J. Aust. Math. Soc. 77 (2004), 335–348

HOMOMORPHISMS OF NONZERO DEGREE BETWEEN PD_n-GROUPS

JONATHAN A. HILLMAN

(Received 21 March 2003; revised 6 August 2003)

Communicated by S. Gadde

Abstract

We give algebraic proofs of some results of Wang on homomorphisms of nonzero degree between aspherical closed orientable 3-manifolds. Our arguments apply to PD_n-groups which are virtually poly-Z or have a Kropholler decomposition into parts of generalized Seifert type, for all n.

2000 *Mathematics subject classification*: primary 20J05; secondary 57N10, 57P10.
Keywords and phrases: atoroidal, cohopficity, degree, Kropholler decomposition, Poincaré duality group, poly-Z, Seifert type, volume condition.

1. Introduction

Aspherical closed 3-manifolds may be partitioned into eight classes, according to the nature of the geometric pieces of the JSJ decomposition. Wang defined a directed graph Γ whose vertices correspond to these classes and which has an edge whenever every manifold in the target class is the image of a map of nonzero degree from some manifold in the source class [21]. We shall give purely algebraic proofs for the cases when the atoroidal parts of the domain are of Seifert type, and our arguments apply to Poincaré duality groups in all dimensions. (In higher dimensions we partition PD_n-groups into ten classes, in terms of properties of the Kropholler decomposition [13].) In many cases we may find degree 1 homomorphisms between such groups. We shall also comment briefly on some related issues considered by Wang and others: cohopficity [7, 22, 23], the volume condition [24] and commensurability [16, 17]. However our observations here are confined to PD_n-groups which are either virtually poly-Z or of generalized Seifert type.

2. PD_n-groups and group pairs

Let G', ζG, $\sqrt{G}$ and $E(G)$ be the commutator subgroup, centre, Hirsch-Plotkin radical and maximal normal elementary amenable subgroup of the group G, respectively. A group G *virtually* has some property (inherited by subgroups of finite index) if it has a subgroup of finite index with that property. If G is virtually solvable let $h(G)$ be its Hirsch length. If S is a subset of G the normal closure of S in G is $\langle\langle S\rangle\rangle$, the intersection of the normal subgroups containing S.

A group G is a *PD_n-group* if it is FP, $H^p(G;\mathbb{Z}[G]) = 0$ for $p \neq n$ and $H^n(G;\mathbb{Z}[G]) \cong Z$. The 'dualizing module' $D = H^n(G;\mathbb{Z}[G])$ is a right $\mathbb{Z}[G]$-module; let $w_G : G \to \mathrm{Aut}(D) = \{\pm 1\}$ be the 'orientation character' determining the action. The group G is *orientable* (or is a *PD_n^+-group*) if $w_G = 1$, that is, if $D \cong \mathbb{Z}$, the augmentation module. (See [1].) If $\phi : G \to H$ is a homomorphism of PD_n-groups such that $w_G = w_H\phi$ the *degree* of ϕ is the induced homomorphism $\deg\phi : H_n(G;\mathbb{Z}^{w_G}) \to H_n(H;\mathbb{Z}^{w_H})$. (If $w_G \neq w_H\phi$ then we set $\deg\phi = 0$.) The absolute value $|\deg\phi|$ is independent of the choice of generators ('orientations') for these groups.

There is a related notion of PD_n-pair $(G, \mathcal{T})$, in which the set of 'boundary components' $\mathcal{T}$ is a finite set of conjugacy classes of embeddings of subgroups (possibly repeated, as in the example given by $(Z, \{Z, Z\})$, the fundamental group system of the annulus $S^1 \times [0, 1]$). If $\mathcal{T}$ is nonempty then $c.d.G = n - 1$ and each $H \in \mathcal{T}$ is a PD_{n-1}-group [2]. (In [4] the notion of PD_n-pair is reformulated in terms of pairs (G, Ω), where Ω is a suitable G-set.)

A PD_n-group pair $(G, \mathcal{T})$ is of *(generalized) Seifert type* if G has a normal, virtually poly-Z subgroup N of Hirsch length $h(N) = n - 2$. The elements of $\mathcal{T}$ are then represented by virtually poly-Z subgroups of Hirsch length $n - 1$. (See the second paragraph of the proof of Theorem 1 below.) The pair $(G, \mathcal{T})$ is *atoroidal* if every virtually poly-Z subgroup of Hirsch length $n - 1$ is conjugate to a subgroup of some element of $\mathcal{T}$. In particular, if G is an atoroidal PD_n-group (with $\mathcal{T}$ empty) then it has no normal poly-Z subgroup of Hirsch length $\geq n - 2$. If G has *max-c* (the property that every strictly increasing sequence of centralizers is finite) there is a reduced G-tree Y such that $G\backslash Y$ is finite, every edge stabilizer is virtually poly-Z of Hirsch length $n - 1$, each such subgroup lies in some vertex stabiliser G_v and there are natural finite families of subgroups $\mathcal{T}_v$ such that $(G_v, \mathcal{T}_v)$ are PD_n^+-pairs of either Seifert or atoroidal type [13]. We shall refer to this as the *Kropholler decomposition* of $(G, \mathcal{T})$. (Kropholler uses the terminology of [4].)

If G is virtually of Seifert type and is not virtually poly-Z it has a normal subgroup H of finite index such that $E(H)$ is virtually poly-Z and $H/E(H)$ is a PD_2-group. Since $E(H)$ is characteristic in H it is normal in G and so G is itself of Seifert type. If G is virtually of Seifert type and virtually poly-Z it is virtually nilpotent, if $n \leq 3$, and is either virtually nilpotent or of Seifert type if $n = 4$ or 5. However there are torsion free virtually poly-Z groups of Hirsch length ≥ 6 which are neither virtually nilpotent nor of Seifert type. (The simplest examples are perhaps the semidirect products $(Z^2 \times G_6) \rtimes_{(\alpha,\beta)} Z$, where G_6 is the Hantzsche-Wendt flat 3-manifold group, $\alpha \in SL(2, \mathbb{Z})$ has infinite order and $\beta \in \mathrm{Aut}(G_6)$ is such that $G_6 \rtimes_\beta Z$ is not of Seifert type, see [11, Section 8.7].)

We shall use frequently the observation that if $G \leq H$ are PD_n-groups then $[H : G]$ is finite [20] and $\chi(G) = [H : G]\chi(H)$. In particular, if a group J is virtually a PD_n-group, so that it has a subgroup G of finite index which is a PD_n-group, the rational Euler characteristic $\chi(J) = \chi(G)/[J : G]$ is well-defined.

LEMMA 1. *Let G and H be PD_n-groups and $\phi : G \to H$ a homomorphism such that* $\deg\phi \neq 0$. *Then $[H : \phi(G)]$ is finite and $c.d.\,\mathrm{Ker}(\phi) < n$.*

PROOF. Let $w = w_H$. Since ϕ factors through $\phi(G)$, $H_n(\phi(G); \mathbb{Z}^w) \neq 0$ and so $c.d.\phi(G) = n$. Therefore $[H : \phi(G)]$ is finite, by [20]. Since $[G : \mathrm{Ker}(\phi)] = |\phi(G)|$ is infinite, the same result implies that $c.d.\,\mathrm{Ker}(\phi) < n$. □

In general, finiteness of the index $[H : \phi(G)]$ does not imply that $\deg\phi \neq 0$. For instance, if $\pi = F(a, b) *_Z F(x, y)$ is the fundamental group of the orientable surface of genus 2, with presentation $\langle a, b, x, y \mid [a, b] = [x, y]\rangle$, the epimorphism $p : \pi \to \pi/\langle\langle b, y, [a, x]\rangle\rangle \cong Z^2$ factors through the free group $F(a, x)$ and therefore has degree 0.

LEMMA 2. *Let G and H be PD_n-groups and $\phi : G \to H$ a monomorphism. Then $[H : \phi(G)]$ is finite and $|\deg\phi| = [H : \phi(G)]$.*

PROOF. The first assertion follows from [20]. It follows also that the restriction of $w = w_H$ to G is w_G. The restriction homomorphism $\mathrm{Res} : H_n(H; \mathbb{Z}^w) \to H_n(G; \mathbb{Z}^w)$ is an isomorphism (see [1, Section 5.3]). As $\deg\phi\,\mathrm{Res}$ is multiplication by the index the second assertion holds also. □

Theorem 9.11 of [1] implies that if $\mathrm{Ker}(\phi) \neq 1$ then it is not FP_{n-1}.

3. Wang's partition

Define ten classes of PD_n-groups as follows

(1) G is atoroidal;
(2) one of the vertex terms (G_v, Ω_v) of the Kropholler decomposition is atoroidal;
(3) the vertex terms of the Kropholler decomposition are of Seifert type, but G is neither of Seifert type nor virtually poly-Z;

(4a) G is virtually poly-Z, but is not virtually of Seifert type;
(4b) G is virtually a product $E \times Z^2$, where E is poly-Z but not virtually nilpotent;
(4c) G is virtually poly-Z and virtually of Seifert type but is not virtually nilpotent nor virtually a product with Z^2;
(5) G is virtually a product $S \times \pi$ of a poly-Z group S with $h(S) = n - 2$ and a surface group π with $\chi(\pi) < 0$;
(6) G is of Seifert type, but is neither virtually such a product nor virtually poly-Z;
(7) G is virtually nilpotent, but not virtually abelian;
(8) G is virtually abelian.

These classes are disjoint, and their union contains all PD_n-groups G such that G has *max-c* [13]. (All 3-manifold groups have this property [14].) The classification is also stable under passage to subgroups of finite index. Classes (4a), (4b), (4c), (7) and (8) consist of the virtually solvable PD_n-groups, and class (4b) is empty if $n \leq 4$, while class (4c) is empty if $n \leq 3$. (It is occasionally convenient to treat

$$(4) = (4a) \cup (4b) \cup (4c)$$

as a single class. The classification could be refined further by considering higher-dimensional geometries.) When $n = 3$ the other eight classes correspond to the eight classes of [21], and we may describe them more explicitly

(1) G is atoroidal (that is, G has no free abelian subgroup of rank 2);
(2) $G = \pi(\mathcal{G})$, where $\mathcal{G}$ is a finite graph of groups with at least one edge, $(G_v, \amalg_{v\in\partial e} G_e)$ is a PD_3-pair for all vertices v, $G_e \cong Z^2$ or $Z \tilde{\times} Z$ for all edges e, and $(G_v, \amalg_{v\in\partial e} G_e)$ is atoroidal, for at least one vertex v;
(3) $\sqrt{G} = 1$ and $G = \pi(\mathcal{G})$, where $\mathcal{G}$ is a finite graph of groups such that $(G_v, \amalg_{v\in\partial e} G_e)$ is a PD_3-pair and $\sqrt{G_v} \neq 1$, for all vertices v;
(4) $\sqrt{G} \cong Z^2$;
(5) $\sqrt{G} \cong Z$ and G is virtually a product;
(6) $\sqrt{G} \cong Z$ but G is not virtually a product;
(7) G is virtually nilpotent, but not virtually abelian;
(8) $\sqrt{G} \cong Z^3$ (that is, G is virtually abelian).

Class (1) contains the fundamental groups of closed hyperbolic 3-manifolds and class (2) contains the groups of aspherical closed 3-manifolds with a nontrivial characteristic variety such that at least one component of the complement is hyperbolic. It follows from [10] that class (3) consists of the groups of aspherical graph manifolds (closed 3-manifolds with a nontrivial characteristic variety in which every component of the complement is Seifert fibred), excepting $\mathbb{S}$ol-manifolds, whose groups are the members of class (4). (It is easy to see that the edge groups G_e in a group of class (3) are all isomorphic to Z^2 or $Z \tilde{\times} Z$. For $c.d.G_v = 2$ [2] and so the edge groups G_e with $v \in \partial e$ all meet $\sqrt{G_v}$ nontrivially.) Classes (5), (6), (7) and (8) consist of the fundamental groups of $\mathbb{H} \times \mathbb{E}$-, $\widetilde{\mathbb{SL}}$-, $\mathbb{N}$il- and flat 3-manifolds, respectively [3]. (For groups G with subgroups of finite index with $\beta_1 > 0$ the simpler argument of [10] may be used instead of [3].) It is not known whether classes (1) and (2) contain any groups which are not 3-manifold groups, nor whether atoroidal 3-manifolds are hyperbolic.

Let Γ_n be the directed graph with vertices $\{1, 2, 3, 4a, 4b, 4c, 5, 6, 7, 8\}$ and with an edge (i, j) if and only if $i \neq j$ and for every group H in class (j) there is a PD_n-group G in class (i) and a homomorphism of nonzero degree from G to H. Wang showed that the corresponding graph Γ for the fundamental groups of aspherical 3-manifolds has edges $(1, n)$ and $(2, n)$ for all n, $(3, m)$ for all $m \geq 3$, $(5, 8)$ and $(6, 7)$. Moreover any homomorphism between 3-manifold groups in classes not connected by an edge in Γ has degree 0 [21]. We shall give algebraic arguments for these results, excepting the existence of edges $(1, 2)$, $(2, 1)$ and $(1, 3)$ and the nonexistence of an edge $(3, 2)$. In verifying these assertions we may pass to subgroups of finite index whenever convenient. In particular, we may assume that all Poincaré duality groups considered are orientable, and hence that the edge groups in graph-of-groups splittings (as in cases (2) and (3)) are also orientable. We may also assume without loss of generality that the vertex groups are nonabelian, for if G_v is abelian then $\{G_e \mid v \in \partial e\}$ has two members, and the inclusions are isomorphisms. (Note however that PD_3-groups of class (4) are virtually HNN extensions with base Z^2.)

We shall see that when $n = 4$ the only nontrivial edges emanating from any of the last five vertices of Γ_4 are $(5, 4c)$, $(5, 8)$ and $(6, 7)$. If $n > 4$ then (4a), (4b), (4c), (7) and (8) are terminal vertices of Γ_n, but there are also edges $(5, 4b)$, $(5, 8)$, $(6, 4c)$ and $(6, 7)$. (Also many groups in class (7) are degree 1 quotients of groups in class (5).)

That Γ has edges $(1, n)$ with $n \geq 3$ follows as in [21] from the fact that every closed orientable manifold has a 2-fold branched cover which is the mapping torus of a pseudo-Apanasov diffeomorphism. The same argument would give an edge $(1, 2)$ if all groups of class (2) were 3-manifold groups. We refer to [21] for the existence of an edge $(2, 1)$ in the 3-manifold case.

4. Homomorphisms with domain a graph of groups

Let $(G, \mathscr{T})$ be a PD_3^+-pair with $\zeta G \cong Z$ and $\mathscr{T} \neq \emptyset$. Then $G \cong F(r) \rtimes Z$, by [1, Theorem 8.8 and Corollary 8.6]. Hence G has a presentation

$$\langle x_1, \dots x_r, t \mid t x_i t^{-1} = \alpha(x_i) \rangle,$$

for some $\alpha \in \mathrm{Aut}(F(r))$. Let $\tilde{G} = \langle x_1, \dots x_r, y, t \mid t x_i t^{-1} = \alpha(x_i), ty = yt \rangle$ and $\widehat{G} = \tilde{G} *_{Z^2} \pi 4_1$, where t and $y \in \tilde{G}$ are identified with a meridian and longitude in the figure-eight knot group $\pi 4_1$, respectively. Then $(\widehat{G}, \mathscr{T})$ is a PD_3-pair, and the natural epimorphism from $\hat{G}$ to $G \cong \hat{G}/\langle\langle (\pi 4_1)' \rangle\rangle$ induces a degree 1 map of pairs. It follows easily that Γ has an edge $(2, 3)$.

If G is a PD_3-group in class (4) we may assume that $\tilde{G} \cong Z^2 \times_\theta Z$, where $\theta \in SL(2, \mathbb{Z})$ has infinite order. The automorphism θ lifts to an automorphism Θ of the free group $F(x, y)$, such that $\Theta([x, y]) = [x, y]$, by a theorem of Nielsen—see [15, Section 3.5]. Let

$$(G_1, Z^2) = (F(a, b) \times Z, Z[a, b] \times Z) \quad \text{and}$$
$$(G_2, Z^2) = (F(x, y) \times_\Theta Z, Z[x, y] \times Z).$$

Then $G_1 *_{Z^2} G_2$ is a group in class (2) which maps onto $\tilde{G}$ via a degree 1 homomorphism. Hence Γ has an edge (2, 4). (We shall see below that Γ has an edge (3,4), so composition gives an edge (2,4), but this construction is simpler.)

The argument excluding (3, 1) as an edge applies in all dimensions.

THEOREM 1. *Let G and H be PD_n-groups such that the vertex terms of the Kropholler decomposition of $G = \pi(\mathcal{G})$ are of Seifert type and H is atoroidal. If $\phi : G \to H$ is a homomorphism then* $\deg \phi = 0$.

PROOF. Let $s_e : G_e \to G_{s(e)}$ and $t_e : G_e \to G_{t(e)}$ be the inclusions of the edge group G_e into the vertex groups corresponding to the source and target of the edge e. We may assume that G is orientable. Let $\overline{\mathcal{G}}$ be the graph of groups with the same underlying graph as $\mathcal{G}$ and with $\overline{G}_v = \phi(G_v)$ and $\overline{G}_e = \phi(G_e)$, for all vertices v and edges e. Then ϕ factors as $\phi = \bar{\phi}\rho$, where $\rho : G \to \overline{G} = \pi(\overline{\mathcal{G}})$ and $\bar{\phi} : \overline{G} \to H$. For each vertex v let $\overline{\mathcal{T}}_v = \{\overline{G}_e \mid s(e) = v\} \amalg \{\overline{G}_e \mid t(e) = v\}$.

If G_v is a vertex group then $c.d.G_v = n - 1$ and so $G_v/E(G_v)$ is virtually free, by [1, Theorem 8.4]. Hence $G_e/G_e \cap s_e^{-1}(E(G_{s(e)}))$ and $G_e/G_e \cap t_e^{-1}(E(G_{t(e)}))$ are also virtually free, for all edges e. Since $c.d.G_e = n - 1$ it follows that $h(G_e \cap E(G_v)) = n - 2$, and since G_e is a PD_{n-1}-group it must in fact be virtually poly-Z. Let $K_v = \mathrm{Ker}(\phi|_{E(G_v)})$. Then either $K_v \neq 1$ or $[\phi(G_v) : \phi(E(G_v))]$ is finite, since H is atoroidal and torsion-free. In the latter case $\phi(G_v)$ is torsion-free and virtually poly-Z, and $h(\phi(G_v)) = n - 2$.

If $\phi(G_v)$ is virtually poly-Z and $h(\phi(G_v)) = n - 2$ for all vertices v then $c.d.\overline{G} \leq n - 1$, and so $\deg \phi = 0$. Suppose there is a vertex v such that $K_v \neq 1$. Let G_v^* be a normal subgroup of finite index in G_v which contains $E(G_v)$ and is such that $G_v^*/E(G_v)$ is a free group. Then conjugation by coset representatives for G_v/G_v^* determines lifts of the embeddings s_e and t_e to embeddings of $s_e(G_e) \cap G_v^*$ and $t_e(G_e) \cap G_v^*$ in G_v^*. Let $\mathcal{T}_v^*$ be the set of G_v^*-conjugacy classes of such lifts. Then $(G_v^*, \mathcal{T}_v^*)$ is a PD_n^+-pair, and the inclusion of G_v^* into G_v has degree $[G_v : G_v^*] \neq 0$ (by the relative version of Lemma 2).

Let $\hat{G}_v = G_v^*/K_v$ and let $\widehat{\mathcal{T}}_v$ be the corresponding set of embeddings of quotients of members of $\mathcal{T}_v^*$. Since $E(G_v)/K_v \cong \phi(E(G_v))$ it is torsion free, and so is a PD_m-group for some $m < h(E(G_v)) = n - 2$. Then $(\hat{G}_v, \widehat{\mathcal{T}}_v)$ is a PD_{m+2}-pair.

https://doi.org/10.1017/S1446788700014476 Published online by Cambridge University Press

Now $\rho|_{G_v^*}$ factors through this pair, and so the homomorphism from $H_n(G_v^*, \mathscr{T}_v^*; \mathbb{Z})$ to $H_n(\overline{G}_v, \overline{\mathscr{T}}_v; \mathbb{Z})$ induced by $\rho|_{G_v^*}$ is 0. This homomorphism is the top row of a commuting square whose bottom row is the homomorphism $H_n(G; \mathbb{Z}) \to H_n(\overline{G}; \mathbb{Z})$ induced by ρ. Since the inclusion of G_v^* into G induces an isomorphism $H_n(G_v^*, \mathscr{T}_v^*; \mathbb{Z}) \to H_n(G; \mathbb{Z})$ [2], it follows that $H_n(\rho; \mathbb{Z}) = 0$. Hence we again have $\deg \phi = 0$. □

COROLLARY 1. *There is no edge* $(3, 1)$ *in* Γ_n.

In the 3-manifold case it follows that there is no edge $(3, 2)$. Can the above argument be adapted to show this is true in general?

Let G be a torsion-free virtually poly-Z group of Hirsch length $n \geq 3$. Then G has a subgroup $\tilde{G}$ of finite index which is an extension of a free abelian group Z^r by a nilpotent normal subgroup N, by Mal'cev's Theorem (see [19, page 35]). If $r = 1$ then $\tilde{G} \cong N \rtimes_\alpha Z$, for some $\alpha \in \mathrm{Aut}(N)$. Since N is nilpotent and $h(N) \geq 2$ there is a subgroup $P < N$ containing N' and such that $N/P \cong Z^2$. Let $F = \langle w, x, y, z \mid w = xyz \rangle$ and $\hat{F} = \langle F, t \mid twt^{-1} = z \rangle$ be the fundamental groups of the quadruply punctured sphere and the twice punctured torus. Let θ be the automorphism of $\hat{F}$ defined by $\theta(f) = f$ for all $f \in F$ and $\theta(t) = zt$, and let

$$H = \langle \hat{F}, s \mid sgs^{-1} = \theta(g)\ \forall g \in \hat{F} \rangle \cong \hat{F} \rtimes_\theta Z.$$

(Thus H is the fundamental group of the mapping torus of the Dehn twist corresponding to θ.) Let $J \cong F \times Z$ be the subgroup of H generated by F and s, and let $J_f \cong Z^2$ be the subgroup of J generated by $\{f, s\}$, for all $f \in F$. Then H is also the HNN extension with base H, associated subgroups J_w and J_z, and stable letter t, since $tst^{-1} = z^{-1}s$ and $twt^{-1} = z$. Moreover $(J, \{J_w, J_x, J_y, J_z\})$ is a PD_3-pair of Seifert type, while $(H, \{J_x, J_y\})$ is a PD_3-pair with $\sqrt{H} = 1$. Let $\gamma : H \to H/\langle\langle, t, z \rangle\rangle \cong Z^2$ be the canonoical epimorphism. Then γ induces a degree 1 homomorphism from $(H, \{J_x, J_y\})$ to $(Z^2, \{Z^2, Z^2\})$. In particular, γ indices isomorphisms $J_x \cong N$ and $J_y \cong N$. Let M be the extension of H by P obtained by pullback over γ, and let M_x and M_y be the preimages in M of J_x and J_y, respectively. Let $\mu_x : M_x \cong N$ and $\mu_y : M_y \cong N$ be the isomorphisms determined by θ. The HNN extension G^* with base M, associated subgroups M_x and M_y and stable letter u acting via $umu^{-1} = \mu_y^{-1}\alpha\mu_x(m)$ for all $m \in M_x$ is a PD_n-group in class (3), and the projection onto $G^*/\langle\langle x, y \rangle\rangle \cong \tilde{G}$ has degree 1. Thus Γ_n has an edge (3, 4a). If $r \geq 2$ then $\tilde{G}$ is of Seifert type, and we shall treat this case in the next paragraph.

Let H be an extension of a PD_2^+-group π of genus g by a torsion-free virtually poly-Z group E of Hirsch length $n - 2$. Let F be the free group with basis $\{a_i, b_i \mid 1 \leq i \leq g\}$ and let $\Pi = \Pi[a_i, b_i]$. Then $\pi \cong F/\langle\langle \Pi \rangle\rangle$ and $H \cong E \rtimes_\alpha F/\langle\langle \Pi w^{-1} \rangle\rangle$, for some homomorphism $\alpha : F \to \mathrm{Aut}(E)$ and element $w \in E$ such that $\alpha(\Pi)(e) = wew^{-1}$. Let $H_1 = E \rtimes_\alpha F$ and let ∂H_1 be the subgroup generated by E and Π. Then

$\partial H_1 \cong E \times Z$, where the second factor is generated by Πw^{-1}. Let $H_2 = E \times F(r, s)$ and let $\partial H_2 = E \times Z$, where the infinite cyclic factor is generated by $[r, s]$. Then $(H_1, \partial H_1)$ and $(H_2, \partial H_2)$ are PD_n-pairs, with isomorphic boundary terms. Define an isomorphism $\psi : \partial H_1 \to \partial H_2$ by $\psi(e) = e[r, s]$ for $e \in E$ and $\psi(\Pi) = [r, s]w$. If we identify the boundaries via $e \mapsto e$ for all $e \in E$ and Πw^{-1} maps to $[r, s]$ we obtain a PD_n-group G with a nontrivial Kropholler decomposition of Seifert type, but which is not itself of Seifert type. The canonical epimorphism from G to $G/\langle\langle r, s\rangle\rangle \cong H$ has degree 1. Since every PD_n-group of Seifert type has such a subgroup H of finite index it follows that Γ_n has edges (3, 4b), (3, 4c), (3, 5) and (3, 6). When $n = 3$ this construction can be adapted to show that every group in class (5) or (6) is the degree 1 quotient of a group in class (3).

5. Homomorphisms with solvable or Seifert domain

If G is a PD_n-group then $E(G)$ is virtually solvable, by [11, Theorem 1.11]. If $E(G) \neq G$ then $h(E(G)) \leq n - 2$. (Suppose that $h(E(G)) \geq n - 1$. If $c.d.E(G) = n - 1$ then $c.d.E(G) = h(E(G))$, so $E(G)$ is a duality group and has a finite $K(E(G), 1)$ complex [12]. A spectral sequence argument then shows that $G/E(G)$ has two ends. Otherwise $c.d.E(G) = n$, so $[\pi : E(G)]$ is finite, by [20]. In either case $G/E(G)$ is virtually solvable, and so $G = E(G)$. See [11, Theorem 8.1] for the case $n = 4$.) If $n = 3$ and $E(G) \neq G$ then $E(G) = \sqrt{G} \cong Z$ or 1 (see [11, Section 2.7]). If $n = 4$ and $h(E(G)) = 2$ then $E(G) \cong Z^2$ or $Z \tilde{\times} Z$ and $G/E(G)$ is virtually a PD_2-group (see [11, Theorems 9.1 and 9.2]). To what extent can this be generalized? If there is a finite $K(E(G), 1)$ complex and $h(E(G)) = n-2$ then the LHS spectral sequence collapses to give $H^{n-2}(E(G); \mathbb{Z}[E(G)]) \cong H^2(G/E(G); \mathbb{Z}[G/E(G)]) \cong Z$, so $E(G)$ is virtually poly-Z and $G/E(G)$ is virtually a PD_2-group [3]. If $G/E(G)$ is virtually a PD_2-group must $E(G)$ be virtually poly-Z?

LEMMA 3. *Let G and H be PD_n-groups and $\phi : G \to H$ a homomorphism such that* $\deg \phi \neq 0$. *If K is a virtually poly-Z normal subgroup of G and $v.c.d.(G/K)$ is finite then $\phi|_K$ is a monomorphism and $\phi(K) \leq E(H)$. In particular, if G is virtually poly-Z then so is H, and G is virtually abelian (respectively, nilpotent) if and only if H is.*

PROOF. The image $\phi(K)$ is torsion-free and virtually poly-Z, since H is torsion free and K is virtually poly-Z. If $\mathrm{Ker}(\phi|_K) \neq 1$ then ϕ factors through $\bar{G} = G/\mathrm{Ker}(\phi|_K)$, and $c.d.\bar{G} = h(\phi(K)) + v.c.d.(G/K) < n = c.d.G = h(K) + v.c.d(G/K)$, by [1, Theorem 5.6], since K and $\phi(K)$ are FP and $v.c.d.(G/K)$ is finite. Thus ϕ is a monomorphism if $\deg \phi \neq 0$. Since $\phi(E(G))$ is a characteristic subgroup of G and

$\phi(G)$ has finite index in H it follows easily that $\phi(E(G)) \leq E(H)$. In particular, $\phi(K) \leq E(H)$. The final assertion is clear. □

COROLLARY 2. *There are no edges emanating from the vertices* (4a), (4b), (4c), (7) *or* (8) *in* Γ_n.

LEMMA 4. *Let G and H be PD_n-groups and $\phi : G \to H$ a homomorphism such that* $\deg \phi \neq 0$, *and suppose that $G/E(G)$ is virtually a PD_2-group. Then*

(i) *if G is virtually a product of a solvable group with a PD_2-group so is H.*

(ii) *G is virtually such a product if $\phi|_{E(G)}$ is a monomorphism and H is either virtually abelian or virtually a product of a solvable group with a nonsolvable PD_2-group.*

PROOF. Let $\tilde{G} \leq G$ be a subgroup of finite index in G such that $\phi(\tilde{G})$ is normal in H and $\pi = \tilde{G}/E(\tilde{G})$ is a PD_2-group. Note that $E(G/E(G)) = 1$, so π, $\tilde{G}$ and G are not solvable.

If $\tilde{G} \cong E(\tilde{G}) \times \pi$, where π is a PD_2-group, then $E(G)$ is a PD_{n-2}-group, and hence is virtually poly-Z, by [1, Theorems 9.11 and 9.23], respectively. Hence $\phi|_{E(G)}$ is a monomorphism, by Lemma 3. As $\phi(\tilde{G}) \cong E(G) \times \phi(\pi)$ has finite index in H the latter group is also virtually a product, and $\phi(\pi)$ is a PD_2-group.

If H is virtually abelian then on passing to subgroups of finite index we may assume that $\tilde{H}$ is abelian, hence free of finite rank, and that $\phi(E(\tilde{G}))$ is a direct factor of $\tilde{H}$. If H has a subgroup $\tilde{H} \cong S \times \sigma$ of finite index, with S solvable and σ a nonsolvable PD_2-group, then $S = E(\tilde{H})$, S is virtually poly-Z and $h(S) = n - 2$. Moreover $\phi(E(\tilde{G})) \leq S$. In either case composition of ϕ with projection onto a factor splits the inclusion of $E(\tilde{G}) \cap \phi^{-1}(\tilde{H})$ into $\phi^{-1}(\tilde{H})$, and so G is virtually such a product. □

If a PD_n-group is a nontrivial direct product its factors are PD_m-groups for suitable $m < n$.

COROLLARY 3. *Assume that $n = 3$ or 4. Then* (5, 4c), (5, 8) *and* (6, 7) *are the only edges emanating from the vertices* (5) *and* (6) *in* Γ_n.

PROOF. Let G and H be PD_n-groups such that $G/E(G)$ is virtually a PD_2-group and let $\phi : G \to H$ be homomorphism such that $\deg \phi \neq 0$. Then $E(H) \neq 1$, by Lemma 3, so H is not in classes (1), (2) or (3).

If $n = 3$ then $E(G) \cong Z$ and so $\phi(G)$ is virtually nilpotent. Hence H is virtually nilpotent. If $n = 4$ then $h(E(G)) = 2$, so $E(G) \cong Z^2$ or $Z \tilde{\times} Z$. Therefore if H is not virtually nilpotent it is of Seifert type and not virtually a product with Z^2. The other exclusions follow easily from Lemma 4. □

Groups in classes (7) and (8) are extensions of flat 2-orbifold groups by an infinite cyclic normal subgroup. Given such a group G we may construct a group of type (6) or (5), respectively, and a degree 1 homomorphism to G by pulling back the extension over an epimorphism corresponding to a degree 1 map from a hyperbolic 2-orbifold. (This construction could be paraphrased in purely algebraic terms, but at somewhat greater length.)

If $n > 4$ then (4a), (4b), (4c), (7) and (8) are terminal vertices of Γ_n, but there are also edges (5, 4b), (5, 8), (6, 4c) and (6, 7), and many groups in class (7) are degree 1 quotients of groups in class (5).

6. Endomorphisms and subgroups of finite index

A group G is *hopfian* if surjective endomorphisms of G are automorphisms, and is *cohopfian* if injective endomorphisms are automorphisms. The *volume condition* holds for G if whenever H_1 and H_2 are isomorphic subgroups of finite index then $[G : H_1] = [G : H_2]$. If G is a PD_n-group and satisfies the volume condition then G is cohopfian (since subgroups of infinite index in PD_n-groups cannot be PD_n-groups [20]). On the other hand, finitely generated nonabelian free groups satisfy the volume condition but are not cohopfian.

If $\pi = Z^2$ or $Z\tilde{\times}Z$ then π is hopfian, by Lemma 3. The hopficity of the other PD_2-groups follows from the next lemma, which is based on a variation of the argument given for [5, Theorem A].

LEMMA 5. *Let π, σ be PD_2-groups with $\chi(\sigma) \leq \chi(\pi) < 0$ and let $\theta : \pi \to \sigma$ be a homomorphism. Then the following are equivalent*:

(i) $H_1(\theta; \mathbb{F}_2)$ *is an epimorphism*;

(ii) $\theta^* w_\sigma = w_\pi$ *and* $\deg\theta \neq 0$;

(iii) θ *is an isomorphism.*

PROOF. If $H_1(\theta; \mathbb{F}_2)$ is an epimorphism it is an isomorphism, since $\chi(\pi) \geq \chi(\sigma)$, and so $\theta^* w_\sigma = w_\pi$ and $\deg\theta \neq 0$, by the nondegeneracy of Poincaré duality with coefficients $\mathbb{F}_2$ and the Wu relation $x^2 = x \cup w_G$, for $x \in H^1(G; \mathbb{F}_2)$ and $G = \pi$ and $G = \sigma$. In particular, $\sigma \cong \pi$. Hence $\theta(\pi)$ is not a free group, so $[\sigma : \theta(\pi)] < \infty$ and $\theta(\pi)$ is a PD_2-group [20]. Since

$$\beta_1(\theta(\pi); \mathbb{F}_2) \leq \beta_1(\pi; \mathbb{F}_2) = \beta_1(\sigma; \mathbb{F}_2) \quad \text{and}$$
$$\chi(\theta(\pi)) = [\sigma : \theta(\pi)]\chi(\sigma) < 0$$

it follows that $[\sigma : \theta(\pi)] = 1$, so θ is onto.

If $\theta^* w_\sigma = w_\pi$ and $\deg\theta \neq 0$ then $\sigma \cong \pi$, $[\sigma : \theta(\pi)] < \infty$ and θ is onto, as before.

We shall henceforth fix an isomorphism $\sigma \cong \pi$ and view θ as an endomorphism of π. If θ is onto then $H_1(\theta;\mathbb{Z})$ is an isomorphism, since $H_1(\pi;\mathbb{Z}) = \pi/\pi'$ is finitely generated. The induced homomorphism $\theta' : \pi' \to \pi'$ is also onto, and induces an onto endomorphism of π'/π''. The latter group is finitely generated as a module over the noetherian ring $\mathbb{Z}[\pi/\pi']$. Since the kernel K_n of the endomorphism induced by θ^n is a normal subgroup of π/π'' it is a $\mathbb{Z}[\pi/\pi']$-submodule. The increasing sequence of submodules K_n must stabilize, since π'/π'' is noetherian. Hence $K_n = 0$ for all n and so $H_1(\theta';\mathbb{Z})$ is also an isomorphism. Now $c.d.\pi' \leq 1$ [20] and so π' is free. Hence the endomorphisms induced on the nilpotent quotients $\pi'/\pi'_{[n]}$ by θ are isomorphisms for all $n \geq 1$ [19]. Hence $\mathrm{Ker}(\theta) \leq \bigcap_{n\geq 1} \pi'_{[n]}$. Since π' is free it is residually a finite p-group [15]. Therefore $\mathrm{Ker}(\theta) = \mathrm{Ker}(\theta') = 1$, so θ is an automorphism. Thus (i) and (ii) each imply (iii); the converse is clear. □

The fundamental groups of 3-manifolds with hyperbolic atoroidal parts are residually finite [9]. Hence they are hopfian, and so degree 1 self maps of such groups are automorphisms.

The cohopficity of surface groups other than Z^2 or $Z\tilde{\times}Z$ is an easy consequence of the multiplicativity of the Euler characteristic in finite extensions. This extends to all hyperbolic 2-orbifold groups.

Let ρ be a group which is virtually a PD_2-group and which has no nontrivial finite normal subgroup, and let κ be the normal subgroup of ρ generated by all its elements of finite order. Then ρ is the orbifold fundamental group of an aspherical 2-orbifold [6], and $\bar{\rho} = \rho/\kappa$ is the fundamental group of the surface obtained by deleting neighbourhoods of the singular points and replacing them with discs. Hence $\chi(\rho) \leq \chi(\bar{\rho})$, with equality only if $\kappa = 1$. (In particular, if $\chi(\rho) < 0$ then $\chi(\rho) \leq -1$.) We shall use this fact (for which we do not have a simple algebraic proof) in the following lemma.

LEMMA 6. *Let ρ, σ be groups which are virtually PD_2-groups and such that ρ has no nontrivial finite normal subgroup. If $\chi(\sigma) \leq \chi(\rho) < 0$ and $\theta : \rho \to \sigma$ is a homomorphism such that $[\sigma : \theta(\rho)] < \infty$ then θ is an isomorphism.*

PROOF. We may assume that σ is a PD_2-group and that θ is an epimorphism. Let κ be the normal subgroup of ρ generated by all its elements of finite order. Then θ factors through $\bar{\rho} = \rho/\kappa$, and $\chi(\sigma) \leq \chi(\rho) \leq \chi(\bar{\rho})$, by the observation in the above paragraph. Therefore θ induces an isomorphism $\bar{\rho} \cong \sigma$, by Lemma 5. Hence $\chi(\rho) = \chi(\bar{\rho})$, so ρ is a PD_2-group and θ is an isomorphism. □

In particular, ρ is both cohopfian and hopfian. Lemmas 3 and 5 together imply that virtually poly-Z groups and PD_n-groups of Seifert type are hopfian. On the other hand, it is easy to see that many groups in classes (4), (5), (7) and (8) are not cohopfian.

THEOREM 2. *Let G be a PD_n-group which is virtually poly-Z or is of Seifert type, and let $\theta : G \to G$ be an endomorphism such that* $\deg\theta \neq 0$. *Then θ is a monomorphism.*

PROOF. We may assume that G is of Seifert type and that $\theta|_{E(G)}$ is a monomorphism, by Lemma 3. The quotient $\bar{G} = G/E(G)$ is virtually a PD_2-group and has no nontrivial finite normal subgroup, and the induced homomorphism $\bar{\theta} : \bar{G} \to \bar{G}$ has image of finite index. Therefore $\bar{\theta}$ is an automorphism, by Lemma 6, and so θ is a monomorphism □

COROLLARY 4. *An endomorphism θ is an automorphism if and only if* $|\deg\theta| = 1$.

PROOF. If $|\deg\theta| = 1$ then θ is onto, by Lemma 2, and hence is an automorphism, by the theorem. The other implication is clear. □

In [18] it is shown that every sequence of degree 1 maps between geometric 3-manifolds eventually becomes a sequence of homotopy equivalences. This remains true for sequences of homomorphisms between PD_n-groups which contain a term from one of the classes (4)–(8), by Lemmas 3–4 and Corollary 4, and the fact that Euler characteristics of hyperbolic 2-orbifolds are bounded above by -1. Rong handles the other cases in dimension 3 using a measure of complexity based on the Gromov norm and the number of Seifert parts.

Wang showed that every endomorphism of nonzero degree of the group of an aspherical 3-manifold with hyperbolic atoroidal parts is a monomorphism, and established the cohopficity of such groups in classes (1), (2), (3) and (6). He uses the Gromov norm to handle classes (1) and (2); class (3) is the most demanding [22]. We shall verify only that PD_3-groups in class (6) are cohopfian.

THEOREM 3. *Let G be a PD_3-group such that $\sqrt{G} \cong Z$ and which is not virtually a product. Then G is cohopfian.*

PROOF. Let $\phi : G \to G$ be a monomorphism and let $\bar{G} = G/\sqrt{G}$. Then the induced endomorphism $\bar{\phi} : \bar{G} \to \bar{G}$ is an automorphism, by Theorem 2. Since $\sqrt{G} \cong Z$ the quotient $\bar{G}$ has a normal subgroup $\bar{H}$ of finite index which is a PD_2^+-group such that $\chi(\bar{H}) < 0$, and such that $\sqrt{G}$ is a central subgroup of the preimage $H \leq G$. Since the automorphism $\bar{\phi}$ permutes the finitely many (torsion-free) subgroups of $\bar{G}$ of index $[\bar{G} : \bar{H}]$, there is an $n \geq 1$ such that $\bar{\phi}^n(\bar{H}) = \bar{H}$. Hence $\phi^n(H) \leq H$. Such extensions are classified by elements $e \in H^2(\bar{H}; \mathbb{Z}) \cong \mathbb{Z}$, and it is not hard to see that we must have

$$e = \left[\sqrt{G} : \phi^n(\sqrt{G})\right] e.$$

Hence either $\sqrt{G} = \phi(\sqrt{G})$, so ϕ is an automorphism, or $e = 0$, in which case H is a product and G is in class (5). $\square$

The Euler class e is used in a similar way in [7, 22, 23]. It is easy to see that no PD_3-group in class (4) is cohopfian, and a similar result is proven in [7] for classes (5), (7) and (8). (See also [8] for cohopficity of groups of bounded 3-manifolds.) In higher dimensions, it is not clear whether there are any cohopfian PD_n groups in classes (4)–(8). (Central extensions of PD_2-groups by free abelian groups of rank > 1 are never cohopfian, and it is probable that no virtually poly-Z group is cohopfian.)

Groups of aspherical geometric 3-manifolds in classes (1), (2), (3) and (6) satisfy the volume condition [24]. Again this may be verified using the Gromov norm for classes (1) and (2), and class (3) presents the most difficulty. (Class (6) can be handled as in Theorem 3.) Since PD_3-groups in the other classes are not cohopfian they do not satisfy the volume condition.

Groups G and H are *commensurable* if there are subgroups $G_1 \leq G$ and $H_1 \leq H$ of finite index such that $G_1 \cong H_1$. All PD_3-groups in any one of the classes (5), (6), (7) or (8) are commensurable, while the commensurability classes of groups of type (4) correspond to the real quadratic extensions of $\mathbb{Q}$. The commensurability classification of hyperbolic 3-manifolds and 3-manifolds with nontrivial geometric decompositions (corresponding to classes (1), (2) and (3), with hyperbolic atoroidal parts) is considerably more delicate. See [16] and [17].

References

[1] R. Bieri, *Homological dimension of discrete groups*, Queen Mary College Mathematics Notes, Mathematics Department (Queen Mary College, London, 1976).

[2] R. Bieri and B. Eckmann, 'Relative homology and Poincaré duality for group pairs', *J. Pure Appl. Algebra* **13** (1978), 277–319.

[3] B. H. Bowditch, 'Planar groups and the Seifert conjecture', preprint (University of Southampton, 1999).

[4] W. Dicks and M. J. Dunwoody, *Groups acting on graphs*, Cambridge Stud. Adv. Math. 17 (Cambridge University Press, Cambridge, 1989).

[5] E. Dyer and A. T. Vasquez, 'Some properties of two-dimensional Poincaré duality groups', in: *Algebra, Topology and Category Theory (a collection of papers in honour of Samuel Eilenberg)* (Academic Press, New York, 1976) pp. 45–54.

[6] B. Eckmann and H. Müller, 'Plane motion groups and virtual Poincaré duality of dimension two', *Invent. Math.* **69** (1982), 293–310.

[7] F. Gonzalez-Acuña, R. Litherland and W. Whitten, 'Cohopficity of Seifert bundle groups', *Trans. Amer. Math. Soc.* **341** (1994), 143–155.

[8] F. Gonzalez-Acuña and W. Whitten, *Imbeddings of three-manifold groups*, Mem. Amer. Math. Soc. 474 (American Mathematical Society, Providence, 1992).

[9] J. Hempel, 'Residual finiteness for 3-manifolds', in: *Combinatorial group theory and topology* (eds. S. M. Gersten and J. R. Stallings), Annals of Math. Stud. 111 (Princeton University Press, Princeton, 1987) pp. 379–396.

[10] J. A. Hillman, 'Seifert fibre spaces and Poincaré duality groups', *Math. Z.* **190** (1985), 365–369.

[11] ———, *Four-manifolds, geometries and knots*, Geometry and Topology Monographs 5 (Geometry and Topology Publications, Coventry, 2002).

[12] P. H. Kropholler, 'Cohomological dimensions of soluble groups', *J. Pure Appl. Algebra* **43** (1986), 281–287.

[13] ———, 'An analogue of the torus decomposition theorem for certain Poincaré duality groups', *Proc. London Math. Soc.* **60** (1990), 503–529.

[14] ———, 'A note on centrality in 3-manifold groups', *Math. Proc. Cambridge Philos. Soc.* **107** (1990), 261–266.

[15] W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, 2nd edition (Dover, New York, 1976).

[16] W. D. Neuman, 'Commensurability and virtual fibration for graph manifolds', *Topology* **36** (1997), 355–378.

[17] W. D. Neumann and A. W. Reid, 'Arithmetic of hyperbolic manifolds', in: *Topology'90 (Columbus, OH, 1990)* (eds. B. Apanasov, W. D. Neumann, A. W. Reid and L. Siebenmann), Ohio State Univ. Math. Res. Inst. Publ. 1 (Walter de Gruyter, New York, 1992) pp. 273–310.

[18] Y. Rong, 'Degree one maps between geometric 3-manifolds', *Trans. Amer. Math. Soc.* **332** (1992), 411–436.

[19] D. Segal, *Polycyclic groups*, Cambridge Tracts in Mathematics 82 (Cambridge University Press, Cambridge, 1983).

[20] R. Strebel, 'A remark on subgroups of infinite index in Poincaré duality groups', *Comment. Math. Helv.* **52** (1977), 317–324.

[21] S. Wang, 'The existence of maps of nonzero degree between aspherical 3-manifolds', *Math. Z.* **208** (1991), 147–160.

[22] ———, 'The π_1-injectivity of self-maps of nonzero degree on 3-manifolds', *Math. Ann.* **297** (1993), 171–189.

[23] S. Wang and Y. Q. Wu, 'Covering invariants and co-hopficity of 3-manifold groups', *Proc. London Math. Soc.* **68** (1994), 203–224.

[24] S. Wang and F. Yu, 'Covering degrees are determined by graph manifolds involved', *Comment. Math. Helv.* **74** (1999), 238–247.

School of Mathematics and Statistics
The University of Sydney
Sydney, NSW 2006, Australia
e-mail: jonh@maths.usyd.edu.au