Glasgow Math. J. **44** (2002) 349–351. © 2002 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089502020177. Printed in the United Kingdom

SPLITTING OFF FREE SUMMANDS OF TORSION-FREE MODULES OVER COMPLETE DVRS

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(Received 24 October, 2000; accepted 11 June 2001)

Abstract. If R is a complete discrete valuation ring and M is a reduced, torsion-free R-module of rank κ, where $\aleph_0 \leq \kappa < 2^{\aleph_0}$, we show that $M \cong \oplus_{\aleph_0} R \oplus C$ for some R-module C. As a consequence, it must be the case that $M \cong M \oplus (\oplus_\alpha R)$, where $\alpha \leq \aleph_0$, and $\mathrm{End}_R M/\mathrm{Fin}\, M$ has rank at least $2^{\aleph_0}$, where $\mathrm{Fin}\, M$ denotes the set of endomorphisms of M with finite rank image.

2000 *Mathematics Subject Classification*. Primary 13C05, 13C10, 20K20, 20K25. Secondary 13F30.

Throughout this note we let R denote a complete discrete valuation ring and p its unique prime and let M be a reduced, torsion-free R-module.

It is well known that if M is countably generated as an R–module, then M is free (see [**3**, p. 48]). Examples of modules M with rank $\lambda^{\aleph_0}$, for any cardinal $\lambda \geq \aleph_0$, have been constructed such that M is essentially indecomposable, that is, if $M = M_1 \oplus M_2$ is any decomposition of M, then M_1 or M_2 has finite rank (see [**1**, p. 462]). In particular there are modules M of rank $2^{\aleph_0}$, where M has no direct summand of countably infinite rank (see [**2**]). However it is a general fact that for any module M, there exists an R-module C_0 such that $M \cong R \oplus C_0$. It follows that if the rank of M is uncountable, then for all $n < \omega$, there exists an R-module C_n such that $M \cong \oplus_n R \oplus C_n$. These results naturally lead us to ask the following question.

If $\aleph_0 \leq \kappa < 2^{\aleph_0}$ and the rank of $_R M$ is κ, does M have a direct summand with countably infinite rank, that is, is $M \cong \oplus_{\aleph_0} R \oplus C$ for some R-module C?

In this note we show that M has such a decomposition (Theorem 2). As a consequence of this, we also obtain that

(i) $\mathrm{End}_R M/\mathrm{Fin}\, M$ has rank at least $2^{\aleph_0}$, where $\mathrm{Fin}\, M$ denotes the ideal of $\mathrm{End}_R M$ consisting of endomorphisms with finite rank image (cf. [**2**, Prop. 1])

(ii) $M \cong M \oplus (\oplus_\alpha R)$, where $\alpha \leq \aleph_0$.

Recall that an R-module M has a basic submodule $\oplus_{e\in B} Re$ and that their completions, with respect to the p-adic topology, are equal. Since $\oplus_{e\in B} Re$ is pure in $\prod_{e\in B} Re$, it follows that $\widehat{M} \subseteq \prod_{e\in B} Re$, where $\widehat{M}$ denotes the p-adic completion of M.

This work is supported by Project No. G-545-173.06/97 of the German-Israeli Foundation for Scientific Research & Development.

Thus we can view an element x of $\widehat{M}$ as a sequence $x = (r_e)_{e\in B}$ $(r_e \in R)$ of elements of R indexed by B. In this context, it makes sense to define the support $[x]_B$ of $x \in \widehat{M}$ with respect to B as

$$[x]_B = \{e \in B : r_e \neq 0\}.$$

It is clear that if $[x]_B$ is finite, then $x \in M$.

PROPOSITION 1. *Let M be an R-module with rank κ, where $\aleph_0 \leq \kappa < 2^{\aleph_0}$. If S is a countably infinite, pure independent subset of M, then there exists a countably infinite subset X of S such that $\widehat{\oplus_{e\in X} Re} \bigcap M$ has countably infinite rank.*

Proof. Let S be a countably infinite, pure independent subset of M and extend S to a maximal, pure independent subset B of M, so that $\oplus_{e\in B} Re$ is a basic submodule of M. There exists a family $\mathcal{F}$ of $2^{\aleph_0}$ countably infinite, almost disjoint subsets of S, that is, if $X_1, X_2 \in \mathcal{F}$, then $X_1 \cap X_2$ has finite cardinality. For each $X \in \mathcal{F}$, consider the submodule

$$I_X = \widehat{\oplus_{e\in X} Re} \cap M.$$

If each I_X has uncountable rank, then for each $X \in \mathcal{F}$, there exists $g_X \in I_X$ such that $[g_X]_B$ is an infinite subset of X. Since the elements of $\mathcal{F}$ are almost disjoint, it is clear that $\{g_X : X \in \mathcal{F}\}$ is an R-independent subset of M with cardinality $2^{\aleph_0}$. This implies that the rank of M is at least $2^{\aleph_0}$, which contradicts our assumption on κ. Thus there exists a countably infinite subset $X \subseteq S$ such that the rank of I_X is countable. $\square$

Recall that the completion of a direct sum $M_1 \oplus M_2$ is the direct sum of their respective completions. Hence if $\oplus_{e\in B} Re$ is a basic submodule of M and $Y \subseteq B$, then $\widehat{\oplus_{e\in Y} Re}$ is a direct summand of $\widehat{M}$ and the projection

$$\pi_Y : \widehat{M} \to \widehat{\oplus_{e\in Y} Re}$$

is an idempotent of the ring $\mathrm{End}_R \widehat{M}$.

THEOREM 2. *Let $_R M$, S and X be as in Proposition 1. Then there exists a countably infinite subset $Y \subseteq X$ such that $\pi_Y|_M \in \mathrm{End}_R M$. Thus $M \cong \oplus_{\aleph_0} R \oplus C$, for some R-module C.*

Proof. Consider a family $\mathcal{H}$ of $2^{\aleph_0}$ countably infinite, almost disjoint subsets of X. For each $Y \in \mathcal{H}$, define the projections

$$\pi_Y : \widehat{M} \to \widehat{\oplus_{e\in Y} Re}.$$

Suppose that $\pi_Y(M) \nsubseteq M$ for all $Y \in \mathcal{H}$. Then there exists $g_Y \in M$ such that $\pi_Y(g_Y) \notin M$. Note that $[\pi_Y(g_Y)]_B$ is an infinite subset of Y, for otherwise $\pi_Y(g_Y)$ would be an element of M. Thus the sets $[\pi_Y(g_Y)]_B$ are almost disjoint subsets of X. It follows that $\{g_Y : Y \in \mathcal{H}\}$ is an R-independent subset of M with cardinality $2^{\aleph_0}$, which contradicts the assumption on the rank of M. Therefore there exists $Y \in \mathcal{H}$ such that $\pi_Y(M) \subseteq M$, and so π_Y is an idempotent in $\mathrm{End}_R M$. Since $\pi_Y(M)$

https://doi.org/10.1017/S0017089502020177 Published online by Cambridge University Press

$\subseteq \widehat{\oplus_{e\in Y} Re} \bigcap M$, and $\widehat{\oplus_{e\in X} Re} \bigcap M$ has countable rank by Proposition 1, it follows that $\pi_Y(M) \cong \oplus_{\aleph_0} R$ and so $M \cong \oplus_{\aleph_0} R \oplus C$, for some R-module C. □

The ideal Fin M is defined to be the set of all R-endomorphisms of M with finite rank image. As a consequence of Theorem 2, we obtain the following corollary (cf. [**2**, Proposition 1]).

COROLLARY 3. *If M has rank κ and $\aleph_0 \leq \kappa < 2^{\aleph_0}$, then*

(i) the rank of $\operatorname{End}_R M/\operatorname{Fin} M$ *is at least* $2^{\aleph_0}$,

(ii) $M \cong M \oplus \oplus_\alpha R$, *where* $\alpha \leq \aleph_0$.

Proof. By Theorem 2, there exists a decomposition $M = \oplus_{i<\omega} Re_i \oplus C$.

(i) If X is an infinite subset of ω, define the projection $\pi_X : M \to \oplus_{i\in X} Re_i$. Clearly $\pi_X \in \operatorname{End}_R M \setminus \operatorname{Fin} M$. This gives rise to $2^{\aleph_0}$ R-independent elements of $\operatorname{End}_R M/\operatorname{Fin} M$.

(ii) Let $A = \oplus_{i<\omega} Re_i \oplus (\oplus_\alpha R)$, for some $\alpha \leq \aleph_0$. Since $A \cong \oplus_{i<\omega} Re_i$, this isomorphism can be extended to an isomorphism of $A \oplus C$ and M by defining it to be the identity on C. □

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