\}} N_v \cdot ([S_n]_v, [S_n]_v)_{\mu_v} \\ &= \bigg(\frac{|S_n| - 1}{|S_n|}\bigg)\hat{h}_\mu(S_n) - \sum_{v \in \mathcal{M}_k''} N_v \cdot ([S_n]_v, [S_n]_v)_{\mu_v} \\ &\quad - \sum_{v \in \mathcal{M}_k'} N_v \cdot ([S_n]_v, [S_n]_v)_{\mu_v}. \end{aligned}$$

This in turn, upon using (3.5), implies

$$\begin{aligned} N_{v_0} \cdot ([S_n]_{v_0}, [S_n]_{v_0})_{\mu_{v_0}} &\leq \bigg(\frac{|S_n| - 1}{|S_n|}\bigg)\hat{h}_\mu(S_n) \\ &\quad - \sum_{v \in \mathcal{M}_k''} N_v \cdot ([S_n]_v, [S_n]_v)_{\mu_v} + \bigg(\frac{|S_n| - 1}{|S_n|}\bigg) \cdot \epsilon. \end{aligned}$$

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Since the set $\mathcal{M}_k''$ contains only finitely many places and $\limsup_{n\to\infty} \hat{h}_\mu(S_n) \le 0$, by the lower bound in (3.1), taking the superior limit in the above inequality yields

$$\begin{aligned}\limsup_{n\to\infty} N_{v_0}\cdot([S_n]_{v_0},[S_n]_{v_0})_{\mu_{v_0}} &\le \limsup_{n\to\infty}\Bigg(\bigg(\frac{|S_n|-1}{|S_n|}\bigg)\hat{h}_\mu(S_n)\\ &\qquad - \sum_{v\in\mathcal{M}_k''} N_v\cdot([S_n]_v,[S_n]_v)_{\mu_v}+\epsilon\Bigg)\\ &\le \limsup_{n\to\infty}\hat{h}_\mu(S_n)+\epsilon\le\epsilon, \end{aligned}\tag{3.6}$$

as claimed. This finishes the proof of the lemma. □

The proof of Theorem 1.1 now follows as in [6, Theorem 4.9]; see also [29, Propositions 2.11 and 4.12]. The set of probability measures on $\mathbb{P}_v^{1,an}$ is compact in the weak topology. Hence, to show that $[S_n]_v$ converges weakly to μ_v as $n\to\infty$, it suffices to show that any convergent subsequence of $[S_n]_v$ converges to μ_v. Passing to a subsequence if necessary, we may assume that $[S_n]_v$ converges to some ν_v,

$$\lim_{n\to\infty}[S_n]_v=\nu_v.$$

We have to show that $\nu_v=\mu_v$. By Lemma 3.1, the μ_v-energy $I_{\mu_v}(\nu_v)$ of ν_v satisfies

$$\begin{aligned}0 &= \lim_{n\to\infty}([S_n]_v,[S_n]_v)_{\mu_v} = \lim_{n\to\infty}\frac{1}{2}\iint_{\mathbb{P}_v^{1,an}\times\mathbb{P}_v^{1,an}\setminus\mathrm{Diag}} g_{\mu_v}(x,y)\,d[S_n]_v(x)\,d[S_n]_v(y)\\ &\ge \frac{1}{2}\iint_{\mathbb{P}_v^{1,an}\times\mathbb{P}_v^{1,an}} g_{\mu_v}(x,y)\,d\nu_v(x)\,d\nu_v(y) \quad \text{by [6, Lemma 3.26]}\\ &= I_{\mu_v}(\nu_v).\end{aligned}$$

Note, our assumption that the measures μ_v have continuous potentials implies that they are log-continuous in the sense of [6, Definition 3.20]. By the energy minimization principle [6, Theorem 3.25], where μ_v is the unique probability measure on $\mathbb{P}_v^{1,an}$ minimizing the μ_v-energy function $I_{\mu_v}(\cdot)$ and $I_{\mu_v}(\mu_v)=0\ge I_{\mu_v}(\nu_v)$, we get that $I_{\mu_v}(\mu_v)=I_{\mu_v}(\nu_v)$ and $\nu_v=\mu_v$. This finishes the proof of Theorem 1.1. □

3.2. *Proof of Theorem 2.2.* First, we show that $\mathbb{M}=\{M_v\}_{v\in\mathcal{M}_k}$ is quasi-adelic implies that $\mu=\{\mu_{M_v}\}_{v\in M_k}$ is quasi-adelic. Assume that $\mathbb{M}=\{M_v\}_{v\in\mathcal{M}_k}$ is a quasi-adelic set. For any $r>0$, let

$$rM_v := \{(\alpha x,\alpha y):(x,y)\in M_v,\alpha\in\mathbb{C}_v \text{ with } |\alpha|_v\le r\}.$$

From the definition of the capacity, we have $\mathrm{Cap}(rM_v)=r^2\,\mathrm{Cap}(M_v)$. Since G_{M_v} is a homogeneous potential for μ_{M_v}, the normalized homogeneous potential $G_{\mu_{M_v}}$ is given by

$$G_{\mu_{M_v}}(x,y)=G_{M_v}(x,y)+\tfrac{1}{2}\log\mathrm{Cap}(M_v),\tag{3.7}$$

and $M_{\mu_{M_v}}=(1/\sqrt{\mathrm{Cap}(M_v)})M_v$. As a consequence,

$$r_{\mathrm{in}}(\mu_{M_v})=\frac{r_{\mathrm{in}}(M_v)}{\sqrt{\mathrm{Cap}(M_v)}},\quad r_{\mathrm{out}}(\mu_{M_v})=\frac{r_{\mathrm{out}}(M_v)}{\sqrt{\mathrm{Cap}(M_v)}}.\tag{3.8}$$

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Moreover, as $\mathrm{Cap}(\bar{D}^2(0, r)) = r^2$,

$$r_{\mathrm{in}}(M_v) \leq \sqrt{\mathrm{Cap}(M_v)} \leq r_{\mathrm{out}}(M_v). \tag{3.9}$$

Then by (3.8),

$$\frac{r_{\mathrm{in}}(M_v)}{r_{\mathrm{out}}(M_v)} \leq r_{\mathrm{in}}(\mu_{M_v}) \leq 1 \leq r_{\mathrm{out}}(\mu_{M_v}) \leq \frac{r_{\mathrm{out}}(M_v)}{r_{\mathrm{in}}(M_v)}.$$

As $\mathbb{M} = \{M_v\}_{v\in\mathcal{M}_k}$ is quasi-adelic, the products of inner and outer radii converge strongly. Then the above inequalities imply that the products of the inner and outer radii of $\mu = \{\mu_{M_v}\}_{v\in\mathcal{M}_k}$ converge, that is, $\mu = \{\mu_{M_v}\}_{v\in\mathcal{M}_k}$ is quasi-adelic.

Now assume that $\bar{D}^2(0, r_v') \subset M_v \subset \bar{D}^2(0, r_v)$ and $\prod_{v\in\mathcal{M}_k} {r_v'}^{N_v}$, $\prod_{v\in\mathcal{M}_k} {r_v}^{N_v}$ converge strongly. Then the products $\prod_{v\in\mathcal{M}_k} r_{\mathrm{out}}(M_v)^{N_v}$ and $\prod_{v\in\mathcal{M}_k} r_{\mathrm{in}}(M_v)^{N_v}$ converge strongly, since

$$r_v' \leq r_{\mathrm{in}}(M_v) \leq r_{\mathrm{out}}(M_v) \leq r_v.$$

Hence, $\mathbb{M} = \{M_v\}_{v\in\mathcal{M}_k}$ is quasi-adelic. Moreover, by (3.9), the product of the capacities converges strongly. Then the last formula for the canonical height is clear from (2.8) and (3.7). This finishes the proof of Theorem 2.2. □

3.3. *A finiteness property.* The following proposition is the analog of the last assertion of [**29**, Theorem 1] for quasi-adelic measures. It will be useful in the last section in proving the equidistribution of parameters t with small height with respect to $\hat{h}_{f_{c(t)}}(c(t))$.

PROPOSITION 3.2. *Let k be a product formula field. Suppose μ is a quasi-adelic measure. Then for any $\delta > 0$, there are at most finitely many $x \in k^{\mathrm{sep}}$ with*

$$\hat{h}_\mu(x) < -\delta.$$

Proof. Assume to the contrary that there are infinitely many $x_i \in k^{\mathrm{sep}}$ with $\hat{h}_\mu(x_i) < -\delta$. Let $S_n = \bigcup_{i=1}^n \mathrm{Gal}(k^{\mathrm{sep}}/k) \cdot x_i$. Then S_n is $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-invariant and $|S_n| \to \infty$. By (2.8), we get $\hat{h}_\mu(S_n) \leq -\delta$. Moreover, from (3.1), we see that for each $v \in \mathcal{M}_k$,

$$\liminf_{n\to\infty}([S_n]_v, [S_n]_v)_{\mu_v} \geq 0.$$

Replacing $\hat{h}_\mu(S_{n_j})$ in (3.6) by $\hat{h}_\mu(S_n)$ and letting ϵ tend to zero, we get

$$\limsup_{n\to\infty}([S_n]_v, [S_n]_v)_{\mu_v} \leq -\delta,$$

which is a contradiction. □

4. *Non-adelic dynamical pairs (f, c) are typical*

In this section, we aim to prove Theorems 1.3 and 4.5. We first introduce some notation and terminology.

4.1. *A dynamical pair on* $\mathbb{P}^1$. Let k be a product formula field and let $K = k(t)$. Recall that a dynamical pair on $\mathbb{P}^1$ is a pair $(f, c) \in K(z) \times K$ with $d := \deg_z f \geq 2$. We say that the pair (f, c) is *isotrivial* if there is a family of Möbius transformations $M_t(z) \in \overline{K}(z)$

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such that both $M_t \circ f_t \circ M_t^{-1}(z)$ and $M_t(c(t))$ are independent of t. Moreover, we say that (f, c) is *preperiodic* if the starting point $c \in K$ is preperiodic under $f \in K(z)$, that is, if there are integers $m > n \geq 0$ with $f^n(c) = f^m(c) \in K$.

Recall further that we have the *Call–Silverman canonical height* function associated to $f \in K(z)$, denoted by $\hat{h}_f : \mathbb{P}^1(\overline{K}) \to \mathbb{P}^1(\overline{K})$, determined uniquely by the properties $\hat{h}_f(f(c)) = d \cdot \hat{h}_f(c)$ and $\hat{h}_f(c) = h(c) + O(1)$. Alternatively, for $c \in K$, we can compute the canonical height as

$$\hat{h}_f(c) := \lim_{n\to\infty} \frac{\deg_t f^n(c)}{d^n}.$$

We note that $\hat{h}_f(c) \geq 0$ and equality holds if and only if (f, c) is either isotrivial or preperiodic; see [2, 17].

4.2. *Homogenization.* Let $(f, c) \in K(z) \times K$ be a dynamical pair with degree $d := \deg_z f \geq 2$. In what follows, we choose lifts of $f_t(z) = P_t(z)/Q_t(z)$ and $c(t) = A(t)/B(t)$. For the lift of f, we write

$$F_{t_1,t_2}(z, w) = (P_{t_1,t_2}(z, w), Q_{t_1,t_2}(z, w)),$$

where P_{t_1,t_2}, Q_{t_1,t_2} are homogeneous polynomials in (z, w) of degree d, with the coefficients of homogeneous polynomials in (t_1, t_2) of the same degree that are relatively prime. For a lift of c, we write

$$C(t_1, t_2) = (A(t_1, t_2), B(t_1, t_2)),$$

where A and B are homogeneous polynomials in $k[t_1, t_2]$ of the same degree and have no common linear factor in $\bar{k}[t_1, t_2]$. Moreover, we define

$$(A_{C,n}(t_1, t_2), B_{C,n}(t_1, t_2)) := F^n_{t_1,t_2}(C(t_1, t_2)).$$

That is,

$$A_{C,0}(t_1, t_2) = A(t_1, t_2) \text{ and } B_{C,0}(t_1, t_2) = B(t_1, t_2),$$

while for all $n \geq 0$,

$$\begin{aligned} A_{C,n+1}(t_1, t_2) &= P_{t_1,t_2}(A_{C,n}(t_1, t_2), B_{C,n}(t_1, t_2)), \\ B_{C,n+1}(t_1, t_2) &= Q_{t_1,t_2}(A_{C,n}(t_1, t_2), B_{C,n}(t_1, t_2)). \end{aligned} \tag{4.1}$$

Note that $f_t^n(c(t)) = A_{C,n}(t, 1)/B_{C,n}(t, 1)$. We often work with dehomogenized coordinates and when doing so, we identify $\mathbb{P}^1(\bar{k})$ with $\mathbb{A}^1(\bar{k}) \cup \{\infty\}$ as follows. A point $[t_1 : t_2] \in \mathbb{P}^1(\bar{k})$ is seen as $t_1/t_2 \in \mathbb{A}^1(\bar{k})$ when $t_2 \neq 0$ and as ∞ when $t_2 = 0$.

We say that $f \in K(z)$ *degenerates* at $t \in \mathbb{A}^1(\bar{k}) \cup \{\infty\}$ if $\deg_z(f_t) < d$ and write $\mathrm{Sing}(f)$ for the set of degenerating parameters. Notice that the set $\mathrm{Sing}(f)$ is precisely the zero locus of the resultant of the homogeneous polynomials P_{t_1,t_2}, Q_{t_1,t_2} in (z, w), which we denote by

$$\mathrm{Res}(F_{t_1,t_2}) := \mathrm{Res}_{(z,w)}(P_{t_1,t_2}, Q_{t_1,t_2}) \in k[t_1, t_2] \setminus \{0\}.$$

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We remark here that $\mathrm{Res}(F_{t_1,t_2})$ is a homogeneous polynomial in t_1, t_2. Thus,

$$\mathrm{Sing}(f) = \{[\alpha_1 : \alpha_2] \in \mathbb{P}^1(\bar{k}) : \mathrm{Res}(F_{\alpha_1,\alpha_2}) = 0\}.$$

We also work with a lift of $f^n(c)$ defined by coprime homogeneous polynomials. To write the greatest common divisor of $F^n_{t_1,t_2}(C(t_1, t_2))$, for each $\alpha \in \mathbb{P}^1(\bar{k})$, we let

$$u_\alpha(t_1, t_2) = \begin{cases} t_1 - \alpha t_2 & \text{if } \alpha \in \mathbb{A}^1(\bar{k}), \\ t_2 & \text{if } \alpha = \infty. \end{cases}$$

Moreover, we let $m_{C,n}(\alpha)$ be the maximal integer $m \in \mathbb{N}$ with $u_\alpha^m | A_{C,n}$ and $u_\alpha^m | B_{C,n}$. Then,

$$\gcd(F^n_{t_1,t_2}(C(t_1, t_2))) = \prod_{\alpha \in \mathrm{Sing}(f)} u_\alpha(t_1, t_2)^{m_{C,n}(\alpha)}.$$

We point out here that for each $\alpha \in \mathrm{Sing}(f)$, the sequence $\{m_{C,n}(\alpha)/d^n\}_{n\in\mathbb{N}}$ converges as $n \to \infty$ since $\{m_{C,n+1}(\alpha) - d \cdot m_{C,n}(\alpha)\}_{n\in\mathbb{N}}$ is uniformly bounded. We associate a lift of $f^n(c)$ with lifts F_{t_1,t_2} and C of f and c respectively, defined by

$$F_{C,n}(t_1, t_2) := F^n_{t_1,t_2}(C(t_1, t_2))/\gcd(F^n_{t_1,t_2}(C(t_1, t_2)).$$

This is given by coprime homogeneous polynomials in the variables (t_1, t_2) and $\deg F_{C,n}(t_1, t_2) = \deg_t f_t^n(c(t))$.

Next, we introduce measures associated with each dynamical pair.

4.3. *Measure associated to a dynamical pair: a family of rescaled 'bifurcation' measures.* Let k be a product formula field and $K = k(t)$ as before. In this section, we introduce the canonical measure $\mu_{f,c} = \{\mu_{f,c,v}\}_{v\in\mathcal{M}_k}$ associated to a non-isotrivial and non-preperiodic dynamical pair (f, c). Recall that for each $v \in \mathcal{M}_k$ and $z, w \in \mathbb{C}_v$, we write $\|(z, w)\|_v = \max\{|x|_v, |y|_v\}$. Since (f, c) is not preperiodic and non-isotrivial, one has $\hat{h}_f(c) \neq 0$ and in particular $\deg F_{C,n} \neq 0$ for sufficiently large n. To see this, in the case of f being non-isotrivial, we refer to [2, Theorem 1.6]; however, when f is isotrivial but (f, c) is non-isotrivial, an easy argument following [7, Lemma 10.1] indicates that $\hat{h}_f(c) > 0$. For n sufficiently large, we let

$$G_{F,C,n,v}(t_1, t_2) := \frac{\log \|F_{C,n}(t_1, t_2)\|_v}{\deg F_{C,n}}.$$

By a telescopic argument as in [8] or [30], this sequence converges locally uniformly on $\mathbb{C}_v^2 \setminus (\{(0, 0)\} \cup \pi^{-1}(\mathrm{Sing}(f)))$. We denote its limit by

$$G_{F,C,v}(t_1, t_2) := \lim_{n\to\infty} G_{F,C,n,v}(t_1, t_2).$$

Note that for non-archimedean places, $G_{F,C,v}$ extends to the product of Berkovich affine lines $\mathbb{A}_v^{1,an} \times \mathbb{A}_v^{1,an} \setminus \{(0, 0)\}$; see [7, Ch. 10]. By slight abuse of notation, we also denote its extension by $G_{F,C,v}$. Let $\pi : \mathbb{C}_v^2 \setminus \{(0, 0)\} \to \mathbb{P}^1(\mathbb{C}_v)$ be the standard projection map.

We define a measure $\mu_{f,c,v}$ on $\mathbb{P}^1(\mathbb{C}_v) \setminus \mathrm{Sing}(f)$ by

$$\mu_{f,c,v} := \pi_* dd^c_{(t_1,t_2)} G_{F,C,v}(t_1, t_2). \tag{4.2}$$

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When v is archimedean so that $\mathbb{C}_v \simeq \mathbb{C}$ and $\mathbb{P}^{1,an}_v \simeq \mathbb{P}^1(\mathbb{C})$, this measure is a rescaled bifurcation measure. Indeed, the bifurcation measure associated to (f, c) is defined on $\mathbb{P}^1(\mathbb{C}_v)\backslash\mathrm{Sing}(f)$ by

$$\pi_* dd^c_{(t_1,t_2)}\bigg(\lim_{n\to\infty} \frac{1}{d^n} \log \|F^n_{t_1,t_2}(C(t_1, t_2))\|_v\bigg).$$

It is an important measure in dynamics as it is supported exactly at the set of non-degenerating parameters $t = \lambda$ at which (f, c) is unstable, that is, the sequence of holomorphic functions $\{t \mapsto f^n_t(c(t))\}$ is not normal near λ. Recall here that the family f_t is *stable* at λ if the Julia set moves holomorphically for a small perturbation of t at λ, or equivalently if the dynamical pairs (f, c) (upon passing to a finite branched cover of $\mathbb{P}^1$) are stable at λ for each critical point c; see [41, 45]. We refer the reader to [13, 14, 24] for more details.

In the following proposition, we show that the definition of $\mu_{f,c,v}$ extends naturally to give a probability measure on $\mathbb{P}^{1,an}_v$ for each $v \in \mathcal{M}_k$. When $\deg F_{C,n}$ is not zero, we write

$$\mu_{f,c,n,v} := \pi_* dd^c_{(t_1,t_2)} G_{F,C,n,v}(t_1, t_2)$$

to denote the probability measure on $\mathbb{P}^{1,an}_v$ associated to $F_{C,n}$ for $v \in \mathcal{M}_k$. This measure is independent of the choice of lifts for f and c.

PROPOSITION 4.1. *Let $(f, c) \in K(z) \times K$ be a non-isotrivial and non-preperiodic dynamical pair. For each $v \in \mathcal{M}_k$, the sequence of measures $\mu_{f,c,n,v}$ on $\mathbb{P}^{1,an}_v$ converges weakly to a probability measure $\mu_{f,c,v}$ on $\mathbb{P}^{1,an}_v$ as $n \to \infty$. Moreover, $\mu_{f,c,v}$ has continuous potentials if and only if $G_{F,C,v}$ extends continuously to $\mathbb{C}^2_v\backslash\{(0, 0)\}$.*

Proof. First we assume that $v \in \mathcal{M}_k$ is an archimedean place, so that $\mathbb{C}_v \simeq \mathbb{C}$ and $\mathbb{P}^{1,an}_v \simeq \mathbb{P}^1(\mathbb{C})$. Thus, we may work on $\mathbb{P}^1(\mathbb{C})$. Notice that $G_{F,C,n,v}(t_1, t_2)$ is a plurisubharmonic function on $\mathbb{C}^2\backslash\{(0, 0)\}$ and recall that the sequence $G_{F,C,n,v}$ converges locally uniformly on $\mathbb{C}^2 \setminus (\{(0, 0)\} \cup \pi^{-1}(\mathrm{Sing}(f)))$. Hence, the sequence of probability measures $\mu_{f,c,n,v}$ converges weakly to the rescaled bifurcation measure $\mu_{f,c,v}$ on $\mathbb{P}^1(\mathbb{C})\backslash\mathrm{Sing}(f)$. Since the space of probability measures on $\mathbb{P}^1(\mathbb{C})$ is compact in the weak topology, to show that $\mu_{f,c,n,v}$ has a unique limit, it suffices to prove that for any convergent subsequence of $\mu_{f,c,n,v}$, the limit admits no point mass on $\mathrm{Sing}(f)$. Without loss of generality, we may assume that $0 \in \mathrm{Sing}(f)$. We have to show that for any $\epsilon > 0$, there is a radius $r > 0$ and an integer $N > 0$, such that for all $n \geq N$, $\mu_{f,c,n,v}(D(0, r)) < \epsilon$. Suppose that this is not the case. Then we may find integers $n_j \to \infty$ and a sequence of radii $r_{n_j} \to 0$ such that

$$\mu_{f,c,n_j,v}(D(0, r_{n_j})) \to \epsilon_0 > 0,$$

as $j \to \infty$. Let $P_{n_j}(t)$ be a potential function of $\mu_{f,c,n_j,v}|_{D(0,r_{n_j})}$. We have

$$P_{n_j}(t) := \int \log |t - s|_v \, d(\mu_{f,c,n_j,v}|_{D(0,r_{n_j})}) \to \epsilon_0 \log |t|_v$$

locally uniformly on a punctured disk centered at 0. Hence, the sequence of subharmonic functions $G_{F,C,n_j,v}(t, 1) - P_{n_j}(t)$ converges locally uniformly to a subharmonic function

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$G_{F,C,v}(t,1) - \epsilon_0 \log|t|_v$ on a punctured disk. So we can find some $L_0 > 0$ and $r_0 > 0$, such that for all big n_j, we have

$$\sup_{|t|=r_0} (G_{F,C,n_j,v}(t,1) - P_{n_j}(t)) < L_0.$$

From [17, Proposition 3.1], one has $G_{F,C,v}(t,1) = o(\log|t|_v)$. Then for very small t, we can find n_j big enough such that

$$G_{F,C,n_j,v}(t,1) - P_{n_j}(t) > -\frac{\epsilon_0}{2}\log|t|_v > L_0,$$

which is a contradiction as the subharmonic function $G_{F,C,n_j,v}(t,1) - P_{n_j}(t)$ achieves its maximal value on the boundary of $D(0,r_0)$. Since all $G_{F,C,n,v}$ are bounded above uniformly near 0, $G_{F,C,v}(t,1)$ is bounded above and subharmonic on the punctured disk centered at 0. By [47, Theorem 3.6.1], $G_{F,C,v}(t,1)$ has a unique extension to a subharmonic function in a disk centered at 0, with $G_{F,C,v}(0,1) := \limsup_{t\to 0} G_{F,C,v}(t,1)$. Because $G_{F,C,v}(t,1) = o(\log|t|_v)$ and $\mu_{f,c,v}(\{0\}) = 0$, the extended subharmonic function is a potential of $\mu_{f,c,v}$ near 0. Hence, $\mu_{f,c,v}$ has continuous potential if and only if $G_{F,C,v}(t_1,t_2)$ can be extended continuously. For properties of subharmonic functions, we refer the reader to [47].

Assume now that $v \in \mathcal{M}_k$ is non-archimedean. Each $F_{C,n,v}$ determines a probability measure $\mu_{f,c,n,v}$ with continuous potential on $\mathbb{P}^{1,an}_v$ defined on $\mathbb{P}^1(\mathbb{C}_v)$ by

$$g_{f,c,n,v}(x) = \log\|\tilde{x}\|_v - G_{F,C,n,v}(\tilde{x}).$$

The measure is given by

$$\mu_{f,c,n,v} := \Delta g_{f,c,n,v} + \lambda_v. \tag{4.3}$$

Here, λ_v is the probability measure supported on the Gauss point. For any neighborhood $U \subset \mathbb{P}^1(\mathbb{C}_v)$ of $\mathrm{Sing}(f)$, the sequence $G_{F,C,n,v}(\tilde{x})$ converges uniformly for $x \in \mathbb{P}^1(\mathbb{C}_v)\backslash U$. Since $\mathbb{P}^1(\mathbb{C}_v)$ is dense in $\mathbb{P}^{1,an}_v$, we also have that for any neighborhood of U^{an} of $\mathrm{Sing}(f)$ in $\mathbb{P}^{1,an}_v$, the function $g_{f,c,n,v}(x)$ converges uniformly on $\mathbb{P}^{1,an}_v\backslash U^{an}$ as $n \to \infty$. Hence, from (4.3), we see that the limit

$$g_v(x) := \lim_{n\to\infty} g_{f,c,n,v}(x) = \log\|\tilde{x}\|_v - G_{F,C,v}(\tilde{x})$$

is an element of $\mathrm{BVD}(\mathbb{P}^{1,an}_v)$ (see [7, Definition 5.11]), with

$$\mu_{f,c,v} - \lambda_v := \Delta g_v.$$

It is clear that the probability measure $\mu_{f,c,v}$ is the unique limit of $\{\mu_{f,c,n,v}\}$ on $\mathbb{P}^{1,an}_v$, with potential g_v. Since the potential function of $\mu_{f,c,v}$ is unique up to a constant, we have that $\mu_{f,c,v}$ has a continuous potential if and only if g_v can be extended continuously to $\mathrm{Sing}(f)$, or equivalently if and only if $G_{F,C,v}$ can be extended continuously on $\mathbb{C}^2_v\backslash\{(0,0)\}$. □

We are indebted to Laura DeMarco for sharing the idea of the following proposition.

PROPOSITION 4.2. *Let* $(f,c) \in K(z)\times K$ *be a non-isotrivial and non-preperiodic dynamical pair and* $v \in \mathcal{M}_k$. *We write* $M_{F,C,v} = \{(\lambda_1,\lambda_2) \in \mathbb{C}^2_v \setminus \{(0,0)\}:$

$G_{F,C,v}(\lambda_1, \lambda_2) \le 0\}$. *Suppose that* $G_{F,C,v}$ *extends continuously to* $\mathbb{C}_v^2 \setminus \{(0,0)\}$. *Then,*

$$\mathrm{Cap}(M_{F,C,v}) \le \liminf_{n\to\infty} |\operatorname{Res} F_{C,n}|_v^{-1/\deg(F_{C,n})^2}.$$

Proof. Let $v \in \mathcal{M}_k$ and write $G_{F,C,n,v}(t_1, t_2) := \log \|F_{C,n}(t_1, t_2)\|_v / \deg F_{C,n}$ and

$$M_{n,v} := \{(s_1, s_2) \in \mathbb{C}_v^2 \setminus \{(0,0)\} : G_{F,C,n,v}(s_1, s_2) \le 0\}.$$

From [23, Proposition 2.1], we have $\mathrm{Cap}(M_{n,v}) = |\operatorname{Res}(F_{C,n})|_v^{-1/\deg(F_{C,n})^2}$. We are going to prove that for any $\epsilon \in |\mathbb{C}_v^*|_v$, there exists an $N \in \mathbb{N}$ such that for any $n \ge N$, we have

$$G_{F,C,n,v}(t_1, t_2) - G_{F,C,v}(t_1, t_2) < \epsilon. \tag{4.4}$$

This will imply that $M_{F,C,v} \subset e^{\epsilon} M_{n,v}$ and hence, by the monotonicity of the homogeneous capacity, $\mathrm{Cap}(M_{F,C,v}) \le e^{2\epsilon}\,\mathrm{Cap}(M_{n,v})$ for all $n \ge N$. Since $|\mathbb{C}_v^*|_v$ is dense in $\mathbb{R}_{\ge 0}$, the proposition follows. Note that $G_{F,C,n,v}(t_1, t_2)$ converges locally uniformly to $G_{F,C,v}(t_1, t_2)$ away from $\pi^{-1}(\mathrm{Sing}(f))$. Hence, it suffices to prove that (4.4) holds in a small neighborhood of $\pi^{-1}(\mathrm{Sing}(f))$. To this end, we may assume without loss of generality that $0 \in \mathrm{Sing}(f)$ and show that there exists an $r > 0$, such that for all $t \in \mathbb{C}_v$ with $|t|_v \le r$, we have $G_{F,C,n,v}(t, 1) < G_{F,C,v}(t, 1) + \epsilon$ for large n. Since $G_{F,C,v}(t, 1)$ is continuous, we can choose r small enough such that for $|t|_v \le r$, we have $|G_{F,C,v}(t, 1) - G_{F,C,v}(0, 1)| < \epsilon/3$. Moreover, enlarging N if necessary, we may further assume that $G_{F,C,n,v}(t, 1) < G_{F,C,v}(t, 1) + \epsilon/3 < G_{F,C,v}(0, 1) + 2\epsilon/3$, when $|t|_v = r$. Then, since $G_{F,C,n,v}(t, 1)$ is subharmonic, by the maximum principle (see [7, Proposition 8.14] when v is non-archimedean), we get

$$G_{F,C,n,v}(t, 1) < G_{F,C,v}(0, 1) + 2\epsilon/3 < G_{F,C,v}(t, 1) + \epsilon$$

for all $t \in \mathbb{C}_v$ with $|t|_v \le r$ and $n \ge N$, as claimed. □

Definition 4.3. We call a non-preperiodic and non-isotrivial dynamical pair (f, c) *adelic* or *quasi-adelic* if the corresponding measure $\mu_{f,c} = \{\mu_{f,c,v}\}_{v\in\mathcal{M}_k}$, defined in Proposition 4.1, is adelic or quasi-adelic respectively.

4.4. *A generic dynamical pair is not adelic.* Let k be a number field or the function field of a smooth projective curve defined over a field (of arbitrary characteristic) and let $\alpha \in \overline{k}$ (or $\alpha = \infty \in \mathbb{P}^1(\overline{k})$). To simplify our notation, for a homogeneous polynomial $H(x, y)$ defined over a ring R and an element $\chi \in R$, we write $H(\chi) := H(\chi, 1)$ and $H(\infty) := H(1, 0)$. In particular,

$$P_\alpha(z, w) := P_{\alpha,1}(z, w) \quad \text{and} \quad Q_\alpha(z, w) := Q_{\alpha,1}(z, w), \text{ when } \alpha \in \overline{k} \text{ and}$$
$$P_\infty(z, w) := P_{1,0}(z, w),\ Q_\infty(z, w) := Q_{1,0}(z, w)$$

We let

$$R_\alpha(z, w) := \gcd(P_\alpha(z, w), Q_\alpha(z, w)),$$

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with $R_\alpha(z, 1)$ being a monic polynomial. Then

$$P_\alpha = R_\alpha \cdot P_\alpha^* \quad \text{and} \quad Q_\alpha = R_\alpha \cdot Q_\alpha^*, \tag{4.5}$$

where $P_\alpha^*(z, w)$, $Q_\alpha^*(z, w)$ are homogeneous polynomials with no common linear factor over $\bar{k}$. Note that $R_\alpha \neq 1$ if and only if $\alpha \in \mathrm{Sing}(f)$. We define the set of α-*holes* of f by

$$\mathcal{H}_{f,\alpha} := \{[\lambda_1 : \lambda_2] \in \mathbb{P}^1(\bar{k}) : R_\alpha(\lambda_1, \lambda_2) = 0\}.$$

Abusing notation slightly, we often view $\mathcal{H}_{f,\alpha}$ as a subset of $\bar{k} \cup \{\infty\}$ identifying $\mathbb{P}^1(\bar{k})$ with $\bar{k} \cup \{\infty\}$. Under this identification, $\mathcal{H}_{f,\alpha}$ consists of $t \in \bar{k}$ such that $R_\alpha(t, 1) = 0$ together with possibly ∞ in the event that $R_\alpha(1, 0) = 0$.

Next, we define our notion of an α-*generic* dynamical pair (f, c) and subsequently state our theorem.

Definition 4.4. Let $(f, c) \in k(t)(z) \times k(t)$ be a dynamical pair. Assume that f is not isotrivial and $\deg_z f \geq 3$. Let $\alpha \in \mathrm{Sing}(f)$. We say that (f, c) is α-*generic* if the following properties are satisfied.

(P1) $\deg_z(f_\alpha) \geq 2$.

(P2) There exists $\rho \in \mathcal{H}_{f,\alpha}$ that is not a totally ramified fixed point of f_α^2.

(P3) For all $n \in \mathbb{N}$, we have $f_\alpha^n(c(\alpha)) \notin \mathcal{H}_{f,\alpha}$.

(P4) $c(\alpha)$ is not preperiodic for f_α.

(P5) If k is a function field, then f_α is not isotrivial.

With this terminology in place, Theorem 1.3 reduces to the following statement.

THEOREM 4.5. *Let k be a number field or the function field of a smooth projective curve. Let $f \in k(t)(z)$ with $\deg_z f \geq 3$ that is not isotrivial. Let $c \in k(t)$ that is not preperiodic for f. If there is an $\alpha \in \mathrm{Sing}(f)$ such that (f, c) is α-generic, then (f, c) is not adelic.*

Before we proceed to the proof of Theorem 4.5, let us see how it implies Theorem 1.3. First we make a useful observation.

Remark 4. A dynamical pair $(f, c) \in k(t)(z) \times k(t)$ is adelic if and only if the pair $(M \circ f^n \circ M^{-1}, M(f^N(c)))$ is adelic for some $n, N \in \mathbb{N}$ and a Möbius transformation $M(z) \in \overline{k(t)}(z)$. This observation allows us to apply Theorem 4.5 in cases when $\deg_z f = 2$, as in §5.

Theorem 4.5 $\Rightarrow$ Theorem 1.3. Let (f, c), $N \in \mathbb{N}$, $g := f^N$, and $\alpha \in \mathrm{Sing}(g)$ be as in Theorem 1.3. Our assumption that $2 \leq \deg(g_\alpha) < d^N$ guarantees that $\deg_z(g) \geq 3$. Our assumptions on (g, c) and α yield that the pair (g, c) is α-generic. Note here that since f is not isotrivial, we also have that $g = f^N$ is not isotrivial. Thus by Theorem 4.5, we get that (f^N, c) is not adelic. In light of Remark 4, this implies that (f, c) is not adelic. Theorem 1.3 follows. □

We now establish some preliminary results toward the proof of Theorem 4.5. Let $S \subset \mathcal{M}_k$ be a finite set containing all the archimedean places (if they exist). We denote the set of S-integers of k by

$$\mathcal{O}_{S,k} := \{\alpha \in k : |\alpha|_v \leq 1 \text{ for all } v \notin S\}.$$

If $\varphi \in k(z)$ is a rational map and $\alpha \in \overline{k}$, we denote the orbit of α under the action of φ as

$$\mathcal{O}_\varphi(\alpha) = \{\varphi^n(\alpha) : n \in \mathbb{N}\}.$$

The following theorem will play a crucial role in our proofs. We thank Patrick Ingram for referring us to it.

THEOREM 4.6. [50, Theorem 2.2] *Let k be a number field. Let $\varphi \in k(z)$ be a rational map of degree at least 2 such that $\varphi^2(z) \notin k[z]$ and let $\alpha \in k$. Let $S \subset \mathcal{M}_k$ be a finite set containing all the archimedean places. Then, $|\mathcal{O}_\varphi(\alpha) \cap \mathcal{O}_{S,k}| < \infty$.*

We point out here that an analog of Theorem 4.6 also holds for function fields of curves; see [11, 38] for the case of characteristic 0 and p. Here we need to assume that φ is not isotrivial.

THEOREM 4.7. [38, Theorem 1] and [11, Theorem 1.4] *Let k be a function field of a smooth projective curve. Let $\varphi \in k(z)$ be a rational map of degree at least 2 that is not isotrivial and is such that $\varphi^2(z) \notin k[z]$. Let $\alpha \in k$. Let $S \subset \mathcal{M}_k$ be a finite set. Then $|\mathcal{O}_\varphi(\alpha) \cap \mathcal{O}_{S,k}| < \infty$.*

LEMMA 4.8. *Let k be either a number field or a function field of a smooth projective curve. Let $\varphi \in k(z)$ be a rational map of degree at least 2 such that $\varphi^2(z) \notin k[z]$. If k is a function field, assume that φ is not isotrivial. Let $(\alpha, \beta) \in k^2 \setminus \{(0,0)\}$ be such that α/β is not preperiodic for φ. Let $\{a_n\}$ and $\{b_n\}$ be the sequences defined as follows. For a choice of coprime homogeneous polynomials $P, Q \in k[z, w]$ such that $\varphi = [P : Q]$, we let*

$$a_0 = \alpha \quad \textit{and} \quad b_0 = \beta, \quad \textit{and for all } n \geq 0,$$
$$a_{n+1} = P(a_n, b_n), \quad b_{n+1} = Q(a_n, b_n).$$

Then there are infinitely many non-archimedean places $v \in \mathcal{M}_k$ such that $|b_n|_v < 1$ for some $n \in \mathbb{N}$.

Proof. We assume that the statement is false and then derive a contradiction. Then there exists a finite set $S \subset \mathcal{M}_k$ containing the archimedean places such that $|b_n|_v \geq 1$ for all $v \notin S$ and all $n \in \mathbb{N}$. We may enlarge the set S, if necessary, to assume that the coefficients of P and Q are in $\mathcal{O}_{S,k}$ and that for all $v \notin S$, we have $\max\{|\alpha|_v, |\beta|_v\} = 1$. Then for all $v \notin S$ and all $n \in \mathbb{N}$, we have $\max\{|a_n|_v, |b_n|_v\} \leq 1$. Recall that by our hypothesis, we have $|b_n|_v \geq 1$. Thus, $|b_n|_v = 1$ and $|a_n|_v \leq 1$ for all $v \notin S$ and all $n \in \mathbb{N}$. Therefore, $\varphi^n(\alpha/\beta) = (a_n/b_n) \in \mathcal{O}_{S,k}$ for all $n \in \mathbb{N}$. Since α/β is not preperiodic for φ and further if k is a function field φ is not isotrivial, in view of Theorems 4.6 and 4.7, we get that $\varphi^2 \in k[z]$. This contradicts our assumption and concludes the proof. □

LEMMA 4.9. *Let $(f, c) \in k(t)(z) \times k(t)$ be a dynamical pair and let $\alpha \in \mathrm{Sing}(f)$ be such that $\deg(f_\alpha) \geq 1$. Assume that for all $n \in \mathbb{N}$, we have $f_\alpha^n(c(\alpha)) \notin \mathcal{H}_{f,\alpha}$. Then,*

$$m_{C,n}(\alpha) = 0$$

for all $n \in \mathbb{N}$. *In particular, for all* $n \in \mathbb{N}$, *we have*

$$\gcd(F^n_{t_1,t_2}(C(t_1,t_2))) = \prod_{\beta \in \mathrm{Sing}(f)\setminus\{\alpha\}} u_\beta(t_1,t_2)^{m_{C,n}(\beta)}.$$

Proof. Let $\alpha \in \mathbb{P}^1(\bar{k})$ be as in the statement of the lemma. Note that

$$m_{C,0}(\alpha) = \min\{\mathrm{ord}_\alpha\, A, \mathrm{ord}_\alpha\, B\} = 0. \tag{4.6}$$

We will show that $m_{C,n}(\alpha) = 0$ for all $n \in \mathbb{N}$. Assume the contrary and let $N \in \mathbb{N}$ be the smallest integer such that $m_{C,N}(\alpha) > 0$. By (4.6), we have $N \geq 1$. Since $m_{C,N}(\alpha) > 0$, (4.1) yields

$$P_\alpha(A_{C,N-1}(\alpha), B_{C,N-1}(\alpha)) = Q_\alpha(A_{C,N-1}(\alpha), B_{C,N-1}(\alpha)) = 0.$$

Hence, by (4.5), we get that either

$$P^*_\alpha(A_{C,N-1}(\alpha), B_{C,N-1}(\alpha)) = Q^*_\alpha(A_{C,N-1}(\alpha), B_{C,N-1}(\alpha)) = 0 \tag{4.7}$$

or

$$R_\alpha(A_{C,N-1,\alpha}(\alpha), B_{C,N-1,\alpha}(\alpha)) = 0. \tag{4.8}$$

If (4.7) is true, then $(A_{C,N-1}(\alpha), B_{C,N-1}(\alpha)) \neq (0,0)$ is a common zero of P^*_α, Q^*_α, contradicting the fact that they do not share a common linear factor.

If however (4.8) holds, then

$$\frac{A_{C,N-1}(\alpha)}{B_{C,N-1}(\alpha)} \in \mathcal{H}_{f,\alpha}. \tag{4.9}$$

Nonetheless, by our assumption, $f^n_\alpha(c(\alpha)) \notin \mathcal{H}_{f,\alpha}$ for all $n \leq N-1$. Moreover, the minimality of N implies that $m_{C,n}(\alpha) = 0$ for all $n \leq N-1$. Hence,

$$f^n_\alpha(c(\alpha)) = \frac{A_{C,n}(\alpha)}{B_{C,n}(\alpha)}$$

for all $n \leq N-1$. Thus, (4.9) yields

$$f^{N-1}_\alpha(c(\alpha)) = \frac{A_{C,N-1}(\alpha)}{B_{C,N-1}(\alpha)} \in \mathcal{H}_{f,\alpha},$$

contradicting our assumption that the orbit of $c(\alpha)$ under iteration by f_α never meets the set $\mathcal{H}_{f,\alpha}$. In both cases, we got a contradiction. Thus, the lemma follows. □

Before stating the next proposition, we recall that ∞ is a totally ramified fixed point of a rational map $\psi \in k(z)$ if and only if $\psi \in k[z]$.

PROPOSITION 4.10. *Let* $(f,c) \in k(t)(z) \times k(t)$ *be a dynamical pair with* $\deg_z f \geq 3$. *Assume that f is not isotrivial. Let* $\alpha \in \mathrm{Sing}(f)$ *be such that* (f,c) *is* α*-generic. Then there are infinitely many* $v \in \mathcal{M}_k$ *such that for some* $n_v \in \mathbb{N}$, *we have*

$$\max\{|A_{C,n_v}(\alpha)|_v, |B_{C,n_v}(\alpha)|_v\} < 1.$$

Proof. Let $\alpha \in \mathrm{Sing}(f)$ be such that (f, c) is α-generic. To simplify the notation, throughout this proof, we write $a_n := A_{C,n}(\alpha)$ and $b_n := B_{C,n}(\alpha)$. Since (P3) holds, by Lemma 4.9 we know that not both of a_n and b_n are zero. We have

$$\begin{aligned} a_{n+1} &= P_\alpha(a_n, b_n) = R_\alpha(a_n, b_n) P^*_\alpha(a_n, b_n), \\ b_{n+1} &= Q_\alpha(a_n, b_n) = R_\alpha(a_n, b_n) Q^*_\alpha(a_n, b_n), \end{aligned} \tag{4.10}$$

for all $n \in \mathbb{N}$. We define auxiliary sequences $\{a^*_n\}$ and $\{b^*_n\}$ as $a^*_0 := a_0$, $b^*_0 := b_0$ and

$$a^*_{n+1} := P^*_\alpha(a^*_n, b^*_n), \quad b^*_{n+1} := Q^*_\alpha(a^*_n, b^*_n) \quad \text{for } n \geq 1.$$

Then, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} a_n &= \prod_{i=0}^{n-1} R_\alpha(a^*_i, b^*_i)^{d^{n-1-i}} \cdot a^*_n, \\ b_n &= \prod_{i=0}^{n-1} R_\alpha(a^*_i, b^*_i)^{d^{n-1-i}} \cdot b^*_n. \end{aligned} \tag{4.11}$$

Let $S \subset \mathcal{M}_k$ be a finite set of places, containing the archimedean ones, such that for all $v \notin S$, the coefficients of P^*_α and Q^*_α are v-adic integers, $|\mathrm{Res}(P^*_\alpha, Q^*_\alpha)|_v = 1$ and $\max\{|a^*_0|_v, |b^*_0|_v\} = 1$. Then invoking [7, Lemma 10.1], for all $n \in \mathbb{N}$ and $v \notin S$, we have

$$\max\{|a^*_n|_v, |b^*_n|_v\} = 1. \tag{4.12}$$

We may enlarge the set S, if necessary, to assume that the elements of $\mathcal{H}_{f,\alpha} \cap \overline{k}$ are v-adic integers for all $v \notin S$. Thus, also the coefficients of R_α are v-adic integers for $v \notin S$. Combining then (4.11) with (4.12), we get that

$$\max\{|a_{n+1}|_v, |b_{n+1}|_v\} \leq |R_\alpha(a^*_n, b^*_n)|_v \leq |u_\rho(a^*_n, b^*_n)|_v \tag{4.13}$$

for all $v \notin S$ and $n \in \mathbb{N}$ and for any $\rho \in \mathcal{H}_{f,\alpha}$. Now let $\rho \in \mathcal{H}_{f,\alpha}$ be as in (P2). We claim that there are infinitely many $v \in \mathcal{M}_k$ such that

$$|u_\rho(a^*_n, b^*_n)|_v < 1 \tag{4.14}$$

for some $n \in \mathbb{N}$. By (4.13), it is clear that this suffices to prove this proposition. To prove (4.14), we use Lemma 4.8. If $\rho = \infty$, our claim follows. Otherwise, let $M_\rho(z, w) = (w + \rho z, z)$ and $(\hat{P}^*_\alpha, \hat{Q}^*_\alpha) = M_\rho^{-1} \circ (P^*_\alpha, Q^*_\alpha) \circ M_\rho$. Consider the morphism $\hat{g}_\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ defined by

$$[z : w] \mapsto [\hat{P}^*_\alpha(z, w) : \hat{Q}^*_\alpha(z, w)].$$

Since $\rho \in \mathcal{H}_{f,\alpha}$ is not a totally ramified fixed point of f^2_α as in (P2), we know that ∞ is not a totally ramified fixed point of $\hat{g}^2_\alpha$. Moreover, by (P4), we have $|\mathcal{O}_{\hat{g}_\alpha}(b_0/a_0 - \rho b_0)| = \infty$ and by (P1), we have $\deg(\hat{g}_\alpha) \geq 2$. Thus, by Lemma 4.8 applied to the rational map $\hat{g}_\alpha$ and $(b_0, a_0 - \rho b_0)$, our claim follows. This finishes our proof. □

We are now ready to prove the main result of this section: In most cases, (f, c) is not adelic.

4.5. *Proof of Theorem 4.5.* Let f and c be as in the statement of the theorem. Let $\alpha \in \mathrm{Sing}(f)$ be such that (f, c) is α-generic and assume to the contrary that (f, c) is adelic. Then there is a finite set $S \subset \mathcal{M}_k$ such that for all $v \notin S$, we have

$$G_{F,C,v}(t_1, t_2) = \log \|(t_1, t_2)\|_v + c_v \tag{4.15}$$

for a constant c_v. Denote by $\mathcal{P} \subset \mathcal{M}_k$ the infinite set of places satisfying the conclusion of Proposition 4.10. In other words, for $v \in \mathcal{P}$, there exists $n_v \in \mathbb{N}$ such that

$$\max\{|A_{C,n_v}(\alpha)|_v, |B_{C,n_v}(\alpha)|_v\} < 1.$$

Enlarging the set S if necessary, we may further assume that for all $v \notin S$, the following hold.

(S1) $|\alpha|_v = 1$ if $\alpha \neq 0, \infty$.

(S2) For all $\beta \in \mathrm{Sing}(f) \setminus \{\alpha\}$, we have $|u_\beta(\alpha)|_v = 1$ and if $\beta \neq 0$ also $|u_\beta(0, 1)|_v = 1$.

(S3) The coefficients of $F_{t_1,t_2}(z, w) \in k[t_1, t_2, z, w]$ are v-adic integers.

(S4) $|\mathrm{Res}_{(t_1,t_2)}(A, B)|_v = 1$ and the coefficients of $C(t_1, t_2) = (A, B)$ are v-adic integers.

(S5) All coefficients of $\mathrm{Res}(F_{t_1,t_2})$ are v-adic units.

We aim to prove that (4.15) does not hold for places in the infinite set $\mathcal{P} \setminus S$, thus leading to a contradiction. To do so, we will evaluate (4.15) at two distinct points that yield distinct values for c_v when $v \in \mathcal{P} \setminus S$.

Let $v \in \mathcal{P} \setminus S$. In the rest of this proof, for $t_0 \in \mathbb{C}_v$, we write

$$T_0 = \begin{cases} (t_0, 1) & \text{if } \alpha \in \mathbb{A}^1(\bar{k}), \\ (1, t_0) & \text{if } \alpha = \infty. \end{cases}$$

View both T_0 and α as elements of $\mathbb{P}^1$. Since by Lemma 4.9 either $A_{C,n_v}(\alpha)$ or $B_{C,n_v}(\alpha)$ is non-zero and v is non-archimedean, we may choose $T_0 \neq (0, 1)$ sufficiently close to α in the v-adic topology to be such that the following hold.

(T1) $0 < |u_\alpha(T_0)|_v < 1$.

(T2) $\max\{|A_{C,n_v}(T_0)|_v, |B_{C,n_v}(T_0)|_v\} \leq \max\{|A_{C,n_v}(\alpha)|_v, |B_{C,n_v}(\alpha)|_v\}$.

Next, we show that evaluating (4.15) at T_0 gives $c_v < 0$. To this end, recall that

$$F_{C,n}(t_1, t_2) = \left(\frac{A_{C,n}(t_1, t_2)}{\gcd(F^n_{t_1,t_2}(C(t_1, t_2))}, \frac{B_{C,n}(t_1, t_2)}{\gcd(F^n_{t_1,t_2}(C(t_1, t_2))} \right), \tag{4.16}$$

where by Lemma 4.9, we have

$$g_{C,n}(t_1, t_2) := \gcd(F^n_{t_1,t_2}(C(t_1, t_2)) = \prod_{\beta \in \mathrm{Sing}(f) \setminus \{\alpha\}} u_\beta(t_1, t_2)^{m_{C,n}(\beta)}.$$

Combining (S2) and (T1), the ultrametric inequality gives $|u_\beta(T_0)|_v = 1$ for all $\beta \in \mathrm{Sing}(f) \setminus \{\alpha\}$. This in turn yields $|g_{C,n}(T_0)|_v = 1$ for all $n \in \mathbb{N}$. Thus, evaluating (4.16) at T_0, we have

$$\|F_{C,n}(T_0)\|_v = \|F^n_{T_0}(C(T_0))\|_v = \|(A_{C,n}(T_0), B_{C,n}(T_0))\|_v. \tag{4.17}$$

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Note that by (S1) and (T1), we have $\|T_0\|_v \leq 1$. Combining this with (S3), (4.17) yields

$$\|F_{C,n+1}(T_0)\|_v \leq \|F_{C,n}(T_0)\|_v^d.$$

An easy argument by induction and (T2) yield that for all $n \geq n_v$, we have

$$\|F_{C,n}(T_0)\|_v \leq \max\{|A_{C,n_v}(\alpha)|_v, |B_{C,n_v}(\alpha)|_v\}^{d^{n-n_v}} < 1,$$

where the last inequality follows from our assumption that $v \in \mathcal{P}$. We now have

$$c_v = \lim_{n\to\infty} \frac{\log \|F_{C,n}(T_0)\|_v}{\deg(F_{C,n})} \leq \frac{\log \max\{|A_{C,n_v}(\alpha)|_v, |B_{C,n_v}(\alpha)|_v\}}{d^{n_v} \cdot \hat{h}_f(c)} < 0, \tag{4.18}$$

as claimed. We point out here that by our assumption f is not isotrivial. Hence, our property (P4) guarantees that $\hat{h}_f(c) \neq 0$; see [2, 17].

However, we can choose $S_0 = (s_0, 1) \in \mathbb{C}_v^2$ to be such that for all $\beta \in \mathrm{Sing}(f)$, we have $|s_0|_v = |u_\beta(S_0)|_v = 1$. Then, upon using (S3), we get that the coefficients of F_{S_0} are v-adic integers. Moreover, by (S5) and since $|u_\beta(S_0)|_v = 1$ for all $\beta \in \mathrm{Sing}(f)$, we have $|\operatorname{Res}_{(z,w)}(F_{S_0})|_v = 1$. Therefore, [7, Lemma 10.1] yields that $\|F_{S_0}(z, w)\|_v = \|(z, w)\|_v^d$. Since by (S4) we have that C has good reduction and moreover $\|S_0\|_v = 1$, another application of [7, Lemma 10.1] yields $\|C(S_0)\|_v = 1$. Thus, an easy argument by induction yields $\|F_{S_0}^n(C(S_0))\|_v = 1$. By our choice of S_0, we have $|g_{C,n}(S_0)|_v = 1$ for all $n \in \mathbb{N}$. Hence, $\|F_{S_0}^n(C(S_0))\|_v = \|F_{C,n}(S_0)\|_v = 1$ for all $n \in \mathbb{N}$. Therefore,

$$c_v = \lim_{n\to\infty} \frac{\log \|F_{C,n}(S_0)\|_v}{\deg(F_{C,n})} = 0.$$

This contradicts (4.18) and finishes the proof of our theorem. □

5. *Quasi-adelicity for almost all starting points*

We study the family $g_{\lambda,t}(z) := \lambda z/(z^2 + tz + 1)$, where λ is an ℓth primitive root of unity for $\ell \geq 2$, and aim to prove Theorem 1.4. We show that for a 'generic' c, the dynamical pair (g_λ, c) is quasi-adelic but is not adelic. In particular, for $c \in \{1, -1\}$ being a critical point of g_λ, the pairs $(g_\lambda, 1)$, $(g_\lambda, -1)$ are quasi-adelic but not adelic. We refer the reader to [9, 46] for the pictures of the bifurcation of $(g_\lambda, \pm 1)$.

5.1. *Homogeneous lifts.*

Throughout the rest of this section, we fix a primitive ℓth root of unity λ with order at least 2. It is more convenient to work with the ℓth iterate of $g_{\lambda,t}(z)$, which we denote by

$$f_t(z) := g_{\lambda,t}^{\ell}(z).$$

It is easy to see that g_λ degenerates at $t = \infty$. Thus, $\mathrm{Sing}(f) = \{\infty\}$. Throughout this section, we write

$$d := 2^\ell, \quad d_1 := 2^{\ell-1} - 1, \quad d_2 := 2^{\ell-1}.$$

At times, we also use notation introduced in §4. We fix a homogeneous lift of $g_{\lambda,t}(z)$ as

$$G_{t_1,t_2}(z, w) := (t_2\lambda zw, t_1 zw + t_2(z^2 + w^2)).$$

The following proposition enables us to show that g_λ^ℓ degenerates to $f_\infty(z) := z/(z^2+1)$ at $t = \infty$.

PROPOSITION 5.1. *Let* λ *be an* ℓ*th primitive root of unity. The* ℓ*th iterate of* G_{t_1,t_2} *is given by*

$$G_{t_1,t_2}^\ell(z,w) = t_2^{d_2} \cdot (\tau \cdot (t_1 zw)^{d_1} zw + t_2(\ldots), \tau \cdot (t_1 zw)^{d_1}(z^2+w^2) + t_2(\ldots)),$$

for some non-zero $\tau \in \mathbb{Z}[\lambda]$. *In particular,* $f = g_\lambda^\ell$ *degenerates to* $f_\infty(z) = z/(z^2+1)$ *at* $t = \infty$.

Proof. We prove this proposition by induction. Notice that, inductively for $2 \le n \le \ell$, one has $G_{t_1,t_2}^n = (P_n, Q_n)$, where

$$\begin{aligned} P_n(z,w) &= t_2^{2^{n-1}}((zw)^{2^{n-1}}\alpha_n t_1^{2^{n-1}-1} + (zw)^{2^{n-1}-1}(z^2+w^2)\eta_n t_2 t_1^{2^{n-1}-2} + t_2^2(\ldots)), \\ Q_n(z,w) &= t_2^{2^{n-1}-1}((zw)^{2^{n-1}}\beta_n t_1^{2^{n-1}} + (zw)^{2^{n-1}-1}(z^2+w^2)\tau_n t_2 t_1^{2^{n-1}-1} + t_2^2(\ldots)), \end{aligned} \tag{5.1}$$

for constants α_n, β_n, η_n, and τ_n depending on λ. From the iteration formula, we get $\alpha_2 = \lambda^2$, $\eta_2 = \lambda^2$, $\beta_2 = 1+\lambda$, $\tau_2 = \lambda + 2$ and for all $n \ge 2$, we have

$$\begin{cases} \alpha_{n+1} = \lambda \cdot \alpha_n \cdot \beta_n, \\ \beta_{n+1} = \beta_n \cdot (\alpha_n + \beta_n), \end{cases} \quad \text{and} \quad \begin{cases} \eta_{n+1} = \lambda \cdot (\alpha_n \cdot \tau_n + \beta_n \cdot \eta_n), \\ \tau_{n+1} = \alpha_n \cdot \tau_n + \beta_n \cdot \eta_n + 2\beta_n \cdot \tau_n. \end{cases}$$

Consequently, for $n \ge 3$, we have

$$\begin{aligned} \alpha_n &= \lambda^n \cdot \prod_{i=1}^{n-2}(1+\lambda+\cdots+\lambda^i)^{2^{n-2-i}}, \\ \beta_n &= (1+\lambda+\cdots+\lambda^{n-1}) \cdot \prod_{i=1}^{n-2}(1+\lambda+\cdots+\lambda^i)^{2^{n-2-i}}. \end{aligned}$$

Since λ is an ℓth primitive root of unity, we have $\alpha_\ell \neq 0$ and $\beta_\ell = 0$. It remains to show that

$$\tau := \tau_\ell = \alpha_\ell. \tag{5.2}$$

Let z_0 be a primitive 6th root of unity and notice that $z_0/(z_0^2+1) = 1$. Since $\beta_\ell = 0$, from the expression of $G_{t_1,t_2}^\ell(z_0, 1)$ in (5.1), we get

$$\lim_{t_1=1, t_2 \to 0} \frac{P_\ell(z_0,1)}{Q_\ell(z_0,1)} \to \frac{\alpha_\ell}{\tau_\ell},$$

or equivalently,

$$\lim_{t\to\infty} g_{\lambda,t}^\ell(z_0) = \frac{\alpha_\ell}{\tau_\ell}. \tag{5.3}$$

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We are going to show that this limit is equal to one, and hence equation (5.2) follows. Notice that

$$g_{\lambda,t}(z_0) = \frac{\lambda z_0}{1 + tz_0 + z_0^2} = \frac{\lambda}{t} \cdot \left(1 - \frac{1}{t} + o\left(\frac{1}{t}\right)\right)$$

for $t \to \infty$. Using the expression of $g_{\lambda,t}(z) = \lambda \cdot z/(1 + t \cdot z + z^2)$, inductively we get

$$g^n_{\lambda,t}(z_0) = \frac{1}{t} \cdot \frac{\lambda^n}{1 + \lambda + \cdots + \lambda^{n-1}} \cdot \left(1 - \frac{1}{1 + \lambda + \cdots + \lambda^{n-1}} \cdot \frac{1}{t} + o\left(\frac{1}{t}\right)\right)$$

for all $1 \le n \le \ell - 1$ as $t \to \infty$. Consequently, we have

$$\begin{aligned}
g^\ell_{\lambda,t}(z_0) &= g_{\lambda,t}(g^{\ell-1}_{\lambda,t}(z_0)) \\
&= g_{\lambda,t}\left(\frac{1}{t} \cdot \frac{\lambda^{\ell-1}}{1 + \lambda + \cdots + \lambda^{\ell-2}} \cdot \left(1 - \frac{1}{1 + \lambda + \cdots + \lambda^{\ell-2}} \cdot \frac{1}{t} + o\left(\frac{1}{t}\right)\right)\right) \\
&= \frac{1/t \cdot \lambda \cdot \lambda^{\ell-1}/(1 + \lambda + \cdots + \lambda^{\ell-2}) \cdot (1 - 1/(1 + \lambda + \cdots + \lambda^{\ell-2}) \cdot 1/t + o(1/t))}{1 + t \cdot 1/t \cdot \lambda^{\ell-1}/(1 + \lambda + \cdots + \lambda^{\ell-2}) \cdot (1 - 1/(1 + \lambda + \cdots + \lambda^{\ell-2}) \cdot 1/t + o(1/t)) + o(1/t)} \\
&= \frac{1/t \cdot 1/(1 + \lambda + \cdots + \lambda^{\ell-2}) \cdot (1 - 1/(1 + \lambda + \cdots + \lambda^{\ell-2}) \cdot 1/t + o(1/t))}{1 + (-1) \cdot (1 - 1/(1 + \lambda + \cdots + \lambda^{\ell-2}) \cdot 1/t + o(1/t)) + o(1/t^2)},
\end{aligned}$$

where in the last equality, we used the fact that $\lambda^\ell = 1$ and $1 + \lambda + \cdots + \lambda^{\ell-1} = 0$. Letting now $t \to \infty$, we get $g^\ell_{\lambda,t}(z_0) \to 1$. Combining this with (5.3), we get (5.2). The proposition follows. □

Let us now fix a lift of f_t in homogeneous coordinates (t_1, t_2) as

$$\begin{aligned}
F_{t_1,t_2}(z, w) :&= (P_{t_1,t_2}(z, w), Q_{t_1,t_2}(z, w)) := G^\ell_{t_1,t_2}(z, w)/(\tau \cdot t_2^{d_2}) \\
&= ((t_1 zw)^{d_1} zw + t_2(\ldots), (t_1 zw)^{d_1}(z^2 + w^2) + t_2(\ldots)).
\end{aligned} \tag{5.4}$$

Notice that at the point at infinity, our lift specializes to the map

$$F_{1,0}(z, w) = ((zw)^{d_1} zw, (zw)^{d_1}(z^2 + w^2)),$$

which is a homogeneous lift of

$$f_\infty(z) = \frac{z}{z^2 + 1}.$$

Keeping the notation as in §4, we have $R_{f,\infty}(z, w) = (zw)^{d_1}$, and hence $\mathcal{H}_{f,\infty} = \{0, \infty\}$. We let k be a number field containing λ, so that $f \in k(t)(z)$. We fix a starting point $c \in k(t)$ satisfying $0, \infty \notin \mathcal{O}_{f_\infty}(c(\infty))$ so that condition (P3) in Definition 4.4 holds for the pair (f, c). We also fix a homogeneous lift of the starting point c, with coefficients in $\mathcal{O}_k$, as

$$C(t_1, t_2) := (A(t_1, t_2), B(t_1, t_2)),$$

and write

$$F_{C,n}(t_1, t_2) = F^n_{t_1,t_2}(C(t_1, t_2))/\gcd(F^n_{t_1,t_2}(C(t_1, t_2))) = (A_{C,n}(t_1, t_2), B_{C,n}(t_1, t_2)).$$

LEMMA 5.2. *For all $n \in \mathbb{N}$, we have* $\gcd(F^n_{t_1,t_2}(C)) = 1$. *Hence,*

$$F_{C,n}(t_1, t_2) = F^n_{t_1,t_2}(C(t_1, t_2)).$$

Proof. As $\mathrm{Sing}(f) = \{+\infty\}$, $\deg(f_\infty) = 2$, and we have assumed that $f_\infty^n(c(\infty)) \notin \mathcal{H}_{f,\infty} = \{0, \infty\}$, this follows from Lemma 4.9. Notice that here,

$$\gcd(F^n_{t_1,t_2}(C(t_1, t_2))) = \prod_{\beta \in \mathrm{Sing}(f)\setminus\{\infty\}} u_\beta(t_1, t_2)^{m_{C,n}(\beta)} = 1,$$

as the product is empty. □

LEMMA 5.3. *We have* $\deg(F_{C,n}) = d_1 \cdot (d^n - 1)/(d - 1) + d^n \cdot \deg(c)$ *for all* $n \in \mathbb{N}$. *In particular, the dynamical pair* (f, c) *is not preperiodic. Furthermore,* $\hat{h}_f(c) = (d_1/(d - 1)) + \deg(c) \neq 0$.

Proof. Since $0, \infty \notin \mathcal{O}_{f_\infty}(c(\infty))$, we get $\deg(A_{C,n}) = \deg(B_{C,n})$ for all $n \in \mathbb{N}$. The lemma now follows inductively from the recursive definition of $F_{C,n}$ as in Lemma 5.2. □

When there is no scope for confusion, we use a_n and b_n to denote $A_{C,n}(1, 0)$ and $B_{C,n}(1, 0)$ respectively. From Lemma 5.2, we see that

$$\begin{aligned} a_{n+1} &= (a_n b_n)^{d_1} \cdot a_n b_n, \\ b_{n+1} &= (a_n b_n)^{d_1} \cdot (a_n^2 + b_n^2) \end{aligned} \tag{5.5}$$

for all $n \in \mathbb{N}$. We also make use of auxiliary sequences $\{a_n^*\}, \{b_n^*\} \subset k$ defined by

$$\begin{aligned} a_0^* &:= a_0, \quad & b_0^* &:= b_0, \\ a_{n+1}^* &:= a_n^* b_n^*, \quad & b_{n+1}^* &:= a_n^{*2} + b_n^{*2} \quad \text{for } n \geq 1. \end{aligned} \tag{5.6}$$

Notice that if we let

$$\alpha_n := \prod_{i=0}^{n-1} (a_i^* b_i^*)^{d_1 \cdot d^{n-i-1}}, \tag{5.7}$$

then for $n \geq 1$, we have $a_n = \alpha_n a_n^*$ and $b_n = \alpha_n b_n^*$.

5.2. *Continuity of the escape rate.* To prove that (f, c) is quasi-adelic, we need to first show that the escape rate $G_{F,C,v}$ is a continuous function.

THEOREM 5.4. *The functions* $\log \|F_{C,n}(t_1, t_2)\|_v / \deg(F_{C,n})$ *converge locally uniformly on* $\mathbb{C}_v^2 \setminus \{(0, 0)\}$ *to the function* $G_{F,C,v}$. *In particular,* $G_{F,C,v}$ *is continuous.*

Before we proceed to the proof of this theorem, we establish some lemmata. First we let

$$F_{C,n}(1, s) = (A_{C,n}(1, s), B_{C,n}(1, s)) = (A_{C,n}(1, 0) + s p_n(s), B_{C,n}(1, 0) + s q_n(s)).$$

LEMMA 5.5. *For each* $v \in \mathcal{M}_k$, *we have* $\gamma_v \in \mathbb{R}$ *satisfying the following:*

- $\gamma_v := \lim_{n\to\infty}(\log |A_{C,n}(1, 0)|_v / d^n) = \lim_{n\to\infty}(\log |B_{C,n}(1, 0)|_v / d^n)$; *and*
- $\limsup_{n\to\infty}(\log |p_n(0)|_v / d^n)$, $\limsup_{n\to\infty}(\log |q_n(0)|_v / d^n) \leq \gamma_v$.

Proof. By [7, Lemma 10.1], the recursive definition of $\{a_n^*\}, \{b_n^*\}$ in (5.6) implies that there is a set of constants $\{L_v : v \in \mathcal{M}_k\}$ and a finite set $S \subset \mathcal{M}_k$ such that $L_v = 1$ for all $v \notin S$

and for all $v \in \mathcal{M}_k$, we have $L_v \geq 1$ and

$$\max\{|a_n^*|_v, |b_n^*|_v\} \leq L_v^{2^n} \quad \text{for all } n \in \mathbb{N}. \tag{5.8}$$

Now let $L = \prod_{v \in \mathcal{M}_k} L_v^{N_v}$. Invoking the product formula, (5.8) yields

$$\min\{|a_n^*|_v, |b_n^*|_v\} \geq \frac{1}{L^{2^n}}. \tag{5.9}$$

Moreover, (5.8) implies

$$\max\{|a_n^*|_v, |b_n^*|_v\} \leq L^{2^n}. \tag{5.10}$$

In particular, (5.9) and (5.10) yield that $\lim_{n\to\infty}(\log|a_n^*|_v/d^n) = \lim_{n\to\infty}(\log|b_n^*|_v/d^n) = 0$ and for $\{\alpha_n\}$ as in (5.7), the sequence

$$\frac{\log|\alpha_n|_v}{d^n} = \sum_{i=0}^{n-1} d_1 \cdot \frac{\log|a_i^* b_i^*|_v}{d^{i+1}}$$

converges. Denoting its limit by γ_v, we have established the following:

$$\gamma_v = \lim_{n\to\infty} \frac{\log|a_n|_v}{d^n} = \lim_{n\to\infty} \frac{\log|b_n|_v}{d^n} = \lim_{n\to\infty} \frac{\log|\alpha_n|_v}{d^n}.$$

The first part of the lemma follows. Now let c_n and e_n be the constant terms of the polynomials $p_n(s)$ and $q_n(s)$ respectively. From the recursive definition of $F_{C,n}$ as in Lemma 5.2, we see that there are homogeneous $\Phi, \Psi \in k[z, w]$ of degree d and $\Phi_i, \Psi_i \in k[z, w]$ for $i = 1, 2$ of degree $d - 1$, such that

$$\begin{aligned} c_{n+1} &= \Phi(a_n, b_n) + c_n \cdot \Phi_1(a_n, b_n) + e_n \cdot \Phi_2(a_n, b_n), \\ e_{n+1} &= \Psi(a_n, b_n) + c_n \cdot \Psi_1(a_n, b_n) + e_n \cdot \Psi_2(a_n, b_n) \end{aligned} \tag{5.11}$$

for all $n \geq 0$. Define auxiliary sequences $\{c_n^*\}, \{e_n^*\} \subset k$ as $c_n = \alpha_n \cdot c_n^*$ and $e_n = \alpha_n \cdot d_n^*$. We will show that

$$\limsup_{n\to\infty} \frac{\log|c_n^*|_v}{d^n}, \ \limsup_{n\to\infty} \frac{\log|e_n^*|_v}{d^n} \leq 0. \tag{5.12}$$

Having proved this, the second part of our lemma will follow, since

$$\limsup_{n\to\infty} \frac{\log|c_n|_v}{d^n}, \ \limsup_{n\to\infty} \frac{\log|e_n|_v}{d^n} \leq \lim_{n\to\infty} \frac{|\alpha_n|_v}{d^n} = \gamma_v.$$

To prove (5.12), first notice that by (5.7), we have $\alpha_n^d/\alpha_{n+1} = 1/(a_n^* b_n^*)^{d_1}$. The recursive formulas in (5.11) can now be written as

$$\begin{aligned} c_{n+1}^* &= \frac{\Phi(a_n^*, b_n^*) + c_n^* \cdot \Phi_1(a_n^*, b_n^*) + e_n^* \cdot \Phi_2(a_n^*, b_n^*)}{(a_n^* b_n^*)^{d_1}}, \\ e_{n+1}^* &= \frac{\Psi(a_n^*, b_n^*) + c_n^* \cdot \Psi_1(a_n^*, b_n^*) + e_n^* \cdot \Psi_2(a_n^*, b_n^*)}{(a_n^* b_n^*)^{d_1}}. \end{aligned} \tag{5.13}$$

Let $L_{n,v} := \max\{|c_n^*|_v, |e_n^*|_v\}$. By (5.8) and (5.9), we get that there is some constant $L_0 \geq 1$ such that

$$\max\left\{|\Phi(a_n^*, b_n^*)|_v, |\Phi_i(a_n^*, b_n^*)|_v, |\Psi(a_n^*, b_n^*)|_v, |\Psi_i(a_n^*, b_n^*)|_v, \frac{1}{|a_n^* b_n^*|_v^{d_1}}\right\} \leq L_0^{2^n}$$

for $i = 1, 2$ and all $v \in \mathcal{M}_k$. Combining this with (5.13), we see that there is some $r \geq 1$ such that

$$L_{n+1,v} \leq r^{2^n} \max\{1, L_{n,v}\}$$

for all $n \in \mathbb{N}$. An easy argument by induction now yields that $L_{n,v} \leq r^{n2^n} \max\{1, L_{0,v}\}$ for all $n \in \mathbb{N}$. Therefore,

$$\limsup_{n\to\infty} \frac{\log L_{n,v}}{d^n} \leq 0,$$

and (5.12) follows. This finishes our proof. □

The next two propositions show that our escape rate function is a locally uniform limit near the degenerate point at $t_2 = 0$.

PROPOSITION 5.6. *Let* $v \in \mathcal{M}_k$. *For every* $\epsilon \in (0, 1)$, *there exist* $\delta > 0$ *and an integer* $N > 0$ *such that*

$$\frac{\log \|F_{C,n}(1, s)\|_v}{\deg(F_{C,n})} - \frac{d-1}{d_1 + (d-1)\deg(c)} \cdot \gamma_v < \epsilon$$

for all $|s|_v < \delta$ *and* $n \geq N$.

Proof. Let $\epsilon \in (0, 1)$. Recall that

$$\lim_{n\to\infty} \frac{d^n}{\deg(F_{C,n})} = \frac{d-1}{d_1 + (d-1)\deg(c)} \tag{5.14}$$

by Lemma 5.3. Thus, it suffices to establish that there exist $\delta > 0$ and an integer $N > 0$ such that

$$\frac{\log \|F_{C,n}(1, s)\|_v}{\deg(F_{C,n})} - \frac{d^n}{\deg(F_{C,n})} \cdot \gamma_v < \epsilon \tag{5.15}$$

for $n \geq N$ and $|s|_v < \delta$. Throughout this proof, we write

$$h := \max\left\{1, \frac{d-1}{d_1 + (d-1)\deg(c)}\right\}.$$

By Lemma 5.5, there exists large $N \in \mathbb{N}$ such that

$$\max\{|A_{C,N}(1, 0)|_v, |B_{C,N}(1, 0)|_v\} < \left(1 + \frac{\epsilon}{4h}\right)^{d^N} e^{\gamma_v \cdot d^N}. \tag{5.16}$$

Moreover by (5.14), we may choose $N \in \mathbb{N}$ large enough such that

$$\left|\frac{d^{N+i}}{\deg(F_{C,N+i})} - \frac{d-1}{d_1 + (d-1)\deg(c)}\right| < \frac{\epsilon}{4} \tag{5.17}$$

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for all $i \geq 0$, and we may further assume that

$$\frac{\log 8}{d^N} < \frac{\epsilon}{4h}.$$

Let $L = 8(1+\epsilon/4h)^{d^N} e^{\gamma_v \cdot d^N}$. By (5.16), we can find some $0 < \delta < 1$ such that for $|s|_v < \delta$, we have

$$\|F_{C,N}(1,s)\|_v < \frac{L}{8}. \tag{5.18}$$

Recall from Lemma 5.2 that $F_{C,n}(1,s) = F_{1,s}^n(C(1,s))$ for all $n \in \mathbb{N}$. From the expression of $F_{1,s} = (z^{d/2}w^{d/2} + s(\ldots), (zw)^{d_1}(z^2+w^2) + s(\ldots))$, shrinking δ if necessary and applying $F_{1,s}$ repeatedly to (5.18), we get

$$\|F_{C,N+i}(1,s)\|_v < \frac{L^{d^i}}{8}$$

for all $i \geq 0$. Therefore, recalling the definition of L, we get

$$\begin{aligned}
&\frac{\log \|F_{C,N+i}(1,s)\|_v}{\deg(F_{C,N+i})} \\
&\quad < \frac{d^i}{\deg(F_{C,N+i})}\log L - \frac{\log 8}{\deg(F_{C,N+i})} \\
&\quad \leq \frac{d^i}{\deg(F_{C,N+i})}\log 8 + \frac{d^{N+i}}{\deg(F_{C,N+i})}\log\left(1+\frac{\epsilon}{4h}\right) + \frac{d^{N+i}}{\deg(F_{C,N+i})}\gamma_v \\
&\quad = \frac{d^{i+N}}{\deg(F_{C,N+i})}\cdot\frac{\log 8}{d^N} + \frac{d^{N+i}}{\deg(F_{C,N+i})}\cdot\frac{\epsilon}{4h} + \frac{d^{N+i}}{\deg(F_{C,N+i})}\gamma_v.
\end{aligned}$$

This inequality combined with our assumptions on $N \in \mathbb{N}$ yield

$$\frac{\log \|F_{C,N+i}(1,s)\|_v}{\deg(F_{C,N+i})} - \frac{d^{N+i}}{\deg(F_{C,N+i})}\gamma_v < \epsilon$$

for all $i \geq 0$ and $|s|_v < \delta$. The proposition follows. □

To show that the convergence is uniform from below, we will first need the following weaker estimate.

LEMMA 5.7. *Let $v \in \mathcal{M}_k$. For all $s \in \mathbb{C}_v$ with $|s|_v \leq 1$ and $(z,w) \in \mathbb{C}_v^2 \setminus \{(0,0)\}$, we have*

$$\frac{\|F_{1,s}(z,w)\|_v}{\|(z,w)\|_v^d} \geq |\tau|_v^{-1}\frac{|s|_v^{3\cdot d_2 - 2}}{4^{d-1}}.$$

Consequently, for all $n > j$ and $s \in \mathbb{C}_v$ with $|s|_v \leq |\tau|_v$, we have

$$\frac{\log \|F_{C,n}(1,s)\|_v}{d^n} \geq \frac{\log \|F_{C,j}(1,s)\|_v}{d^j} + \frac{\log(|s|_v^{3d_2-2})}{d^j} - \frac{\log(|\tau|_v 4^{d-1})}{d^j}.$$

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Proof. Set $G_s(z, w) = (s\lambda zw, sz^2 + zw + sw^2)$. We will first see that for all $s \in \mathbb{C}_v$ with $|s|_v \le 1$, we have

$$\|G_s(z, w)\|_v \ge \frac{|s|_v^2}{4}\|(z, w)\|_v^2. \tag{5.19}$$

Note that by the homogeneity and symmetry of G_s, we may assume that $w = 1$ and $|z|_v \le 1$. Then we have that either $|\lambda sz|_v \ge (|s|_v^2/4)$ or $|z|_v < |s|_v/4$, in which case $|s + z + sz^2|_v \ge |s|_v - (|s|_v/4) - (|s|_v^3/16) \ge \frac{1}{4}|s|_v^2$. In both cases, $\|G_s(z, 1)\|_v \ge (|s|_v^2/4)$, and our claim follows. Note that by the definition of F_{t_1,t_2} in (5.4), we have

$$\|F_{1,s}(z, w)\|_v = \|\tau^{-1}s^{-d_2} \cdot G_s^{\ell}(z, w)\|_v = |\tau|_v^{-1}|s|_v^{-d_2} \cdot \|G_s(G_s^{\ell-1}(z, w))\|_v.$$

Repeated applications of (5.19) now give

$$\|F_{1,s}(z, w)\|_v \ge |\tau|_v^{-1}\frac{|s|_v^{2+2^2+\cdots+2^{\ell}-d_2}}{4^{1+2+2^2+\cdots+2^{\ell-1}}}\|(z, w)\|_v^d = |\tau|_v^{-1}\frac{|s|_v^{3\cdot d_2-2}}{4^{d-1}}\|(z, w)\|_v^d.$$

The first conclusion of the lemma follows. The second conclusion of the lemma follows from the first, once we observe that

$$\left\|\frac{F_{C,n}(1, s)}{F_{C,j}(1, s)^{d^{n-j}}}\right\|_v = \left\|\frac{F_{C,n}(1, s)}{F_{C,n-1}(1, s)^d} \cdot \frac{F_{C,n-1}(1, s)^d}{F_{C,n-2}(1, s)^{d^2}} \cdots \frac{F_{C,j+1}(1, s)^{d^{n-j-1}}}{F_{C,j}(1, s)^{d^{n-j}}}\right\|_v.$$

This concludes the proof of the lemma. □

PROPOSITION 5.8. *Let $v \in \mathcal{M}_k$. For every $\epsilon > 0$, there exists $\delta > 0$ and an integer $N > 0$ such that*

$$\frac{\log\|F_{C,n}(1, s)\|_v}{\deg(F_{C,n})} - \frac{d-1}{d_1 + (d-1)\deg(c)} \cdot \gamma_v > -\epsilon$$

for all $|s|_v < \delta$ and $n \ge N$.

Proof. Recall our notation $F_{C,n}(1, s) = (A_n(s), B_n(s)) = (a_n + sp_n(s), b_n + sq_n(s))$. Let $\epsilon > 0$ be small. Since by Lemma 5.2 we have $F_{C,n+1}(1, s) = F_{1,s}(F_{C,n}(1, s))$, one can find

$$\Phi_3(x, y, z, w), \Psi_3(x, y, z, w) \in k[s][x, y, z, w]$$

depending only on ℓ and homogeneous in $\mathbf{x} := (x, y, z, w)$ of degree d, such that

$$p_{n+1}(s) = \Phi_3(a_n, b_n, p_n(s), q_n(s)) \quad \text{and} \quad q_{n+1}(s) = \Psi_3(a_n, b_n, p_n(s), q_n(s)) \tag{5.20}$$

for all $n \ge 0$. Moreover, one can find a large $L_0 > 0$ such that

$$\|\Phi_3(\mathbf{x})\|_v \le L_0 \cdot \|\mathbf{x}\|_v^d \quad \text{and} \quad \|\Psi_3(\mathbf{x})\|_v \le L_0 \cdot \|\mathbf{x}\|_v^d. \tag{5.21}$$

Enlarging $L_0 \ge 1$ if necessary, we may assume

$$(3d_2 - 2) \cdot d \cdot \log\left(\frac{1 - \epsilon/L_0}{1 + \epsilon/L_0}\right) > -\frac{\epsilon}{4}. \tag{5.22}$$

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In view of Lemma 5.5, we can find a large $N \in \mathbb{N}$ such that

$$\max\{|p_N(0)|_v, |q_N(0)|_v\} < \left(1 + \frac{\epsilon}{L_0}\right)^{d^N} \cdot e^{\gamma_v \cdot d^N} \tag{5.23}$$

and also

$$\left(1 - \frac{\epsilon}{L_0}\right)^{d^{N+i}} \cdot e^{\gamma_v \cdot d^{N+i}} < |a_{N+i}|_v, |b_{N+i}|_v < \left(1 + \frac{\epsilon}{L_0}\right)^{d^{N+i}} \cdot e^{\gamma_v \cdot d^{N+i}}, \tag{5.24}$$

for all $i \geq 0$. By Lemma 5.3, enlarging N if necessary, we may further assume that

$$\frac{d^{N+i}}{\deg(F_{C,N+i})} > \frac{d-1}{d_1 + (d-1)\deg(c)} - \frac{\epsilon}{L_0 \cdot \max\{1, \gamma_v\}} \tag{5.25}$$

for all $i \geq 0$ and that

$$(3d_2 - 2)\left(\frac{\log \epsilon}{d^j} - \frac{\log L_0}{d^{N-1}}\right) - \frac{\log(|\tau|_v 4^{d-1})}{d^j} > -\frac{\epsilon}{4} \tag{5.26}$$

for all $j \geq N$.

Define $L := L_0 \cdot ((1 + \epsilon/L_0)e^{\gamma_v})^{d^N}$. By (5.23) and (5.24), we can find a small $0 < \delta < \min\{1, |\tau|_v\}$ such that if $|s|_v < \delta$, then

$$\max\{|p_N(s)|_v, |q_N(s)|_v\} < \frac{L}{L_0}. \tag{5.27}$$

Combining this with the recursive relations defining $p_n(s), q_n(s)$ given in (5.20) and inequalities (5.21) and (5.24), and since $L_0 \geq 1$, we get inductively that if $|s| < \delta$, then for all $i \geq 0$, we have

$$\max\{|p_{N+i}(s)|_v, |q_{N+i}(s)|_v\} < \frac{L^{d^i}}{L_0}. \tag{5.28}$$

Now choose an integer $N' > N$ such that

$$\delta' := \frac{\epsilon(1 - \epsilon/L_0)^{d^{N'}}}{(1 + \epsilon/L_0)^{d^{N'}} L_0^{d^{N'-N}}} < \delta.$$

We will show that if $|s|_v < \delta'$ and $n \geq N'$, we have

$$\frac{\log \|F_{C,n}(1, s)\|_v}{\deg(F_{C,n})} \geq \frac{d-1}{d_1 + (d-1)\deg(c)}\gamma_v - \epsilon. \tag{5.29}$$

Denote by $\delta_j := \epsilon(1 - \epsilon/L_0)^{d^j}/(1 + \epsilon/L_0)^{d^j} L_0^{d^{j-N}} > 0$. If $j \geq N' > N$ and $|s|_v \leq \delta_j \leq \delta' < \delta$, we have

$$\begin{aligned}\frac{\log \|F_{C,j}(1, s)\|_v}{d^j} &\geq \frac{\log |a_j + sp_j(s)|_v}{d^j} \geq \frac{\log(|a_j|_v - |sp_j(s)|_v)}{d^j} \\ &\geq \frac{1}{d^j} \log\left(\left(1 - \frac{\epsilon}{L_0}\right)^{d^j} \cdot e^{\gamma_v \cdot d^j} - \frac{|s|_v L^{d^{j-N}}}{L_0}\right),\end{aligned} \tag{5.30}$$

where the last inequality follows by (5.24) and (5.28). Therefore, since $|s|_v \leq \delta_j$, we get

$$\frac{\log \|F_{C,j}(1,s)\|_v}{d^j} \geq \log\left(1-\frac{\epsilon}{L_0}\right)+\gamma_v+\log\left(1-\frac{\epsilon}{L_0}\right) \geq -\frac{\epsilon}{4}+\gamma_v, \tag{5.31}$$

when $\epsilon > 0$ is sufficiently small and L_0 is large enough. Combining this with (5.25), we get

$$\frac{\log \|F_{C,j}(1,s)\|_v}{\deg(F_{C,j})} \geq \frac{d-1}{d_1+(d-1)\deg(c)}\gamma_v - \epsilon,$$

and (5.29) holds. If however for $n \geq N'$ we have $\delta_n < |s|_v < \delta'$, then there is some j with $N' \leq j < n$ such that $\delta_{j+1} < |s|_v \leq \delta_j < |\tau|_v$. By Lemma 5.7, we have

$$\frac{\log \|F_{C,n}(1,s)\|_v}{d^n} \geq \frac{\log \|F_{C,j}(1,s)\|_v}{d^j} + \frac{\log(|s|_v^{3d_2-2})}{d^j} - \frac{\log(|\tau|_v 4^{d-1})}{d^j}.$$

This, upon using (5.26) and (5.31), yields

$$\begin{aligned}
&\frac{\log \|F_{C,n}(1,s)\|_v}{d^n}\\
&\geq -\frac{\epsilon}{4}+\gamma_v+\frac{\log(|s|_v^{3d_2-2})}{d^j}-\frac{\log(|\tau|_v 4^{d-1})}{d^j}\\
&\geq -\frac{\epsilon}{4}+\gamma_v+(3d_2-2)\left(\frac{\log\epsilon}{d^j}-\frac{\log L_0}{d^{N-1}}\right)+d\cdot\log\left(\frac{1-\epsilon/L_0}{1+\epsilon/L_0}\right)-\frac{\log(|\tau|_v 4^{d-1})}{d^j}\\
&\geq -\frac{\epsilon}{4}+\gamma_v+(3d_2-2)\cdot d\cdot\log\left(\frac{1-\epsilon/L_0}{1+\epsilon/L_0}\right)-\frac{\epsilon}{4}, \text{ by (5.22)}\\
&\geq \gamma_v-\frac{\epsilon}{4}-\frac{\epsilon}{4}-\frac{\epsilon}{4}=\gamma_v-\frac{3\epsilon}{4}
\end{aligned}$$

for sufficiently small $\epsilon > 0$ and L_0 large enough. Finally, upon using (5.25), we get

$$\frac{\log \|F_{C,n}(1,s)\|_v}{\deg(F_{C,n})} \geq \frac{d-1}{d_1+(d-1)\deg(c)}\gamma_v - \epsilon.$$

In both cases, (5.29) holds. This finishes the proof. □

Proof of Theorem 5.4. By a standard telescoping sum argument, as in [8, Proposition 1.2], we see that the functions $\log \|F_{C,n}(t_1,t_2)\|_v/\deg(F_{C,n})$ converge locally uniformly to the function $G_{F,C,v}$ on $\mathbb{C}_v^2 \setminus \mathbb{C}_v \times \{0\}$. Thus, it suffices to prove that the sequence

$$\frac{\log \|F_{C,n}(1,s)\|_v}{\deg(F_{C,n})}$$

converges locally uniformly in a neighborhood of $s = 0$. This now follows from Propositions 5.6 and 5.8. □

5.3. *Bounds of the radii.* Recall that we have chosen the lift of $c \in k(t)$, denoted by $C(t_1,t_2) = (A(t_1,t_2), B(t_1,t_2))$, so that the coefficients of $C(t_1,t_2)$ lie in $\mathcal{O}_k$. In particular, $\mathrm{Res}(A,B) \in \mathcal{O}_k$. In what follows, we let the following.

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- $S \subset \mathcal{M}_k$ be the finite set consisting of the non-archimedean places of k such that if $v \in S$,
$$|\operatorname{Res}(A, B)|_v < 1 \quad \text{or} \quad |\tau|_v < 1 \quad \text{or} \quad |\operatorname{Res}_{(z,w)}(F_{t,1})|_v < 1,$$
where $\tau \in \mathbb{Z}[\lambda] \subset \mathcal{O}_k$ is defined in Proposition 5.1.
- $\mathcal{M}'_k$ be the set of non-archimedean places in $\mathcal{M}_k$ satisfying the following. If $v \in \mathcal{M}'_k$, then $|\tau|_v = 1$ and there exists $n_v \in \mathbb{N}$ such that
$$\|F_{C,n_v}(1, 0)\|_v = \max\{|A_{C,n_v}(1, 0)|_v, |B_{C,n_v}(1, 0)|_v\} < 1.$$
- $\mathcal{M}'_{k,m}$ the set of places $v \in \mathcal{M}'_k$ for which $m \in \mathbb{N}$ is the smallest integer such that
$$|B_{C,m}(1, 0)|_v < 1.$$

Furthermore, we denote the non-archimedean places of k by $\mathcal{M}^0_k$ and the archimedean ones by $\mathcal{M}^\infty_k$. Finally, note that from (5.5), we get
$$\mathcal{M}'_k = \bigcup_{m \in \mathbb{N}} \mathcal{M}'_{k,m}.$$

LEMMA 5.9. *Let $v \in \mathcal{M}^0_k \setminus S$. If $t \in \mathbb{C}_v$ has $|t|_v \leq 1$, then $\|F_{C,n}(t, 1)\|_v = 1$ for all $n \in \mathbb{N}$. In particular,*
$$G_{F,C,v}(t, 1) = 0.$$

Proof. Let $v \in \mathcal{M}^0_k \setminus S$ and consider $t \in \mathbb{C}_v$ with $|t|_v \leq 1$. Since the coefficients of $C(t_1, t_2)$ are v-adic integers and $v \notin S$, we conclude $|\operatorname{Res}(A, B)|_v = \|(t, 1)\|_v = 1$. Then, [7, Lemma 10.1] yields $\|C(t, 1)\|_v = 1$. Since $v \notin S$ and $\tau \in \mathcal{O}_k$, we have $|\tau|_v = 1$. Now, from the definition of F_{t_1,t_2} in (5.4), we see that all its coefficients are v-adic integers. As furthermore $|\operatorname{Res}_{(z,w)}(F_{t,1})|_v = 1$, using [7, Lemma 10.1] once more, we get $\|F^n_{t,1}(C(t, 1))\|_v = 1$ for all $n \in \mathbb{N}$. This in turn, by Lemma 5.2, implies $\|F_{C,n}(t, 1)\|_v = 1$ for all $n \in \mathbb{N}$ as claimed. □

We write $M_{C,v} := \{(t_1, t_2) \in \mathbb{C}^2_v \setminus \{(0, 0)\} : G_{F,C,v}(t_1, t_2) \leq 0\}$.

PROPOSITION 5.10. *For all places $v \in \mathcal{M}^0_k \setminus (S \cup \mathcal{M}'_k)$, we have*
$$G_{F,C,v}(t_1, t_2) = \log \|(t_1, t_2)\|_v.$$
In particular, $r_{\text{in}}(M_{C,v}) = r_{\text{out}}(M_{C,v}) = 1$.

Proof. Let $v \in \mathcal{M}^0_k \setminus (\mathcal{M}'_k \cup S)$ and $(t_1, t_2) \in \mathbb{C}^2_v \setminus \{(0, 0)\}$. Since $G_{F,C,v}$ scales logarithmically, by Lemma 5.9, we know that the claim holds when $|t_1|_v \leq |t_2|_v$. It suffices to show that for $t_2 \in \mathbb{C}_v$ with $|t_2|_v < 1$, we have $\|F_{C,n}(1, t_2)\|_v = 1$ for all $n \in \mathbb{N}$.

Assume to the contrary that there exist some $t_2 \in \mathbb{C}_v$ with $|t_2|_v < 1$ and $n \in \mathbb{N}$ such that $\|F_{C,n}(1, t_2)\|_v \neq 1$. Notice that since $v \notin S$, we have $|\tau|_v = 1$, which using (5.4) and Lemma 5.2, yields that $F_{C,n}$ has integral coefficients. Hence, our assumption implies
$$\|F_{C,n}(1, t_2)\|_v = \|(A_{C,n}(1, 0) + t_2(\ldots), B_{C,n}(1, 0) + t_2(\ldots))\|_v < 1.$$

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As we also have $|t_2|_v < 1$, this gives

$$\|(A_{C,n}(1,0), B_{C,n}(1,0))\|_v < 1.$$

Since also $|\tau|_v = 1$, we have $v \in \mathcal{M}'_k$, which is a contradiction. This finishes our proof. $\square$

LEMMA 5.11. *Let $m \in \mathbb{N}$ and $v \in \mathcal{M}'_{k,m} \setminus S$. For all $n \geq m+1$, we have*

$$|A_{C,n}(1,0)|_v < |B_{C,n}(1,0)|_v.$$

In particular, for all $n \geq m+1$, we have

$$|A_{C,n}(1,0)|_v^d < |A_{C,n+1}(1,0)|_v, \ |B_{C,n+1}(1,0)|_v < |B_{C,n}(1,0)|_v^d.$$

Moreover, $\{d^{-n} \log |A_{C,n}(1,0)|_v\}_{n \geq m+1}$ is increasing and $\{d^{-n} \log |B_{C,n}(1,0)|_v\}_{n \geq m+1}$ is decreasing.

Proof. Let $m \in \mathbb{N}$ and $v \in \mathcal{M}'_{k,m} \setminus S$. Recall that from (5.5), we have

$$a_{n+1} = a_n^{d_1} b_n^{d_1} \cdot a_n b_n, \ b_{n+1} = a_n^{d_1} b_n^{d_1} (a_n^2 + b_n^2).$$

Since $v \in \mathcal{M}'_{k,m}$, we have $|b_m|_v < 1$ and $|b_n|_v = 1$ for all $n < m$. We will prove the first inequality in the lemma by induction. Using the ultrametric inequality, it is easy to see that if for some $n \in \mathbb{N}$ we have $|a_n|_v < |b_n|_v$, then $|a_{n+1}|_v < |b_{n+1}|_v$. Thus, it remains to prove the base case; that is, $|a_{m+1}|_v < |b_{m+1}|_v$. To this end, we consider cases depending on the value of $m \in \mathbb{N}$.

If $m = 1$, we have $|b_1|_v < 1$ and $|a_0|_v \leq |b_0|_v = 1$. To see that $|a_2|_v = |a_1 b_1|_v^{d_1+1} < |b_2|_v = |(a_1 b_1)^{d_1}(a_1^2 + b_1^2)|_v$, it suffices to show $|a_1 b_1|_v < |a_1^2 + b_1^2|_v$. Note that $|a_1|_v = |a_0|_v^{d_1+1}$ and $|b_1|_v = |a_0|_v^{d_1} |a_0^2 + b_0^2|_v$. If $|a_0|_v < 1$, we have $|a_1|_v < |b_1|_v$; hence, $|a_1 b_1|_v < |a_1^2 + b_1^2|_v$. If $|a_0|_v = 1$, then $|a_1|_v = 1$ and $|a_1 b_1|_v = |b_1|_v < |a_1^2 + b_1^2|_v = 1$ holds as well, since $|b_1|_v < 1$. The base case follows.

If however $m = 0$ or $m \geq 2$, we will prove that $|a_m|_v = 1$. Then $|a_{m+1}|_v = |b_m|_v^{d_2} < |b_m|_v^{d_1} = |b_{m+1}|_v$ and the base case follows. If $m = 0$, so that $|b_0|_v < 1$, our assumption that $v \notin S$ and hence $|\operatorname{Res}(A, B)|_v = 1$ yields $|a_0|_v = 1$, as claimed. If now $m \geq 2$, assume that $|a_m|_v < 1$ to end in a contradiction. Since $v \in \mathcal{M}'_{k,m}$, we have $|b_{m-1}|_v = |b_{m-2}| = 1$, which by (5.5) and our assumption that $|a_m|_v < 1$ implies $|a_{m-1}|_v < 1$ and $|a_{m-2}|_v < 1$. This in turn gives $|b_{m-1}|_v < 1$ contradicting the minimality of $m \in \mathbb{N}$. Hence, $|a_m|_v = 1$ and the base case follows. This completes the proof of the first inequality in the lemma. The rest of the lemma now follows by (5.5). $\square$

The following equalities will be handy later on. They are an immediate consequence of the ultrametric inequality, the definition of $\mathcal{M}'_{k,m}$, and the recursive definitions of a_n and b_n in (5.5).

Remark 5. Let $m \in \mathbb{N}_{\geq 2}$ and $v \in \mathcal{M}'_{k,m}$. We have

$$|A_{C,m}(1,0)|_v = |A_{C,m-1}(1,0)|_v = |B_{C,m-1}(1,0)|_v = 1. \tag{5.32}$$

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In particular,

$$|A_{C,m+1}(1,0)|_v = |B_{C,m}(1,0)|_v^{d_2} = |B_{C,m+1}(1,0)|_v^{d_2/d_1}. \tag{5.33}$$

Before stating the next proposition, recall that from Lemma 5.3, we have $\hat{h}_f(c) \neq 0$.

PROPOSITION 5.12. *Let $m \in \mathbb{N}_{\geq 2}$ and $v \in \mathcal{M}'_{k,m} \setminus S$. We have*

$$\bar{D}^2(0,1) \subset M_{C,v} \subset \bar{D}^2(0, e^{-1/\hat{h}_f(c)((3d_2 \cdot \log |a_{m+1}|_v/d^m)-(\log 4^{d-1}/d^m))}).$$

In particular,

$$0 \leq \log r_{\text{in}}(M_{C,v}) \leq \log r_{\text{out}}(M_{C,v}) \leq -\frac{1}{\hat{h}_f(c)} \cdot \left(\frac{3d_2 \cdot \log |a_{m+1}|_v}{d^m} - \frac{\log 4^{d-1}}{d^m}\right).$$

Proof. Let $m \geq 2$ and $v \in \mathcal{M}'_{k,m} \setminus S$. Since the coefficients of $F_{C,n}$ are v-adic integers for all $n \in \mathbb{N}$, we have $\|F_{C,n}(t_1,t_2)\|_v \leq \|(t_1,t_2)\|_v^{\deg(F_{C,n})}$. Therefore, $\bar{D}^2(0,1) \subset M_{C,v}$ and the first inclusion follows. By Lemma 5.9, we get that if $|t|_v \leq 1$, then $G_{F,C,v}(t,1) = 0$. Therefore, to show the reverse inclusion, it suffices to prove that if $0 < |s|_v < 1$, then

$$G_{F,C,v}(1,s) \geq \frac{1}{\hat{h}_f(c)} \cdot \left(\frac{3d_2 \cdot \log |a_{m+1}|_v}{d^m} - \frac{\log 4^{d-1}}{d^m}\right). \tag{5.34}$$

To this end, let $s \in \mathbb{C}_v$ be such that $0 < |s|_v < 1$. From Lemma 5.11, we know that $\{|b_n|_v\}_{n \geq m+1}$ is a strictly decreasing sequence which converges to zero. Assume first that $0 < |s|_v < |b_{m+1}|_v$. Then there is some $j \geq m+1$ such that $|b_{j+1}|_v \leq |s|_v < |b_j|_v$. As the coefficients of $F_{C,j}(1,s) = (a_j + s(\ldots), b_j + s(\ldots))$ are v-adic integers and since from Lemma 5.11 we have $|a_j|_v < |b_j|_v$ for all $j \geq m+1$, we get $\|F_{C,j}(1,s)\|_v = |b_j|_v$. Therefore, upon using Lemma 5.7 (recall that $|\tau|_v = 1$), we get

$$\begin{aligned}
\frac{\log \|F_{C,n}(1,s)\|_v}{d^n} &\geq \frac{\log \|F_{C,j}(1,s)\|_v}{d^j} + \frac{\log(|s|_v^{3d_2-2})}{d^j} - \frac{\log(|\tau|_v 4^{d-1})}{d^j} \\
&\geq \frac{\log |b_j|_v}{d^j} + d(3d_2-2)\frac{\log |b_{j+1}|_v}{d^{j+1}} - \frac{\log 4^{d-1}}{d^j}
\end{aligned}$$

for all $n > j$. This in turn, using Lemma 5.11, implies

$$\begin{aligned}
\frac{\log \|F_{C,n}(1,s)\|_v}{d^n} &\geq (d(3d_2-2)+1)\frac{\log |b_{j+1}|_v}{d^{j+1}} - \frac{\log 4^{d-1}}{d^j} \\
&\geq (d(3d_2-2)+1)\frac{\log |a_{j+1}|_v}{d^{j+1}} - \frac{\log 4^{d-1}}{d^j} \\
&\geq (d(3d_2-2)+1)\frac{\log |a_{m+1}|_v}{d^{m+1}} - \frac{\log 4^{d-1}}{d^{m+1}} \\
&\geq 3d_2\frac{\log |a_{m+1}|_v}{d^m} - \frac{\log 4^{d-1}}{d^m}.
\end{aligned} \tag{5.35}$$

Finally, assume that $|b_{m+1}|_v \leq |s|_v < 1$. From (5.32), we have $|b_m|_v < |a_m|_v = 1$; thus, $\|F_{C,m}(1,s)\|_v = 1$. Invoking now Lemmata 5.7 and 5.11, we get

$$\begin{aligned}\frac{\log \|F_{C,n}(1,s)\|_v}{d^n} &\geq \frac{\log(|s|_v^{3d_2-2})}{d^m} - \frac{\log 4^{d-1}}{d^m} \geq \frac{\log(|b_{m+1}|_v^{3d_2-2})}{d^m} - \frac{\log 4^{d-1}}{d^m}\\ &\geq \frac{\log(|a_{m+1}|_v^{3d_2-2})}{d^m} - \frac{\log 4^{d-1}}{d^m} \geq 3d_2\frac{\log |a_{m+1}|_v}{d^m} - \frac{\log 4^{d-1}}{d^m}\end{aligned} \tag{5.36}$$

for all $n > m$. Letting $n \to \infty$ in (5.35) and (5.36), (5.34) follows. This finishes our proof. $\square$

LEMMA 5.13. *There exist constants $L_1, L_2 > 0$ such that $\sum_{v\in\mathcal{M}'_{k,m}\setminus S} N_v \leq L_1 \cdot 2^m$ and $\prod_{v\in\mathcal{M}'_{k,m}} |A_{C,m+1}(1,0)|_v^{-N_v} < L_2^{2^m}$ for all $m \geq 2$.*

Proof. We write $T_m := \sum_{v\in\mathcal{M}'_{k,m}\setminus S} N_v$. First we will show that for some $L_1 > 0$, we have $T_m < L_1 \cdot 2^m$. To this end, we define $\mathcal{P}_m := \{v \in \mathcal{M}_k^0 \setminus S : |b_m^*|_v < 1\}$. We claim that $\mathcal{M}'_{k,m}\setminus S \subset \mathcal{P}_m$; hence, it suffices to prove that for some $L_1 > 0$, we have $\sum_{v\in\mathcal{P}_m} N_v < L_1 \cdot 2^m$. To prove that our claim holds, let $m \geq 2$ and $v \in \mathcal{M}'_{k,m}\setminus S$. We recall from (5.32) that $|a_m|_v = 1$. Moreover, from (5.6) and (5.7), we have that a_m and a_m^* are v-adic integers and $|a_m|_v = |a_m^*|_v|\alpha_m|_v = 1$. Hence, $|a_m^*|_v = |\alpha_m|_v = 1$ and

$$|b_m|_v = |\alpha_m|_v \cdot |b_m^*|_v = |b_m^*|_v. \tag{5.37}$$

Our claim follows. Now notice that the recursive definition of $\{b_n^*\}$ in (5.6) allows us to conclude that there is a constant $L_0 > 1$ such that

$$\prod_{v\in\mathcal{M}_k\,:\,|b_m^*|_v>1} |b_m^*|_v^{N_v} \leq L_0^{2^m}. \tag{5.38}$$

Let $r := \sup_{v\in\mathcal{M}_k^0}\{|\alpha|_v : |\alpha|_v < 1 \text{ and } \alpha \in k\} \in (0,1)$. Then, for each $v \in \mathcal{P}_m$, we have

$$|b_m^*|_v^{N_v} \leq r^{N_v}. \tag{5.39}$$

Combining (5.38) and (5.39) and upon using the product formula, we have

$$\prod_{v\in\mathcal{P}_m} r^{N_v} \geq \prod_{v\in\mathcal{P}_m} |b_m^*|_v^{N_v} \geq \prod_{v\in\mathcal{M}_k:|b_m^*|_v<1} |b_m^*|_v^{N_v} = \prod_{v\in\mathcal{M}_k:|b_m^*|_v>1} |b_m^*|_v^{-N_v} \geq L_0^{-2^m}. \tag{5.40}$$

Thus if $L_1 = \log_{1/r} L_0$, we get $T_m \leq L_1 \cdot 2^m$ and the first part of the lemma follows. For the second part of the lemma, first note that (5.33) combined with (5.37) implies

$$|a_{m+1}|_v = |b_m|_v^{d_2} = |b_m^*|_v^{d_2}. \tag{5.41}$$

Using this and the fact that $\mathcal{M}'_{k,m}\setminus S \subset \mathcal{P}_m$, (5.40) yields

$$\prod_{v\in\mathcal{M}'_{k,m}\setminus S} |a_{m+1}|_v^{-N_v} = \prod_{v\in\mathcal{M}'_{k,m}\setminus S} |b_m^*|_v^{-d_2\cdot N_v} \leq \prod_{v\in\mathcal{P}_m} |b_m^*|_v^{-d_2\cdot N_v} \leq L_0^{d_2\cdot 2^m}.$$

Setting $L_2 = L_0^{d_2}$, the lemma follows. $\square$

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We can now control the products of the inner and outer radii as in the following proposition.

PROPOSITION 5.14. *The products*

$$\prod_{v\in\mathcal{M}_k} r_{\text{out}}(M_{C,v})^{N_v}, \qquad \prod_{v\in\mathcal{M}_k} r_{\text{in}}(M_{C,v})^{N_v}, \qquad \textit{and} \qquad \prod_{v\in\mathcal{M}_k} \text{Cap}(M_{C,v})^{N_v}$$

converge strongly.

Proof. We first note that the set $\mathcal{M}_k^\infty \cup S \cup \mathcal{M}'_{k,0} \cup \mathcal{M}'_{k,1}$ is finite. The continuity of the potentials, proved in Theorem 5.4, yields that the products

$$\prod_{v\in\mathcal{M}_k^\infty\cup S\cup\mathcal{M}'_{k,0}\cup\mathcal{M}'_{k,1}} r_{\text{out}}(M_{C,v})^{N_v} \quad \text{and} \quad \prod_{v\in\mathcal{M}_k^\infty\cup S\cup\mathcal{M}'_{k,0}\cup\mathcal{M}'_{k,1}} r_{\text{in}}(M_{C,v})^{N_v}$$

are finite. Hence, invoking Theorem 2.2, and Propositions 5.10 and 5.12, it suffices to prove that the following sum converges:

$$\sum_{m=2}^{\infty} \sum_{v\in\mathcal{M}'_{k,m}} N_v \cdot \left(-\frac{3d_2 \cdot \log|a_{m+1}|_v}{d^m} + \frac{\log 4^{d-1}}{d^m} \right).$$

This in turn follows from Lemma 5.13. □

5.4. *Proof of Theorem 1.4.* Let k be a number field and $c \in k(t)$ be such that $0, \infty \notin \mathcal{O}_{f_\infty}(c(\infty))$. As $f = g_\lambda^\ell$, it suffices to prove the conclusions of our theorem for the pair (f, c) in place of (g_λ, c); see Remark 4. First, we are going to see that the measure $\mu_{f,c}$ is quasi-adelic. Since for each $v \in \mathcal{M}_k$ we have $\mu_{f,c,v} = \mu_{M_{C,v}}$, by Theorem 2.2, it suffices to prove that the set $\{M_{C,v}\}_{v\in\mathcal{M}_k}$ is quasi-adelic. The continuity of the potentials of $M_{C,v}$ is established in Theorem 5.4. Moreover, in view of Proposition 5.14, we know that the products

$$\prod_{v\in\mathcal{M}_k} r_{\text{out}}(M_{C,v})^{N_v} \quad \text{and} \quad \prod_{v\in\mathcal{M}_k} r_{\text{in}}(M_{C,v})^{N_v}$$

converge strongly. Thus, the measure $\mu_{f,c}$ is quasi-adelic. Assume further that $c(\infty)$ is not a preperiodic point for f_∞. Then our assumption on c implies that for $\infty \in \text{Sing}(f)$, the dynamical pair (f, c) is ∞-generic. Note also that f is not isotrivial. By Theorem 4.5, we conclude that (f, c) is not adelic.

6. *Variation of canonical heights and equidistribution on $\mathbb{P}^1$*

Our aim in this section is to prove Theorem 1.6. Let $(f, c) \in k(t)(z) \times k(t)$ be a dynamical pair and chose F and $C = (A, B)$ lifts of f and c respectively. Enlarging the number field k if necessary, we may assume that F and C are defined over k and $\text{Sing}(f) \subset k$. We follow the notation from §4. So we have

$$G_{F,C,v}(t_1, t_2) = \lim_{n\to\infty} \frac{\log \|F_{C,n}(t_1, t_2)\|_v}{\deg F_{C,n}}.$$

We also write $M_{F,C,v} = \{(t_1, t_2) \in \mathbb{C}_v^2 : G_{F,C,v}(t_1, t_2) \le 0\}$.

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In the following proposition, we show that the height associated with a measure $\mu_{f,c}$ for a quasi-adelic pair (f, c) is proportional with the Call–Silverman canonical height; hence both heights have the same small points. The first author of this article is indebted to Laura DeMarco for many ideas in this proof.

PROPOSITION 6.1. *Let k be a number field and let $f \in k(t)(z)$ and $c \in k(t)$ be such that the dynamical pair (f, c) is quasi-adelic. For any $t \in \overline{k} \setminus \mathrm{Sing}(f)$, we have*

$$\hat{h}_{\mu_{f,c}}(t) = \frac{[k : \mathbb{Q}]}{\hat{h}_f(c)} \cdot \hat{h}_{f_t}(c(t)).$$

Proof. We write $d := \deg_z f$. Let $t \in \overline{k} \setminus \mathrm{Sing}(f)$ and write $S := \mathrm{Gal}(\overline{k}/k) \cdot t$. The definition of our height in (2.9) gives

$$\hat{h}_{\mu_{f,c}}(t) = \frac{1}{|S|} \cdot \sum_{x \in S} \sum_{v \in \mathcal{M}_k} \big(N_v \cdot G_{F,C,v}(x, 1) + \tfrac{1}{2} \log \mathrm{Cap}(M_{F,C,v})^{N_v}\big).$$

First we will see that

$$\frac{1}{|S|} \cdot \sum_{x \in S} \sum_{v \in \mathcal{M}_k} N_v \cdot G_{F,C,v}(x, 1) = \frac{[k : \mathbb{Q}]}{\hat{h}_f(c)} \cdot \hat{h}_{f_t}(c(t)). \tag{6.1}$$

To this end, notice that from the definition of the Call–Silverman canonical height, we have

$$\hat{h}_{f_t}(c(t)) = \frac{1}{[k(t) : \mathbb{Q}]} \lim_{n\to\infty} \sum_{x \in S} \sum_{v \in \mathcal{M}_k} N_v \frac{\log \|F_{C,n}(x, 1)\|_v}{d^n}. \tag{6.2}$$

Arguing as in Lemma 5.9, we see that for all but finitely many places $v \in \mathcal{M}_k$, we have $\|F_{C,n}(x, 1)\|_v = 1$ for all $n \in \mathbb{N}$. More specifically, for fixed $x \in S$, this conclusion holds for all places $v \in \mathcal{M}_k^0$ such that the coefficients of F and C are v-adic integers and $|x|_v = |\mathrm{Res}(A, B)|_v = |\mathrm{Res}_{(z,w)}(F_{t,1})|_v = |u_\beta(x, 1)|_v = 1$ for all $\beta \in \mathrm{Sing}(f)$. Therefore, we can interchange the limit with the summation in (6.2) to get

$$\begin{aligned}
\hat{h}_{f_t}(c(t)) &= \frac{1}{[k(t) : \mathbb{Q}]} \sum_{x \in S} \sum_{v \in \mathcal{M}_k} N_v \lim_{n\to\infty} \frac{\log \|F_{C,n}(x, 1)\|_v}{d^n} \\
&= \frac{\hat{h}_f(c)}{[k(t) : \mathbb{Q}]} \sum_{x \in S} \sum_{v \in \mathcal{M}_k} N_v \cdot G_{F,C,v}(x, 1) \\
&= \frac{\hat{h}_f(c)}{[k : \mathbb{Q}]} \cdot \frac{1}{|S|} \cdot \sum_{x \in S} \sum_{v \in \mathcal{M}_k} N_v \cdot G_{F,C,v}(x, 1).
\end{aligned}$$

Thus, we have established (6.1). We now have

$$\hat{h}_{\mu_{f,c}}(t) = \frac{[k : \mathbb{Q}]}{\hat{h}_f(c)} \cdot \hat{h}_{f_t}(c(t)) + \frac{1}{2} \log \prod_{v \in \mathcal{M}_k} \mathrm{Cap}(M_{F,C,v})^{N_v}. \tag{6.3}$$

Since $\mathrm{Cap}(M_{F,C,v})^{N_v} \geq 1$ for all but finitely many $v \in \mathcal{M}_k$, we see that $\prod_{v \in \mathcal{M}_k} \mathrm{Cap}(M_{F,C,v})^{N_v}$ converges strongly. It remains to show that the global logarithmic capacity

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is equal to zero, that is,

$$\log \prod_{v \in \mathcal{M}_k} \mathrm{Cap}(M_{F,C,v})^{N_v} = 0. \tag{6.4}$$

The authors in [22, 23] put a lot of effort into computing the explicit resultant formula for each $F_{C,n}$ to show that the global logarithmic capacity is zero. It is much harder to compute the resultants of $F_{C,n}$ here. Instead, we take a different approach, making use of Proposition 3.2. Laura DeMarco has independently communicated a similar idea with the first author of this article.

Toward the proof of (6.4), we first note that there are infinitely many $t \in \overline{k}$ such that $\hat{h}_{f_t}(c(t)) = 0$; see [17, Theorem 1.6]. Hence from (6.3), we get that for infinitely many $t \in \overline{k}$, we have

$$\hat{h}_{\mu_{f,c}}(t) = \frac{1}{2} \log \prod_{v \in \mathcal{M}_k} \mathrm{Cap}(M_{C,v})^{N_v}.$$

This in turn combined with Proposition 3.2 yields

$$\log \prod_{v \in \mathcal{M}_k} \mathrm{Cap}(M_{F,C,v})^{N_v} \geq 0.$$

We have reduced our claim to proving

$$\log \prod_{v \in \mathcal{M}_k} \mathrm{Cap}(M_{F,C,v})^{N_v} \leq 0. \tag{6.5}$$

From Proposition 4.2, we have

$$\mathrm{Cap}(M_{F,C,v}) \leq \liminf_{n \to \infty} |\operatorname{Res} F_{C,n}|_v^{-1/\deg(F_{C,n})^2}.$$

Note now that there exist a finite subset $S_0 \subset \mathcal{M}_k$ containing all archimedean places of k such that the coefficients of F_{t_1,t_2} and $C(t_1, t_2)$ are S_0-integers and the elements of $\mathrm{Sing}(f)$ are S_0-units. Then we have

$$N_v \cdot \frac{\log |\operatorname{Res}(F_{C,n})|_v}{\deg(F_{C,n})^2} \leq 0 \tag{6.6}$$

for all $n \in \mathbb{N}$ and $v \in \mathcal{M}_k \backslash S_0$. Moreover,

$$\begin{aligned}
&\log \prod_{v \in \mathcal{M}_k} \mathrm{Cap}(M_{F,C,v})^{N_v} \\
&\quad = \sum_{v \in \mathcal{M}_k \backslash S_0} \log \mathrm{Cap}(M_{F,C,v})^{N_v} + \sum_{v \in S_0} \log \mathrm{Cap}(M_{F,C,v})^{N_v} \\
&\quad \leq \sum_{v \in \mathcal{M}_k \backslash S_0} \log \mathrm{Cap}(M_{F,C,v})^{N_v} + \liminf_{n \to \infty} \sum_{v \in S_0} -N_v \cdot \frac{\log |\operatorname{Res}(F_{C,n})|_v}{\deg(F_{C,n})^2} \\
&\quad = \sum_{v \in \mathcal{M}_k \backslash S_0} \log \mathrm{Cap}(M_{F,C,v})^{N_v} + \liminf_{n \to \infty} \sum_{v \in \mathcal{M}_k \backslash S_0} N_v \cdot \frac{\log |\operatorname{Res}(F_{C,n})|_v}{\deg(F_{C,n})^2},
\end{aligned} \tag{6.7}$$

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where in the last equality, we used the product formula. Thus, for any finite subset $\mathcal{M} \subset \mathcal{M}_k \backslash S_0$, we have

$$\liminf_{n\to\infty} \sum_{v\in\mathcal{M}_k\backslash S_0} N_v \cdot \frac{\log|\operatorname{Res}(F_{C,n})|_v}{\deg(F_{C,n})^2} \leq \liminf_{n\to\infty} \sum_{v\in\mathcal{M}} N_v \cdot \frac{\log|\operatorname{Res}(F_{C,n})|_v}{\deg(F_{C,n})^2}$$
$$\leq -\sum_{v\in\mathcal{M}} \log \operatorname{Cap}(M_{F,C,v})^{N_v}. \tag{6.8}$$

Here, for the last inequality, we use the fact $1 \leq \operatorname{Cap}(M_{F,C,v}) \leq \liminf_{n\to\infty} |\operatorname{Res} F_{C,n}|_v^{-1/\deg(F_{C,n})^2}$. Combining (6.7) and (6.8), we get that for any finite set $\mathcal{M} \subset \mathcal{M}_k \backslash S_0$,

$$\log \prod_{v\in\mathcal{M}_k} \operatorname{Cap}(M_{F,C,v})^{N_v} \leq \sum_{v\in\mathcal{M}_k\backslash S_0} \log \operatorname{Cap}(M_{F,C,v})^{N_v} - \sum_{v\in\mathcal{M}} \log \operatorname{Cap}(M_{F,C,v})^{N_v}.$$

We may take an increasing sequence of finite sets $\mathcal{M}_n \subset \mathcal{M}_k \backslash S_0$ such that $\bigcup_{n\geq 1} \mathcal{M}_n = \mathcal{M}_k \backslash S_0$, and apply the previous formula for $\mathcal{M}_n$ in the place of $\mathcal{M}$. Since the global capacity converges strongly and letting n tend to ∞, (6.5) follows. This finishes the proof of this proposition. □

Proof of Theorem 1.6. We combine Propositions 2.3 and 6.1 to get

$$\frac{\hat{h}_f(c)}{[k:\mathbb{Q}]} \cdot \log \prod_{v\in\mathcal{M}_k} r_{\text{in}}(\mu_v)^{N_v} \leq \hat{h}_f(c) \cdot h(t) - \hat{h}_{f_t}(c(t)) \leq \frac{\hat{h}_f(c)}{[k:\mathbb{Q}]} \cdot \log \prod_{v\in\mathcal{M}_k} r_{\text{out}}(\mu_v)^{N_v}$$

for all $t \in \overline{k} \setminus \operatorname{Sing}(f)$. Since the set $\operatorname{Sing}(f)$ is finite, we have that as $t \in \overline{k}$ varies,

$$\hat{h}_{f_t}(c(t)) = \hat{h}_f(c)h(t) + O(1),$$

and the first part of our theorem follows. The equidistribution statement in part 2 now follows directly by combining Theorem 1.1 and Proposition 6.1. □

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References

[1] E. Artin. *Algebraic Numbers and Algebraic Functions*. AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the 1967 original.

[2] M. Baker. A finiteness theorem for canonical heights attached to rational maps over function fields. *J. Reine Angew. Math.* **626** (2009), 205–233.

[3] M. Baker and L. DeMarco. Preperiodic points and unlikely intersections. *Duke Math. J.* **159** (2011), 1–29.

[4] M. Baker and L. DeMarco. Special curves and postcritically-finite polynomials. *Forum Math. Pi* **1** (2013), 35 pp.

[5] M. Baker and L.-C. Hsia. Canonical heights, transfinite diameters, and polynomial dynamics *J. Reine Angew. Math.***585** (2005), 61–92.

[6] M. Baker and R. Rumely. Equidistribution of small points, rational dynamics, and potential theory. *Ann. Inst. Fourier (Grenoble)* **56** (2006), 625–688.

[7] M. Baker and R. Rumely. *Potential Theory and Dynamics on the Berkovich Projective Line (Mathematical Surveys and Monographs, 159)*. American Mathematical Society, Providence, RI, 2010.

[8] B. Branner and J. Hubbard. The iteration of cubic polynomials. I. The global topology of parameter space. *Acta Math.* **160** (1988), 143–206.

[9] X. Buff, A. Epstein and J. Ecalle. Limits of degenerate parabolic quadratic rational maps. *Geom. Funct. Anal.* **23**(2013), 42–95.

[10] G. S. Call and J. H. Silverman. Canonical heights on varieties with morphisms. *Compos. Math.* **89** (1993), 163–205.

[11] A. Carney, W. Hindes and T.-J. Tucker. Integral points in orbits in characteristic p. *Preprint*, 2021, arXiv:2108.03123.

[12] A. Chambert-Loir. Mesures etéquidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.* **595** (2006), 215–235.

[13] L. DeMarco. Dynamics of rational maps: a current on the bifurcation locus. *Math. Res. Lett.* **8** (2001), 57–66.

[14] L. DeMarco. Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity. *Math. Ann.* **326** (2003), 43–73.

[15] L. DeMarco. Iteration at the boundary of the space of rational maps. *Duke Math. J.* **130** (2005), 169–197.

[16] L. DeMarco. The moduli space of quadratic rational maps. *J. Amer. Math. Soc.* **20** (2007), 321–355.

[17] L. DeMarco. Bifurcations, intersections, and heights. *Algebra Number Theory* **10** (2016), 1031–1056.

[18] L. DeMarco, H. Krieger and H. Ye. Uniform Manin–Mumford for a family of genus 2 curves. *Ann. of Math. (2)* **191**(3) (2020), 949–1001.

[19] L. DeMarco, H. Krieger and H. Ye. A uniform bound on the number of common preperiodic points of quadratic polynomials. *Preprint*, 2021, arXiv:1911.02458.

[20] L. DeMarco and N. M. Mavraki. Variation of canonical height and equidistribution. *Amer. J. Math.* **142** (2020), 443–473.

[21] L. DeMarco and N. M. Mavraki. Variation of Canonical height for Fatou points. *Preprint*, 2021, arXiv:2107.05982.

[22] L. DeMarco, X. Wang and H. Ye. Bifurcation measures and quadratic rational maps. *Proc. Lond. Math. Soc. (3)* **111**(1) (2015), 149–180.

[23] L. DeMarco, X. Wang and H. Ye. Torsion points and the Lattès family. *Amer. J. Math.* **138**(3) (2016), 697–732.

[24] R. Dujardin and C. Favre. Distribution of rational maps with a preperiodic critical point. *Amer. J. Math.* **130** (2008), 979–1032.

[25] A. L. Epstein. Bounded hyperbolic components of quadratic rational maps. *Ergod. Th. & Dynam. Sys.* **20**(3) (2000), 727–748.

[26] C. Favre and T. Gauthier. Distribution of postcritically finite polynomials. *Israel J. Math.* **209** (2015), 235–292.

[27] C. Favre and T. Gauthier. Classification of special curves in the space of cubic polynomials. *Int. Math. Res. Not. IMRN* **2018**(2) (2018), 362–411.

[28] C. Favre and T. Gauthier. *The Arithmetic of Polynomial Dynamical Pairs (Annals of Mathematics Studies, 214)*. Princeton University Press, Princeton, NJ, 2022.

[29] C. Favre and J. Rivera-Letelier. Équidistribution quantitative des points de petite hauteur sur la droite projective. *Math. Ann.* **335** (2006), 311–361.

[30] J. E. Fornaess and N. Sibony. Complex dynamics in higher dimensions. *Complex Potential Theory (Montreal, PQ, 1993) (NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 439)*. Kluwer Academic Publishers, Dordrecht, 1994, pp. 131–186.

[31] D. Ghioca and L.-C. Hsia. Torsion points in families of Drinfeld modules. *Acta Arith.* **161**(3) (2013), 219–240.

[32] D. Ghioca, L.-C. Hsia and T. Tucker. Preperiodic points for families of polynomials. *Algebra Number Theory* **7**(3) (2013), 701–732.

https://doi.org/10.1017/etds.2022.49 Published online by Cambridge University Press

[33] D. Ghioca, L.-C. Hsia and T. Tucker. Preperiodic points for families of rational maps. *Proc. Lond. Math. Soc. (3)* **110** (2015), 395–427.
[34] D. Ghioca, H. Krieger and K. Nguyen. A case of the dynamical André–Oort conjecture. *Int. Math. Res. Not. IMRN* **2016**(3) (2016), 738–758.
[35] D. Ghioca, H. Krieger, K. Nguyen and H. Ye. The dynamical André–Oort conjecture: unicritical polynomials. *Duke Math. J.* **166**(1) (2017), 1–25.
[36] D. Ghioca and N. M. Mavraki. Variation of the canonical height in a family of rational maps. *New York J. Math.* **19** (2013), 873–907.
[37] D. Ghioca and H. Ye. A dynamical variant of the André–Oort conjecture. *Int. Math. Res. Not. IMRN* **2018**(8) (2018), 2447–2480.
[38] H.-L. Huang, C. L. Sun and J. Wang. Integral orbits over function fields. *Int. J. Number Theory* **10**(8) (2014), 2187–2204.
[39] P. Ingram. Variation of the canonical height for a family of polynomials. *J. Reine Angew. Math.* **685** (2013), 73–97.
[40] J. Kiwi. Rescaling limits of complex rational maps. *Duke Math. J.* **164** (2015), 1437–1470.
[41] R. Mañé, P. Sad and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. Éc. Norm. Supér. (4)* **16** (1983), 193–217.
[42] D. Masser and U. Zannier. Torsion anomalous points and families of elliptic curves. *Amer. J. Math.* **132** (2010), 1677–1691.
[43] D. Masser and U. Zannier. Torsion points on families of squares of elliptic curves. *Math. Ann.* **352** (2012), 453–484.
[44] D. Masser and U. Zannier. Torsion points on families of products of elliptic curves. *Adv. Math.* **259** (2014), 116–133.
[45] C. McMullen. *Complex Dynamics and Renormalization*. Princeton University Press, Princeton, NJ, 1994.
[46] J. Milnor. Geometry and dynamics of quadratic rational maps. *Exp. Math.* **2** (1993), 37–83. With an appendix by the author and Lei Tan.
[47] T. Ransford. *Potential Theory in the Complex Plane (London Mathematical Society Student Texts, 28)*. Cambridge University Press, Cambridge, 1995.
[48] J. Silverman. Heights and the specialization map for families of abelian varieties. *J. Reine Angew. Math.* **342** (1983), 197–211.
[49] J. Silverman. Variation of the canonical height on elliptic surfaces. I. Three examples. *J. Reine Angew. Math.* **426** (1992), 151–178.
[50] J. Silverman. Integer points, Diophantine approximation, and iteration of rational maps. *Duke Math. J.* **71**(3) (1993), 793–829.
[51] J. Silverman. Variation of the canonical height on elliptic surfaces. II. Local analyticity properties. *J. Number Theory* **48** (1994), 291–329.
[52] J. Silverman. Variation of the canonical height on elliptic surfaces. III. Global boundedness properties. *J. Number Theory* **48** (1994), 330–352.
[53] J. Tate. Variation of the canonical height of a point depending on a parameter. *Amer. J. Math.* **105**(1) (1983), pp. 287–294.
[54] A. Thuillier. Théorie du potentiel sur les courbes en géométrie analytique non archimedienne. Applications à la théorie d'Arakelov. *Thèse*, Université de Rennes 1, 2005.
[55] H. Ye. Quasi-adelic measures and equidistribution on $\mathbb{P}^1$. *Preprint*, 2017, arXiv:1502.04660v1.
[56] X. Yuan. Big line bundles over arithmetic varieties. *Invent. Math.* **173**(3) (2008), 603–649.
[57] U. Zannier. *Some Problems of Unlikely Intersections in Arithmetic and Geometry (Annals of Mathematics Studies, 181)*. Princeton University Press, Princeton, NJ, 2012. With appendixes by D. Masser.
[58] S. Zhang. Positive line bundles on arithmetic varieties. *J. Amer. Math. Soc.* **8** (1995), 187–221.
[59] S. Zhang. Small points and adelic metrics. *J. Algebraic Geom.* **4** (1995), 281–300.