number of units expressed by the denominator. BQ is then drawn perpendicular to AB to meet the circle in Q. OQ is the refracted ray.

The objections to the method are (1) μ must be expressible as a ratio of two simple integers, (2) the trouble of sub-dividing OA.

The construction is simpler if we first make OA μ units in length and $OB = 1$ unit. Draw AP and BQ perpendicular to AB, AP cutting the incident ray in P. With centre O and radius OP describe a circle cutting BQ in Q. Then OQ gives the refracted ray.

Critical Angle.—To find the critical angle for a medium, make $OA = \mu$ units and $OB = 1$ unit.

Describe a circle with radius OA (Fig. 2), and complete the construction as before.

The critical angle can now be measured directly.

Pin method of finding refractive index.—Suppose the directions of the rays OP and OQ (Fig. 1) have been found experimentally, measure $OB = 1$ unit. Draw BQ perpendicular to BA cutting OQ in Q. With centre O and radius OQ describe a circle cutting OP in P. Draw perpendicular PA. Measure OA. This gives the refractive index at once without having a division operation, as in the other method.

WILLIAM MILLER

An Interesting Example in Curve Tracing.—It is proposed to trace the curves represented by the equation

$$y - ax - y^3 + x^3y^2 + x^2y^3 = 0$$

for the values of a (i) $a = 2$, (ii) $a = 1$.

Analysing the equation

$$y - 2x - y^3 + x^3y^2 + x^2y^3 = 0$$

by means of Newton's parallelogram, we obtain as a first approximation to the shape at the origin $y = 2x$, and as a second approximation $y = 2x + 8x^3$.

For a first approximation, when x and y are infinite, we have

$y + x = 0$, and for a second $y = -x + \dfrac{y^3}{x^2y^2}$,

$$= -x - \frac{1}{x}.$$

Hence $y = -x$ is an asymptote.

When x is finite and y infinite, we have as a first approximation

$x^2 = 1$. For a second $x^2 = 1 - \dfrac{x^3y^2}{y^3}$,

$$i.e. \quad x = \pm\sqrt{1 - \frac{x^3}{y}} = \pm\left(1 - \frac{x^3}{2y}\right).$$

Hence $$x = 1 - \frac{1}{2y}, \quad \text{or} \quad x = -1 - \frac{1}{2y}.$$

Thus $x = 1$, $x = -1$ are asymptotes, and the curve approaches them as shown in Fig. 1.

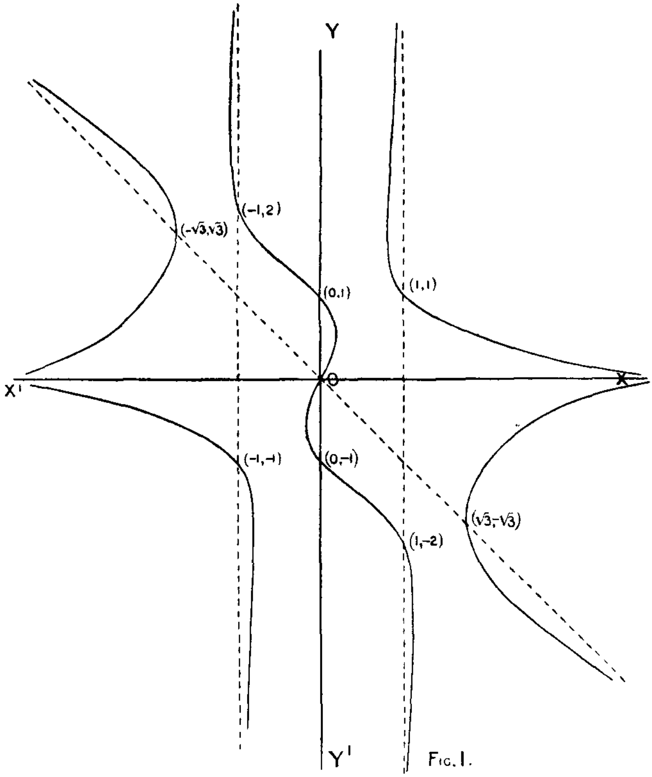

Fig. 1.

When y is finite and x infinite, we have

$$y^2 = \frac{2}{x^2} \quad \text{or} \quad y = \pm \frac{\sqrt{2}}{x}.$$

Hence $y = 0$ is an asymptote, and the curve lies on both sides of this asymptote at each end of it.

Where $y = 0$ cuts the curve we have $x = 0$, and where $x = 0$ cuts the curve $y^3 - y = 0$, whence $y = 0$ or ± 1.

To find where the curve crosses its asymptotes, substitute

(i) $x = 1$, then $y^2 + y - 2 = 0$, whence $y = 1$ or -2;

(ii) $x = -1$, then $y^2 - y - 2 = 0$, whence $y = 2$ or -1;

(iii) $y = -x$, then $x^3 - 3x = 0$, whence $x = 0$ or $\pm\sqrt{3}$.

To examine the shape of the curve at these points, change the origin to each in turn. When the origin is at (0, 1), the equation is

$$2\eta = -2\xi + \xi^2 - 3\eta^2 \ldots,$$

whence the first and second approximations to the shape at the origin are $\qquad \eta = -\xi, \; \eta = -\xi - \xi^2.$

Similarly changing the origin to $(0, -1)$ we obtain for approximations $\qquad \eta = -\xi, \; \eta = -\xi + \xi^2.$

The equation referred to parallel axes through (1, 1) is

$$3\eta + 3\xi + 4\xi^2 + 12\xi\eta + \eta^2 + \ldots = 0,$$

whence we obtain the approximations

$$\eta = -\xi, \; \eta = -\xi - \tfrac{7}{3}\xi^2.$$

Similarly at $(-1, -1)$ we have

$$\eta = -\xi, \; \eta = -\xi + \tfrac{7}{3}\xi^2.$$

Changing the origin to $(1, -2)$ we get the equation

$$3(\eta + 2\xi) = \eta^2 + 12\xi\eta + 4\xi^2 + \ldots,$$

whence the approximations

$$\eta = -2\xi, \; \eta = -2\xi - \tfrac{16}{3}\xi^2.$$

Similarly at $(-1, 2)$ we have

$$\eta = -2\xi, \; \eta = -2\xi + \tfrac{16}{3}\xi^2.$$

When the origin is changed to $(\sqrt{3}, -\sqrt{3})$ the equation becomes $\qquad \eta + 7\xi + 6\sqrt{3}\,\xi^2 - 3\sqrt{3}\,\eta^2 + \ldots = 0.$

and the approximations are

$$\eta = -7\xi, \; \eta = -7\xi + 141\sqrt{3}\,\xi^2$$

Similarly at $(\sqrt{3}, \sqrt{3})$ we have

$$\eta = -7\xi, \; \eta = -7\xi - 141\sqrt{3}\,\xi^2.$$

If the above results are now put together in a diagram, it will be seen that it is not yet clear how they are to be connected, or how the various branches of the curve run. To remove the ambiguities, let us try to find the points of the curve at which the tangent is parallel to the y-axis. We may do this by finding values of k such that the line $x = k$ will cut the curve in coincident points. Where $x = k$ cuts the curve, we have

$$(k^2 - 1)y^3 + k^3y^2 + y - 2k = 0.$$

If this equation has a repeated root, y, then y satisfies the equations

$$3(k^2 - 1)y^2 + 2k^3y + 1 = 0,$$
$$k^3y^2 + 2y - 6k = 0,$$

whence $\qquad \dfrac{y^2}{-(6k^4 + 1)} = \dfrac{2y}{19k^3 - 18k} = \dfrac{1}{-(k^6 - 3k^2 + 3)},$

and therefore $\quad 4(6k^4 + 1)(k^6 - 3k^2 + 3) = (19k^3 - 18k)^2,$

or $\qquad 24k^{10} - 429k^6 + 756k^4 - 336k^2 + 12 = 0.$

If $k^2 \equiv u$, this equation may be written

$$f(u) \equiv 8u^5 - 143u^3 + 252u^2 - 112u + 4 = 0.$$

We have the following table of simultaneous values of u and $f(u)$:—

u	0	$\frac{1}{2}$	1	2	3	$-\infty$
$f(u)$	4	$-\frac{53}{8}$	9	-100	19	$-\infty$

Hence all the roots of $f(u) = 0$ are real, and only one is negative, and therefore our equation in k has four pairs of real roots. The positive roots of $f(u) = 0$ are approximately $\cdot 04$, $\cdot 62$, $1 \cdot 2$, $3 \cdot 0$, and the corresponding values of k are roughly $\pm \cdot 2$, $\pm \cdot 8$, $\pm 1 \cdot 1$, $\pm 1 \cdot 7$. When $x = \pm \cdot 2$ we have $y = \pm \cdot 6$; $x = \pm \cdot 8$, $y = \pm 1 \cdot 5$; $x = \pm 1 \cdot 1$, $y = \pm 3 \cdot 5$; $x = \pm 1 \cdot 7$, $y = \mp 1 \cdot 7$.

The curve is therefore of the shape shown in Fig. 2.

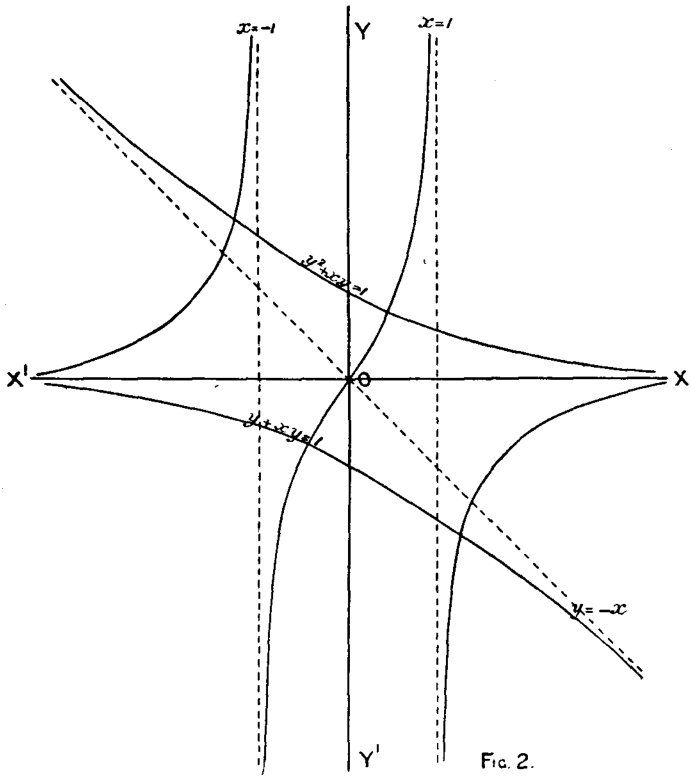

FIG. 2.

When $a = 1$, the equation becomes

$$y - x - y^3 + x^3y^2 + x^2y^3 = 0,$$

$$i.e. \quad (y^2 + xy - 1)(x^2y - y + x) = 0.$$

In this case therefore the curve consists of the hyperbola $y^2 + xy = 1$ and the cubic $x^2y - y + x = 0$, and is shown in Fig. 2.

R. J. T. BELL

Graphical Trisection of Circular Arc.—Let O be the centre of a given circle of radius a; and OA, OB the containing radii of the arc which is to be trisected.

Along OA set off $\mathrm{OE} = \frac{1}{4}\mathrm{OA}$, and draw BF parallel to OA and equal to $\frac{5}{4}a$.

Then a line EP parallel to OF will pass through a point P on the circumference such that $\mathrm{arc\,AP} = \frac{1}{3}\mathrm{arc\,AB}$ approximately.

For, supposing $\mathrm{AOB} = 3\phi$, the equation to EP is

$$y(5 + 4\cos 3\phi) = 4\sin 3\phi(x - \tfrac{1}{4}a).$$

This will be exactly satisfied by $x = a\cos\phi$, $y = a\sin\phi$

if $$5\sin\phi + 4\sin\phi\cos 3\phi = 4\sin 3\phi\cos\phi - \sin 3\phi,$$

or if $$5\sin\phi - 4\sin 2\phi + \sin 3\phi = 0.$$

Now

$$5(\phi - \tfrac{1}{6}\phi^3 + \tfrac{1}{120}\phi^5) - 4(2\phi - \tfrac{8}{6}\phi^3 + \tfrac{32}{120}\phi^5) + (3\phi - \tfrac{27}{6}\phi^3 + \tfrac{243}{120}\phi^5) = 0$$

if $$\phi^5 = 0.$$

Therefore the error is of the fifth order of small quantities if ϕ be small.

For an arc of 60° the error amounts to 2′; and for an arc of 45° or less the error is less than 1′.

[The above was suggested by a construction given by C. S. Bingley, Esq., F.C.I.S., in "Knowledge," November 1911.]

R. F. DAVIS