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THE PROPERTIES OF SOLUTIONS OF WEAKLY SINGULAR INTEGRAL EQUATIONS

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Abstract

We examine the differential properties of the solution of the linear integral equation of the second kind, whose kernel depends on the difference of arguments and has an integrable singularity at the point zero. The derivatives of the solution of the equation have singularities at the end points of the domain of integration, and we derive precise estimates for these singularities.

1. Introduction

As a rule, the rate of convergence of approximate methods depends on the smoothness of the solution of the initial problem. The properties of the solution of the integral equation $u(t) = \int_0^b K(t, s)u(s)\,ds + f(t)$ with a weakly singular kernel $K(t, s)$ have been studied in papers [2]–[7]. Kahane [2] presents conditions under which the solution of the equation is analytical in the interior points of the domain of integration. Richter [5] submits concrete results of practical value concerning the integral equations whose kernel has the form $K(t, s) = \log(|t - s|)$ or $K(t, s) = |t - s|^{-\alpha}$ for $0 < \alpha < 1$. Schneider [6] extends these results to kernels of the form $K(t, s) = m(t, s)\log(|t - s|)$ and $K(t, s) = m(t, s)|t - s|^{-\alpha}$ for $0 < \alpha < 1$, with a smooth function m.

The current paper (see also [7]) deals with kernels of the form $K(t, s) = \kappa(|t - s|)$, where κ and its derivatives have certain algebraic estimations (see (2.3)). The class of kernels allowable here is different to that allowed by

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Schneider, and allows, for example, kernels of the form

$$K(t, s) = m(|t - s|)|t - s|^{-\alpha}(\log(|t - s|))^n,$$

with a smooth function m, $0 < \alpha < 1$ and $n \in N$. We give precise estimations to derivatives of the solution. The main result is contained in Theorem 1. Without any serious difficulties, one can generalize this theorem to the case of kernels $K(t, s) = m(t, s)\kappa(|t - s|)$ or $K(t, s) = m(t, s)\kappa(t - s)$. The details will be treated in another paper of the authors.

We refer also to papers [3], [4] that estimate the behaviour of the first and second derivatives of the solution.

2. Main results

We consider the linear integral equation of the second kind

$$u(t) = \int_0^b \kappa(|t - s|)u(s)\,ds + f(t), \qquad 0 \leqslant t \leqslant b, b < \infty, \tag{2.1}$$

where the functions κ and f are given. We shall assume that

$$f \in C^m[0, b] \qquad \text{and} \qquad \kappa \in C^{m-1}(0, b] \qquad \text{for } m \geqslant 1, \tag{2.2}$$

$$|\kappa^{(k)}(t)| \leqslant \gamma_k t^{-\alpha-k} \qquad \text{for } 0 < t \leqslant b \quad \text{and} \quad k = 0, 1, \ldots, m - 1, \tag{2.3}$$

and

$$|\kappa^{(k)}(t)| \geqslant \gamma_k^0 t^{-\alpha_0-k} \qquad \text{for } 0 < t \leqslant t_0 \quad \text{and} \quad k = 0, 1, \ldots, m - 1, \tag{2.4}$$

where α and α_0 are real constants such that

$$0 < \alpha < 1 \qquad \text{and} \qquad \alpha - (1 - \alpha) < \alpha_0 \leqslant \alpha, \tag{2.5}$$

and γ_k, γ_k^0 and t_0 are some positive constants. By $C^p[0, b]$, where p is a non-negative integer, is denoted the set of p times continuously differentiable real-valued functions on $[0, b]$ (later abbreviate $C^0[0, b]$ by $C[0, b]$) and by $y^{(k)}$ is denoted the kth derivative of the function y: $y^{(k)}(t) = (d^k y/dt^k)(t)$. Our main results are contained in the following theorem.

THEOREM 1. *Suppose that the homogeneous equation corresponding to* (2.1) *has in* $C[0, b]$ *only the trivial solution. If the conditions* (2.2) *and* (2.3) *hold, then equation* (2.1) *has a unique solution* u *and*

$$u \in C[0, b] \cap C^m(0, b),$$

$$|u^{(k)}(t)| \leqslant \eta_k\left[t^{-\alpha-k+1} + (b - t)^{-\alpha-k+1}\right], \qquad 0 < t < b, k = 0, 1, \ldots, m, \tag{2.6}$$

where η_k, $k = 0, 1, \ldots, m$, are positive constants. If the conditions (2.2) *to* (2.4) *hold, then*

$$u^{(k)}(t) = u(0)\kappa^{(k-1)}(t) - u(b)\kappa^{(k-1)}(b-t) + v_k(t), \qquad k = 1, 2, \ldots, m, \tag{2.7}$$

where

$$v_k \in C^{m-k}(0, b),$$
$$\lim_{t\to 0+} \frac{v_k(t)}{\kappa^{(k-1)}(t)} = 0 \quad \textit{and} \quad \lim_{t\to b-} \frac{v_k(t)}{\kappa^{(k-1)}(b-t)} = 0. \tag{2.8}$$

The proof of Theorem 1 is given in Section 6. Sections 4 and 5 contain necessary preliminaries for the proof.

REMARK 1. If (2.4) holds only for some $k = k_0$, then (2.7) holds for $k = k_0 + 1$.

REMARK 2. Theorem 1 is easily modifiable to a case when $\kappa \in C[0, b]$ but a certain derivative has a singularity at the point zero. The corresponding formulation can be guessed by means of equality (2.7).

3. Some specific cases

Many problems of practical interest reduce to the problem of solving the integral equation (2.1) with a kernel of the form

$$\kappa(t) = \log t + \kappa_0(t), \tag{3.1}$$
$$\kappa(t) = t^{-\beta} + \kappa_0(t), \qquad 0 < \beta < 1, \tag{3.2}$$

or, more generally,

$$\kappa(t) = t^{-\lambda}(\log t)^n + \kappa_0(t), \qquad 0 \leqslant \lambda < 1, 1 \leqslant n < \infty, \tag{3.3}$$

where $\kappa_0 \in C^{m-1}[0, b]$ is a smooth function on $[0, b]$ without singularities. For such kernels conditions (2.3) and (2.4) hold, and we can use Theorem 1 to characterize the singularities which appear in the solution of integral equation (2.1). In the case (3.2) we can set $\alpha = \alpha_0 = \beta$; in the cases (3.1) and (3.3) we can take correspondingly $\alpha_0 = 0$ with $\alpha = \varepsilon$ and $\alpha_0 = \lambda$ with $\alpha = \lambda + \varepsilon$ with some $\varepsilon > 0$.

4. Properties of the integral operator

Let T be the integral operator of equation (2.1):

$$(Ty)(t) = \int_0^b \kappa(|t-s|)y(s)\,ds, \qquad 0 \leqslant t \leqslant b.$$

We shall assume throughout that condition (2.3) holds for $k = 0$. Then T is a compact (or completely continuous) linear operator in Banach spaces $C \equiv C[0, b]$ and $L^p \equiv L^p(0, b)$, $1 \leqslant p < \infty$. In addition, the operator T maps L^p into L^q with $p < q < p/(1-(1-\alpha)p)$ and there exists a positive integer l that the composition T^l maps L^1 into C. These facts are well known (see, for example, [1]).

Our objective in this section is to estimate derivatives of the function Ty when the behaviour of the function y and its derivatives is known. The derivatives we shall understand in the sense of distributions.

Let $D^k y$ denote the kth distributional derivative of y. It can be defined as follows: choose the space $\mathfrak{D} \equiv \mathfrak{D}(0, b)$ of all infinitely differentiable on $(0, b)$ functions with the supports in $(0, b)$ and the space $\mathfrak{D}^*$ of continuous linear functionals on $\mathfrak{D}$, and then determine $D^k y$ by insisting that

$$\langle D^k y, \varphi \rangle = (-1)^k \langle y, \varphi^{(k)} \rangle$$

for any $\varphi \in \mathfrak{D}$, where $\langle z, \varphi \rangle$ denotes the value of $z \in \mathfrak{D}^*$ on $\varphi \in \mathfrak{D}$. For the locally integrable function y we can write

$$\langle y, \varphi \rangle = \int_0^b y(t)\varphi(t)\,dt.$$

Each locally integrable function y is differentiable in the sense of distributions (but $D^k y$ need not be a function); if $y \in C^k(0, b)$, then the equality $D^k y = y^{(k)}$ is valid. For more detailed information see, for example, [1].

LEMMA 1. *If the function $y \in C[0, b]$ is such that $Dy \in L^1(0, b)$, then the function Ty has the same properties, and*

$$DTy = TDy + y(0)\kappa - y(b)\kappa_1, \tag{4.1}$$

where

$$\kappa_1(t) = \kappa(b-t).$$

PROOF. Write

$$(Ty)(t) = \int_0^t \kappa(t-s)y(s)\,ds + \int_t^b \kappa(s-t)y(s)\,ds$$
$$= \int_0^t \kappa(\sigma)y(t-\sigma)\,d\sigma + \int_0^{b-t} \kappa(\sigma)y(t+\sigma)\,d\sigma.$$

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From here, because of the differentiability of y, it is easy to conclude the differentiability of Ty. Differentiating, we then obtain (4.1):

$$\begin{aligned}(DTy)(t) &= \int_0^t \kappa(\sigma)\big[(Dy)(t-\sigma)\big]\,d\sigma + \int_0^{b-t} \kappa(\sigma)\big[(Dy)(t+\sigma)\big]\,d\sigma \\ &\quad + y(0)\kappa(t) - y(b)\kappa(b-t) \\ &= (TDy)(t) + y(0)\kappa(t) - y(b)\kappa(b-t),\end{aligned}$$

and so Lemma 1 is proved.

COROLLARY 1. *If* $\varphi \in \mathfrak{D}(0, b)$, *then* $T\varphi \in C^\infty[0, b]$ *and*

$$D^k T\varphi = TD^k\varphi, \qquad k = 0, 1, 2, \ldots. \tag{4.2}$$

LEMMA 2. *Let the function* κ *be such that the conditions* (2.3) *are fulfilled. If* $y \in C^{m-1}(0, b)$ *and*

$$|y^{(k)}(t)| \leqslant a_k\big[t^{-\beta-k} + (b-t)^{-\beta-k}\big], \qquad 0 < t < b,\ 0 < \beta < 1,\ k = 0, 1, \ldots, m-1, \tag{4.3}$$

then for the function $z = Ty$ *we get that* $z \in C^{m-1}(0, b)$ *and*

$$|z^{(k)}(t)| \leqslant a'_k\big[t^{-\beta-k+1-\alpha} + (b-t)^{-\beta-k+1-\alpha}\big], \qquad 0 < t < b,\ k = 0, 1, \ldots, m-1, \tag{4.4}$$

where a_k *and* a'_k *are some positive constants.*

PROOF. It is sufficient to observe a case when

$$\left\{\begin{array}{l} y \in C^{m-1}(0, b], \\ |y^{(k)}(t)| \leqslant a_k t^{-\beta-k}, \quad 0 < t \leqslant b,\ k = 0, 1, \ldots, m-1, \\ y(b) = y'(b) = \cdots = y^{(m-1)}(b) = 0. \end{array}\right\} \tag{4.5}$$

To see this, we take a function r such that $r \in C^\infty[0, b]$, $r(t) = 1$ for $0 \leqslant t \leqslant (b/3)$ and $r(t) = 0$ for $(2b/3) \leqslant t \leqslant b$, and present the function y in the form $y = y_0 + y_1$ with $y_0 = ry$ and $y_1 = (1-r)y$. Then conditions (4.5) for the function y_0 are satisfied. For the function y_1 we get the same conditions after substitutions $t' = b - t$ and $s' = b - s$ in the definition of the operator T.

In the case $k = 0$, the function z is continuous over $(0, b)$ and the inequality (4.4) holds. We shall find a formula for $D^k z$, $k \geqslant 1$. For each $\varphi \in \mathfrak{D}(0, b)$ the equalities

$$\langle D^k z, \varphi\rangle = (-1)^k\langle z, \varphi^{(k)}\rangle = (-1)^k \int_0^b (Ty)(s)\varphi^{(k)}(s)\,ds$$

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are valid. Because of the symmetry of the kernel of the integral operator T and Corollary 1, our chain of equalities can be continued like this:

$$\begin{aligned}\langle D^k z, \varphi\rangle &= (-1)^k \int_0^b y(s) \frac{d^k}{ds^k}\left[(T\varphi)(s)\right] ds \\ &= (-1)^k \int_0^b y(s) \frac{d^k}{ds^k}\left[(T\varphi)(s) - \sum_{i=0}^{k-1} \frac{s^i}{i!}(T\varphi^{(i)})(0)\right] ds.\end{aligned}$$

The function in square brackets belongs to class $C^\infty[0, b]$ and its derivatives of order p, $p \leqslant k - 1$, at the zero point $s = 0$ cancel. This enables us, by means of integration by parts, to transfer the derivatives to the function y, whereby members outside the integral cancel:

$$\langle D^k z, \varphi\rangle = \int_0^b y^{(k)}(s)\left[(T\varphi)(s) - \sum_{i=0}^{k-1} \frac{s^i}{i!}(T\varphi^{(i)})(0)\right] ds.$$

The interior integrals

$$\int_0^b \kappa(t)\varphi^{(i)}(t)\, dt, \qquad i = 1, 2, \dots, k-1,$$

are integrated by parts. Then we obtain

$$\langle D^k z, \varphi\rangle = \int_0^b y^{(k)}(s)\left\{\int_0^b\left[\kappa(|t - s|) + \sum_{i=0}^{k-1}(-1)^{i+1}\frac{s^i}{i!}\kappa^{(i)}(t)\right]\varphi(t)\, dt\right\} ds. \tag{4.6}$$

Now we change the order of integration on the right of equality (4.6). The foundation of this step is given later. We get

$$\langle D^k z, \varphi\rangle = \int_0^b\left\{\int_0^b y^{(k)}(s)\left[\kappa(|t - s|) + \sum_{i=0}^{k-1}(-1)^{i+1}\frac{s^i}{i!}\kappa^{(i)}(t)\right] ds\right\}\varphi(t)\, dt,$$

from which, owing to the arbitrariness of the function $\varphi \in \mathfrak{D}$, we get finally that

$$(D^k z)(t) = \int_0^b y^{(k)}(s)\left[\kappa(|t - s|) + \sum_{i=0}^{k-1}(-1)^{i+1}\frac{s^i}{i!}\kappa^{(i)}(t)\right] ds. \tag{4.7}$$

From here

$$|(D^k z)(t)| \leqslant g_k(t) + h_k(t), \qquad 0 < t < b, \quad k = 1, 2, \dots, m-1,$$

where (see (4.5))

$$g_k(t) = a_k \int_0^t s^{-\beta-\kappa} \left| \kappa(t-s) + \sum_{i=0}^{k-1} (-1)^{i+1} \frac{s^i}{i!} \kappa^{(i)}(t) \right| ds$$

and

$$h_k(t) = a_k \int_t^b s^{-\beta-k} \left[|\kappa(s-t)| + \sum_{i=0}^{k-1} \frac{s^i}{i!} |\kappa^{(i)}(t)| \right] ds.$$

For estaimating the member g_k we use Taylor's formula in the form

$$\kappa(t-s) + \sum_{i=0}^{k-1} (-1)^{i+1} \frac{s^i}{i!} \kappa^{(i)}(t) = \int_t^{t-s} \frac{(t-s-\sigma)^{k-1}}{(k-1)!} \kappa^{(k)}(\sigma)\, d\sigma$$

and assumption (2.3). The result is

$$\begin{aligned} g_k(t) &\leqslant \frac{a_k \gamma_k}{(k-1)!} \int_0^t s^{-\beta-k} \left\{ \int_{t-s}^t [\sigma - (t-s)]^{k-1} \sigma^{-\alpha-k}\, d\sigma \right\} ds \\ &= \frac{a_k \gamma_k}{(k-1)!} t^{-\beta-k+1-\alpha} \int_0^1 r^{-\beta-k} \left\{ \int_{1-r}^1 [\rho - (1-r)]^{k-1} \rho^{-\alpha-k}\, d\rho \right\} dr. \end{aligned}$$

The last integral converges if $\beta < 1$ and $\alpha < 1$; to get the last equality, we introduced new variables $r = st^{-1}$ and $\rho = \sigma t^{-1}$. Using (2.3) again, we get the estimate for member h_k:

$$\begin{aligned} \int_t^b s^{-\beta-k} |\kappa(s-t)|\, ds &\leqslant \gamma_0 \int_t^b s^{-\beta-k} (s-t)^{-\alpha}\, ds \\ &\leqslant \gamma_0 \left[t^{-\beta-k} \int_t^{2t} (s-t)^{-\alpha}\, ds + \int_{2t}^b (s-t)^{-\beta-k-\alpha}\, ds \right], \qquad t \leqslant \frac{b}{2}, \end{aligned}$$

and

$$\int_t^b s^{-\beta-k} s^j |\kappa^{(j)}(t)|\, ds \leqslant \gamma_j t^{-\alpha-j} \int_t^b s^{-(k-j)-\beta}\, ds, \qquad j = 0, 1, \ldots, k-1,$$

and, after finding the integrals, we get that

$$h_k(t) \leqslant (\text{constant})_k \left[t^{-\beta-k+1-\alpha} + (b-t)^{-\beta-k+1-\alpha} \right].$$

Hence, estimate (4.4) is valid for $1 \leqslant k \leqslant m-1$. It should be mentioned here that this estimate justifies the change in the order of integration in formula (4.6) since the function to be integrated is absolutely integrable in the square $0 \leqslant t$, $s \leqslant b$; it is considered here that $\varphi(t)$ equals zero in some neighborhoods of $t = 0$ and $t = b$.

By means of standard discussions based on the idea of the suppression of singularities, we conclude from equality (4.7) that $D^k z$, $k = 1, 2, \ldots, m-1$, is continuous; therefore $D^k z = z^{(k)}$ and $z \in C^{m-1}(0, b)$. Thus Lemma 2 is proved.

According to (2.3) and Lemma 2 we have

$$|(D^iT\kappa)(t)| \leqslant a_i\left[t^{-\alpha-i+1-\alpha} + (b-t)^{-\alpha-i+1-\alpha}\right].$$

Repeated application of Lemma 2 gives

$$|(D^iT^j\kappa)(t)| \leqslant a_{i,j}\left[t^{-\alpha-i+j(1-\alpha)} + (b-t)^{-\alpha-i+j(1-\alpha)}\right]$$

provided $\beta_j \equiv [\alpha - (j-1)(1-\alpha)] > 0$. If $\beta_j < 0$, which occurs for all sufficiently large j, then we substitute the power $-\beta_j - i$ by $-\varepsilon - i$, where $\varepsilon > 0$, and Lemma 2 allows the estimate

$$|(D^iT^j\kappa)(t)| \leqslant a_{i,\varepsilon}\left[t^{-\varepsilon-i+1-\alpha} + (b-t)^{-\varepsilon-i+1-\alpha}\right].$$

Combining these two estimates we get the estimate

$$|(D^iT^j\kappa)(t)| \leqslant a_{i,j,\varepsilon}\left[t^{-i+\min\{1-\alpha-\varepsilon,j-(j+1)\alpha\}} + (b-t)^{-i+\min\{1-\alpha-\varepsilon,j-(j+1)\alpha\}}\right],$$
$$i = 0, 1, \ldots, m-1, \quad j = 0, 1, 2, \ldots.$$

Taking here $j = l \geqslant 1/(1-\alpha-\varepsilon)$, we obtain that

$$|(D^i(DT^l)\kappa)(t)| \leqslant a_{i+1,l,\varepsilon}\left[t^{-i-\alpha-\varepsilon} + (b-t)^{-i-\alpha-\varepsilon}\right], \qquad i = 0, 1, \ldots, m-2,$$

which allows us to use Lemma 2 again and, by means of arguments as above, we get that

$$|(D^iT^j(DT^l)\kappa)(t)| \leqslant a_{i,j,l,\varepsilon}\left[t^{-i+r_{j,\alpha,\varepsilon}} + (b-t)^{-i+r_{j,\alpha,\varepsilon}}\right],$$

where $i = 0, 1, \ldots, m-2, j = 0, 1, 2, \ldots$ and

$$r_{j,\alpha,\varepsilon} = \min\{1-\alpha-\varepsilon, j-(j+1)(\alpha+\varepsilon)\}. \tag{4.8}$$

Again taking $j = l \geqslant 1/(1-\alpha-\varepsilon)$, we get

$$\left|\left(D^i(DT^l)^2\kappa\right)(t)\right| \leqslant a_{i+1,l,l,\varepsilon}\left[t^{-i-\alpha-\varepsilon} + (b-t)^{-i-\alpha-\varepsilon}\right], \qquad i = 0, 1, \ldots, m-3,$$

and so on. The result is formulated as follows.

LEMMA 3. *Let the conditions* (2.3) *be fulfilled, and the inequalities* $j \geqslant 0$, $i \geqslant 0$, $n \geqslant 0$, $i + n \leqslant m - 1$ *and* $l \geqslant 1/(1-\alpha-\varepsilon)$, *with an* $\varepsilon > 0$, *where* $\varepsilon + \alpha < 1$, *be valid. Then*

$$\left|\left(D^iT^j(DT^l)^n\kappa\right)(t)\right| \leqslant a_{i,j,l,n,\varepsilon}\left[t^{-i+r_{j,\alpha,\varepsilon}} + (b-t)^{-i+r_{j,\alpha,\varepsilon}}\right], \tag{4.9}$$

where $r_{j,\alpha,\varepsilon}$ *is defined by* (4.8) *and* $a_{i,j,l,n,\varepsilon}$ *is a positive constant.*

All the derivatives to the left of inequality (4.9) *exist in the classical sense. The same estimate holds if we replace* κ *by* κ_1, *where* $\kappa_1(t) = \kappa(b-t)$.

If we take $i = j = 0$ in inequality (4.9), then we get

$$\left|\left((DT^l)^n\kappa\right)(t)\right| \leqslant a_{0,0,l,n,\varepsilon}\left[t^{-\alpha-\varepsilon} + (b-t)^{-\alpha-\varepsilon}\right].$$

Therefore

$$(DT^l)^n\kappa \in L^1(0, b) \quad \text{and} \quad (DT^l)^n\kappa_1 \in L^1(0, b), \tag{4.10}$$

for $n = 0, 1, \ldots, m - 1$. In the following we consider the integer $l > 1/(1 - \alpha - \varepsilon)$ to be such that T^l maps $L^1(0, b)$ into $C[0, b]$. Therefore

$$T^l(DT^l)^n\kappa \in C[0, b] \quad \text{and} \quad T^l(DT^l)^n\kappa_1 \in C[0, b], \tag{4.11}$$

for $n = 0, 1, \ldots, m - 1$.

5. The first derivative of the solution

LEMMA 4. *Let the kernel κ satisfy the conditions* (2.3) *and let the homogeneous equation $u_0 = Tu_0$ have in the space $C[0, b]$ only the trivial solution $u_0 = 0$. Let $f \in C[0, b]$ and $Df \in L^1(0, b)$. Then the solution u of the equation*

$$u = Tu + f \tag{5.1}$$

has an integrable derivative $v = Du \in L^1(0, b)$ which satisfies the equation

$$v = Tv + Df + u(0)\kappa - u(b)\kappa_1, \qquad \kappa_1(t) = \kappa(b - t). \tag{5.2}$$

PROOF. As mentioned previously, the operator T is completely continuous in spaces C and L^p, $1 \leqslant p < \infty$. If, for $u_0 \in L^p$, the equation $u_0 = Tu_0$ is valid, then $u_0 = T^ju_0$, $j = 1, 2, \ldots$, from which we conclude that $u_0 \in C$ and, according to the assumption of the lemma, $u_0 = 0$. Thus equation (5.1) has a unique solution in both spaces C and L^p, and the solution belongs to the same space as the inhomogeneous term. Hence equation (5.1) has a unique solution $u \in C[0, b]$ and equation (5.2) has a unique solution $v \in L^1(0, b)$. If it were known for the solution u of equation (5.1) that $Du \in L^1$ then, according to Lemma 1, $v = Du$ would satisfy equation (5.2). Approximating the kernel κ by smooth kernels, it is easy to show that Du really belongs to L^1. We omit details. Thus Lemma 4 is proved.

6. Proof of Theorem 1

Let the assumptions of Theorem 1 be fulfilled and let $1 \leqslant k \leqslant m$. Put $S = I + T + T^2 + \cdots + T^{l-1}$ and $I = T^0$; concerning the choice of l, see the

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last part of Section 4. For the solution u of equation (2.1) we shall derive the formula

$$\begin{aligned} D^k u = {} & D^{k-1}S\big[u(0)\kappa - u(b)\kappa_1\big] \\ & + D^{k-2}S\big[u(0)(DT^l)\kappa - u(b)(DT^l)\kappa_1 + u_1(0)\kappa - u_1(b)\kappa_1\big] \\ & + D^0 S\Big[u(0)(DT^l)^{k-1}\kappa - u(b)(DT^l)^{k-1}\kappa_1 + u_1(0)(DT^l)^{k-2}\kappa \\ & \qquad - u_1(b)(DT^l)^{k-2}\kappa_1 + \cdots + u_{k-1}(0)\kappa - u_{k-1}(b)\kappa_1\Big] \\ & + u_k, \end{aligned} \tag{6.1}$$

where $u_1, u_2, \ldots, u_k$ are continuous functions over $[0, b]$ introduced in the course of discussions and $\kappa_1(t) = \kappa(b - t)$.

We start from the fact that $Du \in L^1$ and Du satisfies the equation

$$Du = TDu + f' + u(0)\kappa - u(b)\kappa_1$$

(see Lemma 4). We introduce the function u_1 so that

$$Du = S\big[u(0)\kappa - u(b)\kappa_1\big] + u_1 \tag{6.2}$$

is valid. Then u_1 satisfies the equation $u_1 = Tu_1 + f_1$, where $f_1 = f' + u(0)T^l\kappa - u(b)T^l\kappa_1$. According to the conditions (4.10) and (4.11), we get that $f_1 \in C$ and $Df_1 \in L^1$, and using Lemma 4 results in $u_1 \in C$, $Du_1 \in L^1$ and

$$Du_1 = TDu_1 + f'' + u(0)(DT^l)\kappa - u(b)(DT^l)\kappa_1 + u_1(0)\kappa - u_1(b)\kappa_1$$

being valid. Now we introduce the function u_2 so that

$$Du_1 = S\big[u(0)(DT^l)\kappa - u(b)(DT^l)\kappa_1 + u_1(0)\kappa - u_1(b)\kappa_1\big] + u_2 \tag{6.3}$$

is valid. Then u_2 satisfies the equation $u_2 = Tu_2 + f_2$ where

$$f_2 = f'' + T^l\big[u(0)(DT^l)\kappa - u(b)(DT^l)\kappa_1 + u_1(0)\kappa - u_1(b)\kappa_1\big].$$

Again by means of (4.10) and (4.11) we see that $f_2 \in C$ and $Df_2 \in L^1$, and again using Lemma 4 we get $u_2 \in C$, $Du_2 \in L^1$ and

$$\begin{aligned} Du_2 = {} & TDu_2 + (DT^l)\big[u(0)(DT^l)\kappa - u(b)(DT^l)\kappa_1 + u_1(0)\kappa - u_1(b)\kappa_1\big] \\ & + u_2(0)\kappa - u_2(b)\kappa_1 \end{aligned}$$

and so on. At the kth stage, according to the member Du_{k-1} we introduce the function u_k so that

$$\begin{aligned} Du_{k-1} = S\Big[& u(0)(DT^l)^{k-1}\kappa - u(b)(DT^l)^{k-1}\kappa_1 \\ & + u_1(0)(DT^l)^{k-2}\kappa - u_1(b)(DT^l)^{k-2}\kappa_1 \\ & + \cdots + u_{k-1}(0)\kappa - u_{k-1}(b)\kappa_1\Big] + u_k \end{aligned} \tag{6.4}$$

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is valid. Then u_k satisfies the equation $u_k = Tu_k + f_k$ where

$$f_k = f^{(k)} + T'\Big[u(0)(DT')^{k-1}\kappa - u(b)(DT')^{k-1}\kappa_1 + \cdots + u_{k-1}(0)\kappa - u_{k-1}(b)\kappa_1\Big].$$

Again according to (4.10) and (4.11) we get that $f_k \in C$ and thus also $u_k \in C$. Now we gradually establish by means of connections (6.2) to (6.4) that

$$\begin{aligned} D^k u &= D^{k-1}S\big[u(0)\kappa - u(b)\kappa_1\big] + D^{k-1}u_1 \\ &= D^{k-1}S\big[u(0)\kappa - u(b)\kappa_1\big] \\ &\quad + D^{k-2}S\big[u(0)(DT')\kappa - u(b)(DT')\kappa_1 + u_1(0)\kappa - u_1(b)\kappa_1\big] \\ &\quad + D^{k-2}u_2, \end{aligned}$$

and so on. Finally we get formula (6.1).

From (6.1), by means of Lemma 3, we get that $D^k u = u^{(k)}$ is continuous over $(0, b)$ and statement (2.6) holds.

Really, the dominant member among members with singularities to the right of the equality (6.1) is

$$u(0)D^{k-1}\kappa - u(b)D^{k-1}\kappa_1. \tag{6.5}$$

The next rival member is

$$u(0)D^{k-1}T\kappa - u(b)D^{k-1}T\kappa_1.$$

According to Lemma 3 its bound is

$$(\text{constant})\big[t^{-(k-1)+1-2\alpha-2\varepsilon} + (b-t)^{-(k-1)+1-2\alpha-2\varepsilon}\big]. \tag{6.6}$$

If $\varepsilon > 0$ is small enough, then it follows from $\alpha - (1-\alpha) < \alpha_0$, see (2.5), that

$$-(k-1) - \alpha_0 < -(k-1) + 1 - 2\alpha - 2\varepsilon.$$

Thus the singularities in (6.6) are milder than the ones of

$$|\kappa^{(k-1)}(t)| \geqslant \gamma_k^0 t^{-(k-1)-\alpha_0}$$

and

$$|\kappa^{(k-1)}(b-t)| \geqslant \gamma_k^0 (b-t)^{-(k-1)-\alpha_0},$$

see (2.4). Therefore the member (6.5) is the dominant member in (6.1); the other members to the right in (6.1) have still milder singularities. The proof of Theorem 1 is completed.

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