This result is very like that given by M. Bonnet.

Again, we have

$$Q = \sum_{r=0}^{r=m} A_0B_0 = \sum_{r=1}^{r=m}(A_{r-1} - A_r)P_{r-1} + A_mP_m$$

and, as before, $\sum_{r=1}^{r=m}(A_{r-1} - A_r)P_{r-1}$ lies between $M(A_0 - A_m)$ and $N(A_0 - A_m)$

$$\therefore\ Q = [N + \theta(M - N)](A_0 - A_m) + A_mP_m$$

$$\therefore \int_{x_0}^{x_m} f(x)\phi(x)dx = [N + \theta(M - N)](f(x_0) - f(x_m)) + f(x_m)\int_{x_0}^{x_m}\phi(x)dx$$

$$= (f(x_0) - f(x_m))\int_{x_0}^{\xi}\phi(x)dx + f(x_m)\int_{x_0}^{x_m}\phi(x)dx$$

$$= f(x_0)\int_{x_0}^{\xi}\phi(x)dx + f(x_m)\int_{\xi}^{x_m}\phi(x)dx$$

which is the ordinary form of the "Second Theorem of the Mean."

Lastly, we may note that the theorem of "Integration by parts" is virtually given in Abel's theorem, for we have

$$Q = A_0P_0 + \sum_{r=1}^{r=m} A_r(P_r - P_{r-1}) = \sum_{r=1}^{r=m}(A_{r-1} - A_r)P_r + A_mP_m$$

Let $A_r = f(x_r)$ and $P_r = \phi(x_r)$

$$\therefore \qquad A_{r-1} - A_r = -\frac{df(x_r)}{dx}h_r, \quad P_r - P_{r-1} = \frac{d\phi(x_r)}{dx_r}h_r$$

$$\therefore \quad f(x_0)\phi(x_0) + \int_{x_0}^{x_m} f(x)\phi'(x)dx = f(x_m)\phi(x_m) - \int_{x_0}^{x_m} f'(x)\phi(x)dx$$

$$i.e., \int_{x_0}^{x_m} f(x)\phi'(x)dx = f(x_m)\phi(x_m) - f(x_0)\phi(x_0) - \int_{x_0}^{x_m} f'(x)\phi(x)dx.$$

On the inscription of a triangle of given shape in a given triangle.

By R. E. Allardice, M.A.

§ 1. *To inscribe in a triangle ABC a triangle similar to the triangle DEF, and having its sides parallel to those of DEF.*

In order to inscribe in the triangle ABC (fig. 21), a triangle having its sides parallel to those of DEF, through D, E, F, draw lines parallel to the sides of ABC, and then reduce the figure A′B′C′ in the ratio BC : B′C′.

Thus an infinite number of triangles may be inscribed in a given triangle, similar to another given triangle.

A direct construction may also be given, as follows:

In BC (fig. 22) take any point G: draw GH parallel to DE; HK parallel to EF; KL parallel to FD. Then we may easily calculate the ratio A′L : A′G where A′ is the vertex of the required triangle that lies in BC.

For,
$$\frac{A'L}{A'G} = \frac{A'L}{C'K} \cdot \frac{C'K}{B'H} \cdot \frac{B'H}{A'G}$$
$$= \frac{LB}{BK} \cdot \frac{KA}{AH} \cdot \frac{HC}{CG} \text{ which is known.}$$

[Dr Mackay suggests a modification of this method, depending on the fact that A, A′ and the point of intersection of KL and HG are collinear.]

The theorems of §§ 2, 3, and 4 are required further on in this paper.

§ 2. *To find the condition that the perpendiculars to the sides of a triangle ABC, drawn at the points D, E, F, in the sides, be concurrent.*

Let the perpendiculars at D, E, F, (fig. 23) meet in the point O.

Since $AO^2 - BO^2 = AF^2 - BF^2$, the necessary and sufficient condition for concurrence is obviously

$$AF^2 - BF^2 + BD^2 - CD^2 + CE^2 - AE^2 = 0$$

or,
$$AF^2 + BD^2 + CE^2 = BF^2 + CD^2 + AE^2.$$

§ 3. *If one triangle is inscribed in another triangle, and if the perpendiculars from the vertices of one triangle on the sides of the other triangle are concurrent, then the perpendiculars from the vertices of the second triangle on the sides of the first are also concurrent.*

Let DEF (fig. 24) be inscribed in ABC; and let the perpendiculars at D, E, F, to the sides BC, CA, AB, meet in the point O; then the perpendiculars from A, B, C, on the sides of EF, FD, DE, will also meet.

Let the perpendiculars from A, B, C, meet the sides of DEF in A′, B′, C′.

Since the perpendiculars at D, E, F, are concurrent,

$$\therefore \qquad AF^2 - BF^2 + BD^2 - CD^2 + CE^2 - AE^2 = 0.$$

But $\qquad AF^2 - AE^2 = A'F^2 - A'E^2$, etc.

$$\therefore \qquad A'F^2 - B'F^2 + B'D^2 - C'D^2 + C'E^2 - A'E^2 = 0\,;$$

which proves the theorem.

§ 4. *The lines joining the points O and P of last paragraph to the vertices are equally inclined to the bisectors of the angles of the triangle.*

Let O and P (fig. 24) be the two points.

Then, $\angle FAO = \angle FEO =$ complement of $\angle FEA = \angle EAP$.

In the recent geometry of the triangle, lines equally inclined to the bisector of an angle are called *isogonal* lines; and the points of concurrence of one set of three lines passing through the vertices and of the three isogonal lines are called *inverse* points. Thus OA and PA, OB and PB, OC and PC, are pairs of isogonal lines; and O and P are inverse points.

§ 5. The problem that first suggested itself, and that led to this paper, was as follows:

To find a point P within a triangle such that the images Q, R, S, of P in the three sides shall be the vertices of an equilateral triangle.

This is obviously the same problem as the following:

To inscribe, in a given triangle, an equilateral triangle, such that the perpendiculars to the sides of the given triangle, drawn through the vertices of the equilateral triangle, shall be concurrent.

First Method.

This problem may be solved by finding the point inverse (in the sense mentioned above) to the point of intersection of the perpendiculars through the vertices of the required equilateral triangle.

The corresponding point is, in fact, the point of concurrence of the circles circumscribing the equilateral triangles, described on the sides of the given triangle.

Suppose DEF (fig. 24) to be equilateral; then

$$\angle PAC = \angle OAF = \angle OEF,$$
$$\angle PCA = \angle OCB = \angle OED\,;$$
$$\therefore \quad \angle PAC + \angle PCA = \angle DEF = 60^\circ, \quad \therefore \quad \angle APC = 120^\circ.$$

Thus the problem is solved.

§ 6. Eight systems of three circles may be obtained as the circles circumscribing equilateral triangles described on the sides of a given triangle. The question naturally arises, In how many of these systems do the circles concur in one real point? We shall show that this happens only in the case of two of the systems.

First Case. Consider a triangle ABC with the angle C greater than 120°.

There is obviously a point of concurrence on each of the arcs of 120° described on AB, and none on either of the arcs of 60° described on AB.

Second Case. Consider a triangle ABC, with no angle greater than 120°.

There is always one point of concurrence of circles within the triangle, and one without the triangle. If the triangle has only one angle greater than 60°, the point of concurrence that lies outside the triangle is on the arc of 120° described on the greatest side of the triangle; while if the triangle has two angles greater than 60°, this point of concurrence is on the arc of 120° described on the shortest side of the triangle.

We may also show analytically, by getting the equations to the circles, that in only two of the systems do the three circles meet in one point.

The equation to the circle circumscribing the equilateral triangle described *externally* on the side BC is

$$(a\beta\gamma + b\gamma a + c a\beta)\sin 60^\circ - \gamma(a\alpha + b\beta + c\gamma)\sin(60^\circ + \mathrm{A}) = 0\,; \qquad (1)$$

while the equation to the circle circumscribing the equilateral triangle described *internally* on the same side (that is, so that the remaining vertex of the equilateral triangle and the vertex A are on the same side of BC), is

$$(a\beta\gamma + b\gamma a + c a\beta)\sin 60^\circ - \gamma(a\alpha + b\beta + c\gamma)\sin(60^\circ - \mathrm{A}) = 0. \qquad (2)$$

Now, it may be shown that the circle (1) and the other two corresponding circles meet in a point, and that the same is true of the circle (2) and the two other circles corresponding to it; and that these are the only two systems containing three concurrent circles.

§ 7. Another construction may be given for the points P of § 6, that is, for the points of concurrence of the circles circumscribing the equilateral triangle described on the sides of the given triangle.

It may, in fact, easily be shown that if equilateral triangles BCD, CAE, ABF, (fig. 25) are described on the sides of the triangle ABC,

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the vertices A and D being on opposite sides of BC, B and E on opposite sides of CA and C and F on opposite sides of AB, then the lines AD, BE, CF are concurrent; and that the point of concurrence is the point of concurrence of the circles circumscribing the triangles BCD, CAE, ABF.

The same result follows if the equilateral triangles are described so that A and D are on the same side of BC, B and E on the same side of CA and C and F on the same side of AB.

§ 8. Second Method.

Let ABC (fig. 24) be the given triangle, DEF the required inscribed equilateral triangle; then, by the formula for the chord of a circle, in terms of the angle it subtends at the circumference and the diameter of the circle,

$$\mathrm{EF} = \mathrm{OA}\sin\mathrm{A} = \mathrm{FD} = \mathrm{OB}\sin\mathrm{B} = \mathrm{DE} = \mathrm{OC}\sin\mathrm{C};$$

$$\therefore \quad \mathrm{OA} : \mathrm{OB} : \mathrm{OC} = 1/\sin\mathrm{A} : 1/\sin\mathrm{B} : 1/\sin\mathrm{C} = 1/a : 1/b : 1/c.$$

Hence the point O may be found as follows :—

Let the bisectors of the angles at A meet BC in D and D′; on DD′ as diameter describe a circle. The two points of intersection of this circle, and the circles obtained by taking the other two sides instead of the side BC, are the points required.

§ 9. *Analytical Investigation.*

In fig. 24, let $\mathrm{OD} = \alpha$, $\mathrm{OE} = \beta$, $\mathrm{OF} = \gamma$, $\mathrm{DE} = \mathrm{EF} = \mathrm{FD} = l$; then

$$\mathrm{OE}^2 + \mathrm{OF}^2 + 2\mathrm{OE}.\mathrm{OF}\cos\mathrm{A} = l^2;$$

or,
$$\beta^2 + \gamma^2 + 2\beta\gamma\cos\mathrm{A} = l^2; \tag{1}$$

similarly,
$$\gamma^2 + \alpha^2 + 2\gamma\alpha\cos\mathrm{B} = l^2; \tag{2}$$

$$\alpha^2 + \beta^2 + 2\alpha\beta\cos\mathrm{C} = l^2. \tag{3}$$

On subtracting the second of these equations from the first we get the equation to one of the circles of § 7, namely,

$$\beta^2 - \alpha^2 + 2\gamma(\beta\cos\mathrm{A} - \alpha\cos\mathrm{B}) = 0; \qquad (4).$$

and we get the other two by taking the other two differences.

Equation (4) may be expressed in the form

$$(b^2 - c^2)(a\beta\gamma + b\gamma\alpha + c\alpha\beta) + a(a\alpha + b\beta + c\gamma)(c\beta - b\gamma) = 0. \tag{5}$$

The radical axis of the three circles given by the equations (1), (2), (3), is

$$bc(b^2 - c^2)\alpha + ca(c^2 - a^2)\beta + ab(a^2 - b^2)\gamma = 0;$$

or,
$$\alpha\sin(\mathrm{B} - \mathrm{C}) + \beta\sin(\mathrm{C} - \mathrm{A}) + \gamma\sin(\mathrm{A} - \mathrm{B}) = 0.$$

A simple geometrical construction may be given for this line, namely (fig. 26) make $\angle ABQ = \angle ACQ = A$; let AQ meet BC in P; then P is the point where the radical axis meets BC.

The equation to the radical axis of the circle of equation (5) and the circumcircle of the triangle is obviously $c\beta - b\gamma = 0$. It may easily be shown by means of this equation that if the radical axis meets the side AB in O, then $AO : OB = b^2 : a^2$. This may also be proved without the use of the above equations, by calculating the segments AO and OB. It follows from this result that AO is one of the symmedians of the triangle.

§ 10. Generalization.

To find a point O (fig. 27) such that, if OD′, OE′, OF′, be the perpendiculars from O on the sides of ABC, the triangle D′E′F′ shall be similar to any given triangle DEF.

On AB, BC, CA, describe the triangles ABC′, A′BC, AB′C, similar to the triangle DEF (the triangles are named as they correspond to DEF); and let all these triangles be described externally or all internally on the sides of ABC. Then in both cases the circles circumscribing these triangles will be concurrent (in a point P); and the point inverse to this point of concurrence will be the point O required. The proof is almost identical with that given before for the special case when DEF is equilateral.

The point P may also be obtained as the point of concurrence of the lines AA′, BB′, CC′.

It should be noticed that, according to the construction given above, the point corresponding to D will lie in BC, the point corresponding to E in CA, and the point corresponding to F in AB. By making the triangles ABC′, AB′C, A′BC, similar to EFD, the vertices corresponding in this order, we may make the vertex corresponding to E lie in BC, that corresponding to F lie in CA, and that corresponding to E in AB; and by making other variations in the correspondence of the similar triangles, we may make the vertices corresponding to those of DEF lie in whichever sides of ABC we choose.

An analytical investigation, similar to that of § 9, may also be given.