NORMAL STRUCTURE AND FIXED POINT PROPERTY

by J. GARCÍA-FALSET and E. LLORÉNS-FUSTER

Introduction. The most classical sufficient condition for the fixed point property of non-expansive mappings FPP in Banach spaces is the normal structure (see [**6**] and [**10**]). (See definitions below). Although the normal structure is preserved under finite l_p-product of Banach spaces, $(1 < p \leq \infty)$, (see Landes, [**12**], [**13**]), not too many positive results are known about the normal structure of an l_1-product of two Banach spaces with this property. In fact, this question was explicitly raised by T. Landes [**12**], and M. A. Khamsi [**9**] and T. Domínguez Benavides [**1**] proved partial affirmative answers. Here we give wider conditions yielding normal structure for the product $X_1 \otimes_1 X_2$.

Moreover, the permanence of the FPP for the l_1-product of two Banach spaces with this property is still only partially understood. In a recent paper [**11**] T. Kukzumov, S. Reich and M. Schmidt have given sufficient conditions for a product of two Banach Spaces X_1 and X_2 endowed with the l_1-product norm to have FPP. They introduce the semi-Opial property, mainly with the technical role of guaranteeing the FPP for a space $\mathbb{R}_1 \otimes X_2$ provided that the space X_2 has FPP. Thus, if the space X_1 is uniformly convex in every direction UCED and the second space X_2 verifies the semi-Opial property, then the l_1-product $X_1 \otimes_1 X_2$ have the FPP.

Here we also show that if the space X_1 has the generalized Gossez-Lami Dozo property (a known sufficient condition for weak normal structure [**8**]) and the second space X_2 verifies the semi-Opial property, then the l_1-product $X_1 \otimes_1 X_2$ has the FPP. The scope of this last result is different from that of Theorem (1) of [**11**]. For example, a non UCED James space J has the generalized Gossez-Lami Dozo property while the space c_0 has a UCED renorming without the GGLD property.

Definitions and notation. Let $(X_1, \|.\|_1)$, $(X_2, \|.\|_2)$ be Banach spaces. Throughout $X_1 \otimes_1 X_2$ will denote the product space $X_1 \times X_2$ endowed with the norm $\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2$.

Also, let S_X, B_X and X^* denote, respectively, the unit sphere in X, the closed unit ball in X and the conjugate space of X.

For $f \in S_{X^*}$, and $\delta \in (0, 1)$ we hereafter denote by $S(f, \delta)$ the slice

$$S(f, \delta) := \{x \in B_X : f(x) > 1 - \delta\}.$$

In this paper "$\rightharpoonup$" denotes weak convergence.

A subset A of a normed space X is said to have *normal structure* (NS for short) if every bounded, convex, subset C of A with positive diameter d is contained in a ball with radius smaller than d, and center in C:

$$d := \operatorname{diam}(C) > 0 \Rightarrow r(C) = \inf\{\sup\{\|x - y\| : y \in C\}, x \in C\} < d$$

A Banach space X is said to have weakly normal structure (WNS for short) if every weakly compact convex subset of X has NS.

[1] Partially supported by a grant from the University of Valencia (683.2-1004).

For a Banach space $(X, \|.\|)$ and a fixed element $z \in S_X$, let the modulus of convexity of X in the direction z be the function $\delta_z:[0,2]\to[0,1]$ defined by

$$\delta_z(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1, \|y\| \leq 1 \; \|x - y\| \geq \varepsilon, x - y = tz, t \in \mathbb{R}\right\}.$$

If $\delta_z(\varepsilon) > 0$ for all $\varepsilon > 0$ and all such z, then X is called *uniformly convex in every direction.*

A Banach space X is said to have the *semi-Opial property*, SO for short, if for any bounded non-constant sequence (x_n) with $x_{n+1} - x_n \to 0$ there exists a subsequence (x_{n_k}) such that $x_{n_k} \rightharpoonup x \in X$ and

$$\lim_{k\to\infty} \|x_{n_k} - x\| < \operatorname{diam}(\{x_n\}).$$

In a recent paper ([**8**]), the author introduced the *generalized Gossez-Lami Dozo property* (GGLD in short), (see [**7**]) for a Banach space X as follows: X is said to have GGLD whenever

$$D[(x_n)] > 1$$

for all weakly null sequence (x_n) in X such that

$$\lim_n \|x_n\| = 1,$$

where

$$D[(x_n)] := \limsup_n \left(\limsup_m \|x_m - x_n\|\right).$$

He also defined the coefficient $\beta(X)$ of X by

$$\beta(X) := \inf\{D((x_n)) : x_n \rightharpoonup 0, \|x_n\| \to 1\}.$$

This property GGLD is a strengthening of another sufficient condition for normal structure due to Tingley (see [**14**]). In [**4**] the authors proved that GGLD is a weaker form of the weak uniform normal structure and that the coefficient $\beta(X)$ is equal to Bynum's coefficient, $WCS(X)$.

Notice that if (x_n) is a weakly null sequence in X, with $l = \lim\|x_n\| > 0$, then the sequence $y_n = (1/l)x_n$ is weakly null also and,

$$1 < D(y_n)$$

whenever X has the GGLD property. Hence

$$\lim_n \|x_n\| < \limsup_n \left(\limsup_m \|x_m - x_n\|\right).$$

l_1-product and normal structure. In infinite dimensional Banach spaces normal structure is a consequence of the fact that some subsets of its unit sphere are "nearly compact." (See [6]). For example, $(X, \|.\|)$ has the *weak uniform Kadek-Klee property* WUKK if there exist $\varepsilon \in (0,1)$ and $\delta \in (0,1)$ such that, for every weakly compact convex subset C of B_X with $\alpha(C) > \varepsilon > 0$, then $\operatorname{dist}(0, C) \leq 1 - \delta$. Here $\alpha(C)$ is the Kuratowski measure of noncompactness of the set $C \subset X$, i.e.

$$\alpha(C) := \inf\{r > 0 : C \text{ has a finite } r\text{-cover}\}.$$

In [3] the authors prove that WUKK$\Rightarrow$WNS. In [4] [5] the authors give sufficient conditions for weakly normal structure in terms of some "measures" of the *slices* of the unit ball B_X.

So, a Banach space X is said to have the property α' whenever there exists $\delta \in (0,1)$ such that

$$\alpha(S(f, \delta)) < 1$$

for every $f \in S_X$.

THEOREM 1. *Let $(X_1, \|.\|_1)$ be a Banach space with the GGLD property. If $(X_2, \|.\|_2)$ is a Banach space with the α' property, then*

$$X_1 \otimes_1 X_2 \text{ has WNS.}$$

Proof. Suppose, for a contradiction, that $X_1 \otimes_1 X_2$ does not have WNS. In this case there exists a sequence (x_n) in $X_1 \otimes_1 X_2$ such that

$$\begin{cases} (x_n) = (x_{n1}, x_{n2}) \rightharpoonup (0,0) \\ \|x_n\| \leq 1, \lim_{n\to\infty} \|x_n\| = 1 \\ \lim_{n\to\infty} \operatorname{dist}(x_{n+1}, \operatorname{conv}\{x_1, \ldots, x_n\}) = 1 \end{cases}$$

Without loss of generality we may assume that the following limits exist

$$\lim_{j\to\infty}\left(\lim_{s\to\infty} \|x_{s1} - x_{j1}\|_1\right) \lim_{j\to\infty}\left(\lim_{s\to\infty} \|x_{s2} - x_{j2}\|_2\right)$$

$$\lim_{s\to\infty} \|x_{s1}\|_1 \lim_{s\to\infty} \|x_{s2}\|_2$$

and then

$$\lim_{j\to\infty}\left(\lim_{s\to\infty} \|x_{s1} - x_{j1}\|_1\right) + \lim_{j\to\infty}\left(\lim_{s\to\infty} \|x_{s2} - x_{j2}\|_2\right)$$
$$= \lim_{j\to\infty}\left[\lim_{s\to\infty} \|x_{s1} - x_{j1}\|_1 + \lim_{s\to\infty} \|x_{s2} - x_{j2}\|_2\right] = \lim_{j\to\infty}\left[\lim_{s\to\infty} \{\|x_s - x_j\|\}\right] = 1.$$

If we suppose that $\lim_{s\to\infty} \|x_{s1}\|_1 \neq 0$ by the GGLD condition we have

$$\lim_{s\to\infty} \|x_{s1}\|_1 < \lim_{j\to\infty}\left(\lim_{s\to\infty} \|x_{s1} - x_{j1}\|_1\right)$$

https://doi.org/10.1017/S0017089500031220 Published online by Cambridge University Press

and therefore we obtain the following contradiction:

$$1 = \lim_{s\to\infty} \|x_s\| = \lim_{s\to\infty} \|x_{s1}\|_1 + \lim_{s\to\infty} \|x_{s2}\|_2 \le \lim_{s\to\infty} \|x_{s1}\|_1 + \lim_{j\to\infty}\lim_{s\to\infty} \|x_{s2} - x_{j2}\|_2$$

$$< \lim_{j\to\infty}\left(\lim_{s\to\infty} \|x_{s1} - x_{j1}\|_1\right) + \lim_{j\to\infty}\left(\lim_{s\to\infty} \|x_{s2} - x_{j2}\|_2\right) = 1.$$

On the other hand, if $\lim_{s\to\infty} \|x_{s1}\|_1 = 0$, we also obtain a contradiction. In fact, since X_2 has the (α') property, there exists $\delta > 0$ such that

$$\alpha(S(f, \delta)) < 1.$$

for every $f \in S_{X_2^*}$. It is obvious that

$$\lim_{n\to\infty} \|x_{n2}\|_2 = 1$$

and then there exists a positive integer n_0 such that

$$\|x_{n_0 2}\|_2 > 1 - \delta.$$

Let $f_0 \in S_{X_2^*}$ such that

$$f_0(x_{n_0 2}) = \|x_{n_0 2}\| > 1 - \delta.$$

Since $x_{n_{02}} - x_{n2} \rightharpoonup x_{n_0 2}$, there exists a positive integer $n_1 \ge n_0$ such that

$$x_{n_0 2} - x_{n2} \in S(f_0, \delta)$$

for all $n \ge n_1$. If we prove that

$$\alpha(\{x_{n_0 2} - x_{n2} : n \ge n_1\}) = 1$$

then we shall have a contradiction because $\alpha(S(f_0, \delta)) < 1$.

In fact, if $r \in (0, 1)$ we can choose $\varepsilon > 0$ such that $r + \varepsilon < 1$. The sequence (x_n) is diametral, and hence there exists $n_2 \ge n_1$ such that

$$\|x_n - x_m\| > r + \varepsilon$$

for $n, m \ge n_2$. On the other hand, there exist $n_3 \ge n_2$ such that

$$\|x_{n1}\|_1 < \frac{\varepsilon}{2}$$

for all $n \ge n_3$. Now, for $n, m \ge n_3$, we have

$$r + \varepsilon < \|x_n - x_m\| = \|x_{n1} - x_{m1}\|_1 + \|x_{n2} - x_{m2}\|_2 < \varepsilon + \|x_{n2} - x_{m2}\|_2,$$

and therefore

$$r < \|x_{n2} - x_{m2}\|_2$$

for $m, n \ge n_3$. This effectively shows that

$$\alpha(\{x_{n_0 2} - x_{n2} : n \ge n_1\}) = 1.$$

and the proof is complete.

COROLLARY. *Let* $(X_1, \|.\|_1)$ *be a Banach space with the GGLD property. If* $(X_2, \|.\|_2)$ *is a reflexive Banach space with the WUKK property, then* $X_1 \otimes_1 X_2$ *has WNS.*

Proof. It is straightforward from the fact that a reflexive Banach with WUKK has the (α') property. (See [5]).

It should be noticed that the reflexivity condition in the previous corollary can nevertheless, be dropped, by just following a similar argument to that of theorem 1.

While T. Landes in [13] proved that WNS is not preserved under finite l_1-product of Banach spaces with this property, next we show that the GGLD property is stable under finite l_1-products.

THEOREM 2. *Let* $(X_1, \|.\|_1)$ *and* $(X_2, \|.\|_2)$ *be Banach spaces with the GGLD property. Then* $X_1 \otimes_1 X_2$ *has GGLD.*

Proof. Suppose, for a contradiction, that $X_1 \otimes_1 X_2$ does not have GGLD. In this case there exists a sequence (x_n) in $X_1 \otimes_1 X_2$ such that

$$\begin{cases} a) \ (x_n) = (x_{n1}, x_{n2}) \rightharpoonup (0,0) \\ b) \ \lim_{n\to\infty} \|x_n\| = 1 \\ c) \ D(\{x_n\}) = 1 \end{cases}$$

It is easy to see that each subsequence of this sequence (x_n) satisfies the three above conditions. Hence, without loss of generality we may assume that the following limits exist

$$\lim_{j\to\infty}\left(\lim_{s\to\infty} \|x_{s1} - x_{j1}\|_1\right) \lim_{j\to\infty}\left(\lim_{s\to\infty} \|x_{s2} - x_{j2}\|_2\right)$$

$$\lim_{n\to\infty} \|x_{n1}\|_1 \lim_{n\to\infty} \|x_{n2}\|_2$$

By the same argument of Theorem 1, if we suppose that $\lim_{s\to\infty} \|x_{s1}\|_1 \neq 0$ or $\lim_{s\to\infty} \|x_{s2}\|_2 \neq 0$ by the GGLD condition we obtain a contradiction with above condition (c). Otherwise we have

$$\lim_{s\to\infty} \|x_{s1}\|_1 = \lim_{s\to\infty} \|x_{s2}\|_2 = 0,$$

which contradicts (b).

REMARKS. 1. T. Domínguez Benavides have proved, ([1], Corollary 1), that if X_1, X_2 are reflexive Banach spaces, then

$$WCS(X_1 \otimes_1 X_2) = \min\{WCS(X_1), WCS(X_2)\}$$

The condition $WCS(X) > 1$ can be considered as an (stronger) uniform version of GGLD property. Thus the above theorem is a partial generalization of this Corollary 1 of [1].

2. M. A. Khamsi [9] has introduced, for every Banach space X with a finite dimensional Schauder descomposition, the coefficient $\beta_p(X)$ defined for $p \in [1, \infty)$ as the infimum of the set of numbers λ such that

$$(\|x\|^p + \|y\|^p)^{1/p} \leq \lambda \, \|x + y\|$$

for every $x, y \in X$ with $\text{supp}(x) < \text{supp}(y)$. He showed that if (X_i) is a sequence of Banach spaces such that $\sup_i\{\beta_1(X_i)\} < 2$ then $\oplus_1 X_i$ has weakly normal structure.

It is easy to see that every Banach space X with $\beta_p(X) < 2^{1/p}$ has the GGLD property.

l_1-product and F.P.P. Let K be a nonempty subset of a Banach space X. Recall that a mapping $T: K \to X$ is said to be *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for every $x, y \in K$. We say that the Banach X has the *fixed point property* (FPP for short) if every nonexpansive selfmapping T of every weakly compact convex subset K of X has a fixed point. If the space X fails to have FPP, then there exist a nonempty weakly compact and convex subset K of X, and a nonexpansive fixed point free mapping $T: K \to K$. We can assume that K is *minimal* for T, and $\text{diam}(K) > 0$. (Such a sequence is called *approximate fixed point sequence for* T). A well known property of the minimal sets is the following Goebel-Karlovitz lemma.

LEMMA. *Let K be a minimal weakly compact convex subset for a nonexpansive mapping T, and let (x_n) be an a.f.p.s. for T. Then for all $x \in K$*

$$\lim_n \|x_n - x\| = \text{diam}(K).$$

THEOREM 3. *Let $(X_1, \|.\|_1)$ be a Banach space with the GGLD property. If $(X_2, \|.\|_2)$ is a Banach space with the SO property, then*

$X_1 \otimes_2 X_2$ has the FPP.

Proof. Suppose, to get a contradiction, that $X_1 \otimes_1 X_2$ fails to have the FPP. Then there exist a nonempty weakly compact and convex minimal subset C of $X_1 \otimes_1 X_2$, with $\text{diam}(C) > 0$, and a nonexpansive fixed point free mapping $T: C \to C$. Let $(x_n) = (x_{n1}, x_{n2})$ be an almost fixed point sequence with

$$x_{n+1} - x_n \to 0$$

Then $x_{n+1,2} - x_{n2} \to 0$, and since X_2 has the SO property, there exists (x_{n_k}) subsequence of (x_n), such that

$$x_{n_k 2} \rightharpoonup x_2 \in X_2 \quad \text{and} \quad \lim_k \|x_{n_k 2} - x_2\| < \text{diam}(x_{n_k 2}).$$

By the weak compactness of the set C, we can assume that

$$x_{n_{k_s}} = (x_{n_{k_s} 1}, x_{n_{k_s} 2}) \rightharpoonup (x_1, x_2) =: x \in C$$

Let $K := C - (x_1, x_2)$. We define

$$\tilde{T}: K \to K,$$
$$y - (x_1, x_2) \mapsto T(y) - (x_1, x_2).$$

Then the sequence

$$y_s := (x_{n_{k_s} 1}, x_{n_{k_s} 2}) - (x_1, x_2)$$

is an a.f.p.s. for the nonexpansive mapping $\tilde{T}$ with $y_s \rightharpoonup (0,0) \in K$. Moreover, K is a minimal set for $\tilde{T}$.

By the Goebel-Karlovitz Lemma, for every $(z_1, z_2) \in K$ we have

$$\lim_{s\to\infty} \|y_s - z\| = \operatorname{diam}(K) = \operatorname{diam}(C).$$

It is well known that it suffices to formulate fixed point problems in a separable setting (see [**6**]). Hence we can suppose that the set K is separable, and by passing to subsequences if it is necessary, we can also assume that the following limits exist

$$\lim_s \|y_{s1} - z_1\|_1, \lim_s \|y_{s2} - z_2\|_2$$

for every $(z_1, z_2) \in K$, and then without loss of generality we may suppose that the following limits exist also

$$\lim_{m\to\infty}\left(\lim_{s\to\infty} \|y_{s1} - y_{m1}\|_1\right) \lim_{m\to\infty}\left(\lim_{s\to\infty} \|y_{s2} - y_{m2}\|_2\right)$$

therefore

$$\lim_{m\to\infty}\left(\lim_{s\to\infty} \|y_{s1} - y_{m1}\|_1\right) + \lim_{m\to\infty}\left(\lim_{s\to\infty} \|y_{s2} - y_{m2}\|_2\right)$$

$$= \lim_{m\to\infty}\left[\lim_{s\to\infty} [\|y_s - y_m\|]\right] = \lim_{m\to\infty} [\operatorname{diam}(K)] = \operatorname{diam}(K).$$

If we suppose that $\lim_{s\to\infty} \|y_{s1}\|_1 \neq 0$ by the GGLD condition we obtain

$$\lim_{s\to\infty} \|y_{s1}\|_1 < \lim_{j\to\infty}\left(\lim_{s\to\infty} \|y_{s1} - y_{m_j 1}\|_1\right)$$

and then we have the following contradiction:

$$\operatorname{diam}(C) = \lim_{s\to\infty} \|y_s\| = \lim_{s\to\infty} \|y_{s1}\|_1 + \lim_{s\to\infty} \|y_{s2}\|_2$$

$$\leq \lim_{s\to\infty} \|y_{s1}\|_1 + \lim_{j\to\infty}\lim_{s\to\infty} \|y_{s2} - y_{m_j 2}\|_2$$

$$< \lim_{j\to\infty}\left(\lim_{s\to\infty} \|y_{s1} - y_{m_j 1}\|_1\right) + \lim_{j\to\infty}\left(\lim_{s\to\infty} \|y_{s2} - y_{m_j 2}\|_2\right) = \operatorname{diam}(C).$$

On the other hand, if $\lim_{s\to\infty} \|y_{s1}\|_1 = 0$, then

$$\operatorname{diam}(C) = \lim_{s\to\infty} \|(y_{s1}, y_{s2})\| = \lim_{s\to\infty} \|y_{s2}\|_2$$

$$= \lim_{s\to\infty} \|x_{n_{k_s}2} - x_2\|_2 = \lim_{k\to\infty} \|x_{n_k 2} - x_2\|_2 < \operatorname{diam}(X_{n_k 2}) \leq \operatorname{diam}(C).$$

which is also a contradiction.

Next we shall see that for reflexive Banach spaces, SO condition is more general than GGLD property.

PROPOSITION. *Every reflexive Banach space X with the GGLD property satisfies the SO condition.*

Proof. Let (x_n) be a bounded sequence in X such that $\|x_{n+1}-x_n\|\to 0$. There exists a subsequence (x_{n_k}) with

$$x_{n_k} \rightharpoonup x \in X.$$

since the space X is reflexive. We can suppose that $l:=\lim_k \|x_{n_k}-x\|>0$, otherwise it follows immediately the result.

Let

$$y_k := \frac{x_{n_k}-x}{l}.$$

From the GGLD property of the space X we obtain

$$1 < D((y_k))$$

therefore

$$l < \limsup_k \left(\limsup_s \|x_{n_k}-x_{n_s}\|\right) \le \operatorname{diam}(x_{n_k}) \le \operatorname{diam}(x_n).$$

which completes the proof.

REMARK. The Banach space $X_\beta := (l_2, \|.\|_\beta)$ where

$$\|x\|_\beta = \max\{\|x\|_2, \beta\ \|X\|_\infty\}$$

have the SO property for $1<\beta<2$, but if $\sqrt{2}<\beta$, X_β does not have NS and hence X_β cannot have GGLD.

As a direct consequence of this proposition and the Theorem 1 of [**11**] we obtain the following.

COROLLARY. *If $(X_1, \|.\|_1)$ is a UCED Banach space and the second space $(X_2, \|.\|_2)$ is a reflexive Banach space with the GGLD property, then*

$$X_1 \otimes_1 X_2 \text{ has the FPP.}$$

REMARKS. 1. In the paper [**11**], it is shown that if the space X_1 is UCED and the second space X_2 verifies SO then the l_1-product $X_1 \oplus_1 X_2$ has the FPP. Although theorem 3 above and theorem 1 of [**11**] are both closely related, neither one of these results implies the other. We consider James space J which consists of the sequences $x=(x_n)\in c_0$ such that

$$\|x\| = \sup\{(x_{p_1}-x_{p_2})^2 + (x_{p_2}-x_{p_3})^2 + \ldots + (x_{p_{n-1}}-x_{p_n})^2\}$$

is finite, where the supremum is taken for every increasing sequence of positive integers (p_i). This space J fails UCED (in fact, does not have NS), but J satisfies the GGLD property. (See [**6**] and [**2**]).

On the other hand, in the classical space of sequences c_0 we can define the norm

$$\|x\| = \sqrt{\|x\|_\infty^2 + \sum_{i=1}^{\infty} \frac{x_i^2}{2^i}},$$

and it is known that $(c_0, \|.\|)$ is UCED (see **[6]**), but it is easy to see that this space fails to have GGLD.

2. The proofs of all theorems in this paper equally work if the l_1-product norm in the product space $X_1 \times X_2$ is replaced by the norm

$$\|(x_1, x_2)\| := |(\|x_1\|_1, \|x_2\|_2)|$$

where $|(\alpha, \beta)|$ is any monotonic norm in $\mathbb{R}^2$, i.e. $|(\alpha_1, \beta_1) \leq |(\alpha_2, \beta_2)|$ when $0 \leq \alpha_1 \leq \alpha_2$ and $0 \leq \beta_1 \leq \beta_2$.

REFERENCES

1. T. Domínguez Benavides, Weak uniform normal structure in direct sum spaces. *Studia Math.* **103,** (3), (1992), 283–290.

2. T. Domínguez Benavides, G. López Acedo and H. K. Xu, Weak uniform normal structure and construction fixed points of nonexpansive mappings. Preprint.

3. D. van Dulst and B. Sims, Fixed points of nonexpansive mappings and Chebyshev centers in Banach space with norms of type (KK) in *Banach Space Theory and Its Applications* (*Bucharest, 1981*), Lecture Notes in Math. 991 (Springer-Verlag, 1983), 35–43.

4. J. García-Falset, A. Jiménez-Melado and E. Lloréns-Fuster, A characterization of normal structure in *Proceedings of Second International Conference on Fixed Point Theory and Appl. Halifax*, *1991*, (World Scientific, Singapore, 1992), 122–129.

5. J. García-Falset, A. Jiménez-Melado and E. Llorens-Fuster, Measures of noncompactness and normal structure in Banach spaces. *Studia Math.* **110,** (1), (1994), 1–8.

6. K. Goebel and W. A. Kirk, *Topics in metric fixed point theory* (Cambridge University Press, 1990).

7. J. P. Gossez and E. Lami Dozo. Some geometrical properties related to the fixed-point theory for nonexpansive mappings, *Pacific J. Math.* **40** (1972), 565–573.

8. A. Jiménez-Melado, Stability of weak normal structure in James quasi reflexive space, *Bull. Austral. Math. Soc.* **46** (1992), 367–372.

9. M. A. Khamsi, Normal structure for Banach spaces with Schauder decomposition, *Canad. Math. Bull.* **32** (3), (1989), 344–351.

10. W. A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly* **72** (1965), 1004–1006.

11. T. Kuczumow, S. Reich and M. Schmidt, A fixed point property of l_1-product spaces, *Proc. Amer. Math. Soc.* **119** (2), (1993), 457–463.

12. T. Landes, Permanence properties of normal structure, *Pacific J. Math.* **110,** 1, (1984), 125–143.

13. T. Landes, Normal structure and the sum property, *Pacific J. Math.* **123** (1986), 127–147.

14. D. Tingley, The normal structure of James quasi reflexive space, *Bull. Austral. Math. Soc.* **42,** (1990), 95–100.

Departamento de Análisis Matemático
Facultad de Matemáticas
Doctor Moliner, 50
46100 Burjasot
Valencia
Spain