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The practical use of the Rayleigh-Ritz method in compressible flow

P. E. Lush and J. W. Stephenson

We show

(i) that the Rayleigh-Ritz method is a practical procedure for obtaining approximations to the velocity potential for compressible flows, and

(ii) how to calculate an estimate of the error in such approximations.

1. Introduction

This paper is concerned with the practical use of the Rayleigh-Ritz method. The particular application discussed is that of compressible flow, but the procedures can be applied in any context where the Rayleigh-Ritz method is appropriate.

In the hydrodynamical context [3], the Rayleigh-Ritz method gives approximations to the velocity potential ϕ by using the property that, under prescribed boundary conditions upon $\partial\phi/\partial n$, ϕ makes stationary an integral of the form

$$\int_R \int f(\nabla\phi)dxdy \quad \left(= I^*[\phi], \text{ say}\right) .$$

The stationary value of $I^*[\phi]$ is a maximum in the case of a subsonic flow, that is $|\nabla\phi| \leq C < c^*$, where c^* is the sonic speed.

The approximations sought are of the form

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$$\phi_n = \phi_0 + \sum_1^n A_i h_i \ ,$$

where $h_1, h_2, \ldots$ are functions (of x, y) suitably chosen, and $A_1, A_2, \ldots, A_n$ are constants to be determined so that $I^*[\phi_n]$ is maximised. Thus, if $I^*[\phi_n]$ is evaluated as an explicit function of $A_1, A_2, \ldots$, the numbers A_i are obtained by solving the (algebraic) equations $\partial I^*/\partial A_i = 0$, $i = 1, \ldots, n$.

The function f contains a term T of the form $\left[1-K(\nabla\phi)^2\right]^\alpha$, [3], where K and α $\left(= \gamma/(\gamma-1) = 3.46 \ldots \text{ for air}\right)$ are given constants. The use of the value $3.46 \ldots$ for α entails approximating to T $\left(= (1-\tau)^\alpha \text{ , say}\right)$ by a polynomial in τ , in order that I^* may be explicitly integrated. Even for the simplest problems (for example, flow past a circular cylinder) this gives rise to an enormous amount of algebra. For more complicated cylinders, or for more complicated problems such as flow through a nozzle, the algebra required is prohibitive. In this paper, we show how to circumvent this difficulty so that we can quickly find Rayleigh-Ritz approximations (to the velocity potential ϕ).

In the case of subsonic flow, an upper bound to $I^*[\phi]$ can be obtained by minimizing an integral $J^*[\psi]$ of the stream function ψ , [2], - the purpose of such an upper bound is that it can be used to estimate the error in the Rayleigh-Ritz approximations to ϕ , [2], [3]. It is not practical to find this upper bound numerically by explicitly minimizing $J^*[\psi]$; however, we show how to find a fairly close estimate of this upper bound.

In [1], Greenspan and Jain have attempted to find ϕ by using the integral $I^*[\phi]$ - they do not exploit fully the stationary character of $I^*[\phi]$ in the usual Rayleigh-Ritz manner, but use instead the ideas of finite differences.

2. The variational principles for the aerofoil problem

It is shown in [3] that the aerofoil problem, that is the 'local'

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deflection of an otherwise uniform stream by a cylinder, can be represented as a variational problem. For a non-circulatory, subsonic flow, the velocity potential ϕ maximizes

$$I^*[\phi] = \int_R \int \left\{p(\phi)-p_\infty+\rho_\infty \nabla\phi_0 . \nabla\left(\phi-\phi_\infty\right)\right\} dxdy \ ,$$

where the pressure is expressed as a function of ϕ by use of Bernoulli's equation. Here R is the (infinite) region occupied by the fluid, p_∞, ρ_∞ are the pressure and density in the free stream, and ϕ_0 is the velocity potential for the corresponding incompressible flow. At infinity

$$\phi \sim Ux + O\left(r^{-1}\right) \ ,$$

where $r = \sqrt{(x^2+y^2)}$ and U is the free stream speed, and, on the cylinder, $\partial\phi/\partial n = 0$. For the adiabatic case, that is $p = A\rho^\gamma$, Bernoulli's equation gives

$$p(\phi) = p_0\left\{1-(\nabla\phi)^2/\left(2\beta c_0^2\right)\right\}^{\beta\gamma} \ , \quad c^2 = dp/d\rho \ ,$$

$$\beta = 1/(\gamma-1) \ , \quad \rho_\infty = \rho_0\left\{1-U^2/\left(2\beta c_0^2\right)\right\}^\beta \ ,$$

where the zero subscript refers to stagnation values.

Define

$$J^*[\psi] = \int_R \int \left\{p(\psi)-p_\infty+\rho u\left(u-u_\infty\right)+\rho v\left(v-v_\infty\right)\right\} dxdy$$

where, to any ψ of class C_2 , ρ is defined by Bernoulli's equation with $p = p(\rho)$, and thence u, v by

$$\rho u = \frac{\partial\psi}{\partial y} \ , \quad \rho v = -\frac{\partial\psi}{\partial x} \ .$$

The 'boundary' conditions on ψ are that, at infinity,

$$\psi = \rho_\infty U y + O\left(r^{-1}\right) \ ,$$

and $\psi = 0$ on the cylinder. It is shown in [2] that the minimum value of $J^*[\psi]$ (with respect to ψ) is equal to the maximum value of $I^*[\phi]$ (with respect to ϕ).

Finally, the purpose of the terms p_∞ and ϕ_∞ in $I^*[\phi]$ is to ensure that the infinite integral converges - the variational properties of $I^*[\phi]$ would not be affected by replacing p_∞ and ϕ_∞ by $p(\phi_0)$ and ϕ_0. As this proves to be convenient in the subsequent numerical work we define

$$I[\phi] = \int_R\int \left\{p(\phi)-p(\phi_0)+\rho_\infty\nabla\phi_0.\nabla(\phi-\phi_0)\right\}dxdy ,$$

and likewise

$$J[\psi] = \int_R\int \left\{p(\psi)-p(\rho_\infty\psi_0)+\rho u(u-u_0)+\rho v(v-v_0)\right\}dxdy ,$$

where the zero suffix refers to the corresponding incompressible flow, ψ_0 being conjugate to ϕ_0.

3. Outline of the method

It is shown in [3] that the integrand of $I[\phi]$ is $O(r^{-4})$ as $r \to \infty$, so that, for a given set of values of $A_1, A_2, \ldots, A_n$, I can be evaluated numerically to any (practical) accuracy. Knowing I numerically as a function of $A_1, A_2, \ldots, A_n$, we maximize I by finding its largest value in the region of $A_1, A_2, \ldots, A_n$ space that corresponds to subsonic "velocities".

To do this, let P_0 be a given point in $A_1, A_2, \ldots, A_n$ space, and let Q_0 be the quadric obtained from the Taylor expansion of I about the point $\left(P_0, I(P_0)\right)$. If we call P_1 the maximum of this quadric (found by solving $\partial Q_0/\partial A_1 = 0, \ldots$), we fit another quadric Q_1 to I at $(P_1, I(P_1))$ and find its maximum, P_2 say. In this manner, a sequence of points $P_0, P_1, P_2, \ldots, P_m$ is constructed which, we expect, will converge to the maximum of I.

To find the maximum of the quadric Q_m requires solving (for X) a set of simultaneous equations of the form

$$B_m(\mathrm{X}-\mathrm{X}_m) = -\mathrm{b}_m \, .$$

Here the vector X_m is the (known) coordinate of P_m , X is the coordinate of the maximum of Q_m , that is X is the coordinate of P_{m+1} . The vector b_m $(\partial I/\partial A_i)$ is the gradient of the surface I at P_m , and B_m is the matrix of the second derivatives of I , that is $\partial^2 I/\partial A_i \partial A_j$ evaluated at P_m . Since the second variation of I is positive definite for $|\nabla\phi| \le C < c^*$ where c^* is the sonic speed, [3], it follows in an analogous manner to the argument in [2] that $I(P_{m+1}) > I(P_m)$, that is we have a monotonic increasing sequence bounded above (by $J[\psi]$). Further, since B_m and its minors are bounded from zero, it follows that if $\mathrm{b}_m \to 0$, $\mathrm{X}_{m+1} - \mathrm{X}_m \to 0$ and X_m converges to the point at which $I[\phi]$ has its maximum value.

For a given set of values of $A_1, A_2, \ldots$ the values of I , $\partial I/\partial A_i$, $\partial^2 I/\partial A_i \partial A_j$ were obtained by numerical integration. These integrals need to be computed to a fairly high degree of accuracy, and since the precise criteria for convergence of the sequence $P_0, P_1, \ldots$, is somewhat vague, we need to be able to adjust quickly our integration process to cope with any accuracy of integration that may be required. In these circumstances, it is appropriate to treat the double integrals as repeated integrals - each integration being effected by Gaussian quadrature. [4, p. 38.]

4. Details of the calculation of ϕ

For illustrative purposes, consider the case of non-circulatory flow past a circular cylinder which, without loss of generality, we may take to be $r = 1$. For this case, [3],

$$\phi_0 = V\surd(2\beta)c_0\left(r + \frac{1}{r}\right)\cos\theta$$

$$\phi = \phi_0 + V\surd(2\beta)c_0\left\{f_1(\theta)\left(\frac{1}{r} - \frac{1}{3r^3}\right) + f_2(\theta)\left(\frac{1}{3r^3} - \frac{1}{5r^5}\right)\right\} ,$$

with

$$f_1(\theta) = A_1\cos\theta + A_2\cos 3\theta + A_5\cos 5\theta ,$$

$$f_2(\theta) = A_3\cos\theta + A_4\cos 3\theta + A_6\cos 5\theta .$$

Here $r = \surd(x^2+y^2)$, θ is the polar angle and $V\surd(2\beta)c_0$ is the (prescribed) free stream velocity. To compare our results with those in [3], we choose V so that the free stream Mach number is 0.4 , that is

$$V = \surd\{0.16/(2\beta+0.16)\} ,$$

and we compute ϕ_2, ϕ_4, ϕ_6 . Putting $u = r^{-1}$ and $\phi = V\surd(2\beta)c_0\Phi$, $\left(\phi_0 = V\surd(2\beta)c_0\Phi_0\right)$, I is computed in the form

$$4p_0 \int_0^1 \int_0^{\pi/2} \left[\left\{1-V^2(\nabla\Phi)^2\right\}^{\beta\gamma} - \left\{1-V^2\left(\nabla\Phi_0\right)^2\right\}^{\beta\gamma}\right.$$
$$\left. + 2\beta\gamma V^2\left(1-V^2\right)^{\beta}\nabla\Phi_0 . \nabla\left(\Phi-\Phi_0\right)\right]u^{-3}du d\theta .$$

If we put $\Phi = \Phi_0 + \sum_1^n A_i H_i$,

$$\frac{\partial I}{\partial A_i} = 8\beta\gamma p_0 \int_0^1 \int_0^{\pi/2} \left[\left\{1-V^2(\nabla\Phi)^2\right\}^{\beta} V^2\left(-\nabla\Phi . \nabla H_i\right) + V^2\left(1-V^2\right)^{\beta}\nabla\Phi_0 . \nabla H_i\right]u^{-3}du d\theta$$

and $\partial^2 I/\partial A_i \partial A_j$ then follows readily.

Our aim was to determine $A_1, \ldots, A_6$ correct to 4 decimal places $\left(|A_i| < 1\right)$. To do this, the method of integration used needed to be sufficiently accurate to detect, with significance, changes in I due to changes in the 5th decimal place of $A_1, \ldots, A_6$. Moreover, the equations $\partial I/\partial A_i = 0$ are ill-conditioned, [3], so that, in certain directions, the surface $I(A_1, \ldots, A_6)$ is very "flat" near its maximum. To be conservative, we estimated that changes in the 5th decimal place of the A_i would lead to changes in the 12th decimal place of $I[\phi]$. For changes in the value of $I[\phi]$ to give reliable information about the surface, $I[\phi]$ should be computed correct to about 17 decimal places.

To be consistent, the quadrature formula used should be accurate to about 10^{-17}. The error in evaluating $\int_{-1}^{1} f(x)dx$ by an n-point Gaussian formula is

$$E(n) = \frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi) , \quad -1 < \xi < 1 ,$$

where $f^{(2n)}(x)$ is the $2n$-th derivative of $f(x)$. For $f(x) \doteq (1-x)^{3.46\ldots}$ (which resembles the term $p(\phi)$ in $I[\phi]$), $E(20) = 0.8 \times 10^{-18}$ and so a 20-point Gaussian formula was used for each integration. By exploiting the symmetries of the flow, the number of points at which the integrands are evaluated is drastically reduced.

As a check upon this rough estimate of the accuracy of the numerical integration, we computed the case of $I[\phi_1]$ with $\gamma = 2$ and $A_1 = 0.23$. In this case, $p(\phi)$ can be expanded exactly in terms of $\nabla\phi$ and the integral $I[\phi]$ can be explicitly evaluated - its value was found correct to 26 decimal places. The same integral was evaluated by numerical integration - the discrepancy between the two determinations of $I[\phi_1]$ was 3×10^{-17}.

The computer that was used - an IBM 1620 - terminated each machine operation by truncation rather than by rounding off. The accumulation of error due to this cause amounted to about 5 figures, so we worked (in floating point form) with 22 figures. (All numbers used in computing I, and I itself, are less than 1.)

Since the total time on the (very slow) computer was quite small, very little attempt was made to work justifiably with either a lesser number of figures, or a lesser number of points at which the integrands were evaluated.

5. Results

The values of $A_1, \ldots, A_6$, $I[\phi_n]$ for the cases ϕ_2, ϕ_4, ϕ_6 are shown in the table - the numbers shown in brackets are the corresponding values obtained in [3]. In each case, the initial point of the sequence $P_0, P_1, \ldots$ was the origin, and the values of A_i shown in the table correspond to P_5.

	A_1	A_2	A_3	A_4	A_5	A_6	$I[\phi_n]/P_0$
ϕ_2	0.2188	-0.1216					0.00519246
ϕ_4	0.2411	-0.1142	-0.0548	-0.0454			0.00816398
ϕ_6	0.2421	-0.1128	-0.0551	-0.0599	0.0023	0.0327	0.00824812
	(0.2428)	(-0.1127)	(-0.0553)	(-0.0594)	(0.0022)	(0.0325)	

The criteria for convergence of the sequence $P_0, P_1, P_2, \ldots, P_m$ was that

(i) the distance $P_{m-1}P_m$ should be sufficiently small;

(ii) the slope of the surface of $I[\phi]$ at P_m should be sufficiently small; and

(iii) the value of $I(P_m)$ should be greater than any other value of I computed in the process.

For the case ϕ_6, the distances P_2P_3, P_3P_4, P_4P_5 are 1×10^{-3}, 1×10^{-7}, 2×10^{-15}, and the values of the slope of I at the points P_3, P_4, P_5 are 2×10^{-4}, 2×10^{-8}, 5×10^{-16}; thus the sequence $P_0, P_1, \ldots$ converges rapidly to the maximum of I.

The values of A_i in the table are 'better' than those obtained in [3], since the value $I[\phi_6]$ shown in the table is significantly greater than the value of $I[\phi_6]$ derived from the values of A_i in [3], namely, 0.00824800 .

6. Calculation of the upper bound $J[\psi_n]$

To any given approximation ϕ_n, we can derive an 'exact' density

$\rho(\phi_n) = \rho_n$, say, by using Bernoulli's equation. The $\nabla\phi_n$ and ρ_n so determined do not describe exactly a fluid motion since they do not satisfy the continuity equation, that is there is no function ψ_n for which

$$\rho_n \frac{\partial \phi_n}{\partial x} = \frac{\partial \psi_n}{\partial y} , \quad \rho_n \frac{\partial \phi_n}{\partial y} = - \frac{\partial \psi_n}{\partial x} .$$

However, we can find a ψ_n that corresponds in some sense to a ϕ_n by taking a form

$$\rho_\infty \psi_0 + B_1 g_1 + B_2 g_2 + \ldots + B_n g_n ,$$

where ψ_0 is the stream function for the incompressible flow, $g_1, g_2, \ldots$ are suitably chosen functions of (x, y) and $B_1, B_2, \ldots$ are constants to be determined by minimizing

$$(1) \qquad \int_R \int \left\{ \left(\frac{\partial \psi_n}{\partial y} - \rho_n \frac{\partial \phi_n}{\partial x} \right)^2 + \left(\frac{\partial \psi_n}{\partial x} + \rho_n \frac{\partial \phi_n}{\partial y} \right)^2 \right\} dx dy .$$

We would expect that such a ψ_n would give a $J^*[\psi_n]$ that is close to the minimum value of $J^*[B_1, \ldots, B_n]$ since $J^* = J^*[(\nabla\psi)^2]$, and so, by the property referred to above, $J^*[\psi_n]$ should be a reasonable good upper bound for the sequence $I^*[\phi_1], I^*[\phi_2], \ldots$.

From ϕ_6 , the following approximation to the stream function was found by minimizing (1):

$$\begin{aligned}
(2) \quad \psi_6 = \rho_\infty V\sqrt{(2\beta)} c_0 \Bigg\{ & \left(r - \frac{1}{r} \right) \sin\theta \\
& + \left(\frac{1}{r} - \frac{1}{r^3} \right) (-0.1422\sin\theta + 0.0505\sin3\theta - 0.0023\sin5\theta) \\
& + \left(\frac{1}{r^3} - \frac{1}{r^5} \right) (0.0013\sin\theta + 0.0064\sin3\theta - 0.0040\sin5\theta) \Bigg\} \\
= \rho_\infty V\sqrt{(2\beta)} c_0 \Psi & , \text{ say.}
\end{aligned}$$

To determine $\rho(\psi_6)$, put $p = p_0(\rho/\rho_0)^\gamma$ in Bernoulli's equation, and set

$\rho_\infty = (1 - V^2)^\beta \rho_0$ to give

$$(3) \qquad V^2(1-V^2)^{2\beta}(\nabla\Psi)^2 + \left(\frac{\rho}{\rho_0}\right)^{\gamma+1} = \left(\frac{\rho}{\rho_0}\right)^2 , \quad 0 \le \rho/\rho_0 \le 1 .$$

Now $X^2 - X^{\gamma+1}$ has a maximum M, say, in $0 \le X \le 1$, that occurs at the sonic speed. For (3), $V^2(1-V^2)^{2\beta}(\nabla\Psi)^2$ is greater than M at six of the points used in the numerical integration that are close to the flanks of the cylinder, so that (3) does not define $\rho(\psi_6)$ at these points. To overcome this, we assigned a value to $|\nabla\psi_6|$ at $r = 1$, $\theta = \pi/2$ that corresponded to the sonic speed, since this is the value of $|\nabla\phi_6|$ at this point. The values used for $|\nabla\psi_6|$ at the 6 points were those obtained by (non-linear) interpolation between the point $r = 1$, $\theta = \pi/2$ and the nearest points for which $V^2(1-V^2)^{2\beta}(\nabla\Psi)^2 < M$. If we were to attempt to determine ψ_n by minimizing $J[\psi]$, the natural starting point, namely $B_1 = B_2 = \ldots = 0$, corresponds to a value of $V^2(1-V^2)^{2\beta}(\nabla\Psi)^2$ that is considerably greater than M, so that we are not able to find the upper bound of $I[\phi]$ by the obvious method.

Using the modified ψ_6, we obtained

$$J[\psi_6] = .00825790p_0$$

so that

$$|I_{max}-I[\phi_6]| < 0.00001p_0 .$$

References

[1] Donald Greenspan and Padam Jain, "Application of a method for approximating extremals of functionals to compressible subsonic fluid flows", *J. Math. Anal. Appl.* 18 (1967), 85-111.

[2] P.E. Lush, "The convergence of Rayleigh-Ritz approximations in hydrodynamics", *J. Austral. Math. Soc.* 3 (1963), 99-103.

[3] P.E. Lush and T.M. Cherry, "The variational method in hydrodynamics", *Quart. J. Mech. Appl. Math.* 9 (1956), 6-21.

[4] A.H. Stroud; Don Secrest, *Gaussian quadrature formulae* (Prentice-Hall, Englewood Cliffs, New Jersey, 1966).

University of New England,
Armidale,
New South Wales.