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RATIONAL ROULETTE*

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Over the years a large number of different betting systems have been proposed for the game of roulette. These systems are reviewed and their merits (if any) assessed. It is shown that some are not without advantages and that additionally they give rise to some unsolved mathematical problems. It is also suggested that these betting systems might usefully be looked at as procedures for testing for deviations from randomness.

1. Introduction

Roulette, allegedly invented by Pascal in 1655, has a long and eventful history, with an extensive but largely anecdotal literature. The search for a winning system has perhaps generated more probabilistic misunderstandings and misrepresentations than most games, and new systems are continually being discovered and rediscovered.

Conceptually roulette is a simple game. It is played for most of the purposes of this paper on a wheel containing 37 compartments numbered from 0 to 36, although in some casinos the wheel has 38 compartments with a double zero compartment in addition. Half of the numbers 1-36 are coloured red and half black, while the zero(s) are of a different colour, often

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green. The wheel rotates in a dish, in which a ball is rotated in the opposite direction; ultimately the ball comes to rest in one of the compartments. Bets may be placed on a single number (including the zero) or on combinations of numbers, and a bet wins if the ball comes to rest in a compartment corresponding to one of the numbers on which the bet has been placed. The possible bets and the odds paid for successful bets are given in Table 1, with both their English (American) and French names. Bets are made by placing counters (chips) on a tableau; the arrangement of the tableau determines what actual combinations of numbers are possible although the details need not concern us here.

TABLE 1

Possible bets in roulette

Bet	American	French	Odds Paid
Single number	Straight	En Plein	35:1
Two numbers	Split	A Cheval	17:1
Three numbers	Street	En Traversale	11:1
Four numbers	Square or 4-line	Carre ou 0 1 2 3	8:1
Six numbers	6-line	Sixaine	5:1
Twelve numbers	Dozen or Column	Douzaine ou Colonne	2:1
Eighteen numbers	Low (1-18) or High (19-36) or Red or Black or Odd or Even	Chances Simples	1:1
Twenty four numbers	2-Dozens or 2-Columns	A Cheval Douzaine ou A Cheval Colonne	1:2

If it is assumed that the ball is equally likely to come to rest in any of the 37 compartments the odds offered are slightly below the "fair" odds and it is this, together with the speed of the game (30-90 spins per hour depending on the type of game), which makes it profitable for the casinos. The casino's "edge" comes from the fact that when the zero occurs all bets, except those on the zero, are lost (or partially lost in the case of even money bets).

It is well known (see, for example, Epstein [7]) that no betting system can be constructed which will give the player an advantage over the bank in a game such as roulette played on an unbiased wheel with a zero (or zeros). As Wilson [17] puts it "the system maker attempts to take a series of plays (games), on each of which there is an expected loss, and somehow add them together to get an expected win. This is clearly impossible".

The purpose of this paper is to examine the variety of systems which have been proposed over the years. Such an examination not only produces a number of unsolved mathematical problems, but also suggests that some systems may not be entirely without merit. In addition the connection between roulette systems and statistical hypothesis testing is briefly discussed.

2. Basic roulette

The usual way in which the bank's "edge" in roulette (or other games of chance) is described is by the expected gain per unit stake, expressed as a percentage. In roulette the single zero gives an "edge" of 2.70 per cent for all the bets other than the *chances simples*. For these even money bets British Gaming Board Regulations say that when the zero occurs only half the stake money is lost, giving an "edge" to the bank of 1.35 per cent. In some casino's the stake can, in these circumstance, be put *en prison* to await the result of the next spin. If this is the case the bank's advantage is slightly higher; a discussion of the rules, and the resulting calculations, is given in Davies and Ross [2].

This "edge" does not represent the percentage of the "drop" (the amount of money changed for chips by the casino) which the casino may expect to win. It represents the average rate at which the "action" (the amount of money actually bet, which will be many times the "drop") is acquired by the casino from the gamblers. It is, however, a long-term effect; in the short term there will be substantial fluctuations about this trend and it is these short-term fluctuations which often encourage system makers to believe that they have successfully beaten the bank. In Downton and Holder [6] it was suggested that the probability of not losing over a relatively short period was a much more realistic criterion by which a gambler judged his performance. It was also suggested (erroneously) that these probabilities were monotonic in the gain ratio (the inverse of the

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coefficient of variation of the amount won, suggested in a different context by Joseph [9]) of the bet. While this monotonicity is achieved asymptotically, these probabilities exhibit a cycling behaviour which leads to it only being true in games of roulette for sequences of bets whose length is nearly a multiple of 36 games when the number of games is not very large. The properties of these probabilities were discussed in Anderson and Fontenot [1] and Downton [4]. What is perhaps surprising is that for a small number of games the probability of not losing approaches or even exceeds ½ for many of the bets in roulette. This may well explain both the attraction of the game itself and also the popularity of systems. If a system has the effect, as it well may have, of increasing the probability of not losing over a short period (at the expense of heavier losses if the gambler loses) then it may be seen by the gambler as a successful system.

Finally, it has become fashionable in recent years to discuss games in terms of "utilities". The probability of not losing in a sequence of n games can be seen as attaching a "utility" of +1 to not losing and of zero to losing in that sequence. More complicated "utilities" would give different criteria and it may be that some such structure could more realistically model a gambler's reaction to the game. Nevertheless unless some positive "utility" is given to losing money it is impossible to overcome the "edge" to the bank and achieve a long run gain in utility from a fair roulette wheel with one or more zeros.

3. Roulette systems

Broadly speaking roulette systems fall into five categories.

(i) Covering systems or pattern betting

In such systems the gambler places bets on a number of different options in a fruitless attempt to obtain a better chance of winning. For example a bet might be placed on Black 6 with an "insurance" bet on Red or Odd. Or bets might be placed on both Odd and Black, so covering 28 (not as might be supposed 27) of the numbers. The distribution of the numbers on the wheel is as follows:

Odd-Black : 8 Odd-Red : 10 Even-Black : 10 Even-Red : 8 .

It may easily be seen that, allowing for losing only half the stake on an

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even money bet when the zero appears, the "edge" to the bank remains 1.35 per cent.

A similar bet is known as the *Cuban* where a unit bet is made on both Black (at odds of 1:1) and on the third column (*colonne*, at odds of 2:1) of the tableau. That column contains the twelve numbers which are multiples of 3 of which only 6, 15, 24 and 33 are black, the remaining 8 numbers being red. (An equivalent bet would be on Red and the middle column of the tableau.) The underlying fallacy here is to suppose that because 26 numbers are covered the gambler somehow has overcome the bank's "edge". It may readily be seen that this is not the case, and only a very naive gambler would follow these systems.

(ii) Compensatory systems

These systems are based on the fallacious belief that the "law of averages" applies in the short run; for example, if a number has not recently appeared then the "law of averages" is alleged to make it more likely to appear in the near future. Squire [14] devotes almost a whole book to systems based on this philosophy, and the contradictory flavour of the approach may be obtained by quoting from that book. He proposes betting for short periods and writes: "The 'short periods' we intend to suggest are simply blocks of numbers, considered in isolation from their predecessors and successors, and a system which limits itself to one particular block, abides by the result, and then turns to a fresh block. What we need to discover is blocks of numbers where a method may be immune to serious misfortune but have a really excellent chance of winning. The 'short periods' we select are columns of 37 spins. If the column is suitable we play; if not we don't".

To find these suitable blocks of numbers concepts such as that of a "sleeper" are introduced. A sleeper is a number which has failed to appear in 74 (or sometimes 111) spins. It is assumed therefore to be more likely to appear in the next 35 spins, so that, since it is paid at odds of 35:1, betting on it should show a profit. Similarly O'Neil-Dunne ([12]) introduces the "sleeping final". A final is the second digit of the numbers on the wheel; 0, 1, 2, 3, 4, 5 and 6 appear four times and 7, 8 and 9 appear three times, and betting on the final 0, say, means placing a bet on each of 0, 10, 20 and 30. A sleeping final is one which has failed to appear in 28 spins, and is regarded as having a good chance of appearing in the next 7 or 8 spins. Betting on sleepers or sleeping finals is sometimes referred to as a *Biarritz* system.

Squire ([*14*]) also introduces the idea of the hypothetical bet. Since the probability of long runs of the *chances simples* is small the gambler operating some bet variation system (see later), based on runs of losses terminating sooner or later, starts the sequence of bets with hypothetical bets on the "principle" that the losing bets are used up without loss and that when real bets are made the losing run will end sooner rather than later. Such compensatory systems are, of course, based on a misunderstanding of the mechanism and long-run nature of the so-called "law of averages".

(iii) Predictive systems

Such systems are based on predicting the region of the wheel where the ball will come to rest, based on the behaviour and habits of the croupier. Squire ([*14*]) refers to such a method, calling it *Les Voisins* and aserts without evidence (page 170) that using the method "can turn the 2.7 per cent advantage for the bank into about a 16 per cent advantage for the punter". (It should perhaps be mentioned that some other authors use the term *Les Voisins* for pattern betting on numbers neighbouring zero on the wheel.) Thorp ([*16*]) refers to the existence of a small computer, which would enable appropriate predictions to be made but this is outside the scope of this paper.

(iv) Stake spreading systems

Stake spreading systems are based on a philosophy directly opposed to that underlying compensatory systems; that is, it is assumed that if a result appears it is more, not less, likely to appear again. The betting tableau is divided up into a number of equal, mutually exclusive and exhaustive (apart from the zero) groups of numbers. If that number is 2 we have two opposing *chances simples*; if 3 either the columns or the dozens; if 6 the 6-lines 1-6, 7-12, 13-18, 19-24, 25-30 and 31-36; if 12 the streets; if 18 suitable arrangements of splits; and if 36 the groups contain only one number and are all the straights. A stake spreading system of this type based on squares is not really feasible. A bet is made on one of the groups of numbers. If it wins that group is bet on again;

if it loses the next bet is on the recently winning group with an "insurance" bet on the group that has lost. This process of betting on the most recently winning group with "insurance" bets on all the groups on which bets have been lost continues until a win (either with the principal or an "insurance" bet) is obtained,when the process starts again with a single bet on the most recently winning group of numbers. On an unbiased wheel such a system cannot affect the "edge" to the bank, although it may affect the variance of the amount won (or lost) and the probability of not losing in the short run. On a biased wheel however such systems have merits and one of these systems will be examined in more detail from this point of view later in the paper.

(v) Stake variation systems

Most stake variation systems are based on betting on the *chances simples*, and will be described here on that basis. The best known of these is the martingale or doubling-up system. A gambler bets on one of the *chances simples*. If he wins he simply repeats the bet; if he loses he doubles his stake. Doubling-up of the stake continues until a win is obtained when the process starts again. The argument underlying the system is that completion of the sequence of bets is certain (the existence of the zero is usually conveniently forgotten), when the gambler will have won an amount equal to his original bet. There is often an underlying attitude, also, that a long run of one particular *chance simple* implies that its complement is more likely to occur, as in the compensatory systems. The fallacy of this argument arises from the limitations in bet size imposed by either the "house limit", the maximum bet size permitted in the casino, or the gambler's capital. Sooner or later a sequence of bets arises in which doubling-up is no longer possible and a massive loss results. Thus a sequence of small wins is counterbalanced by a large loss.

Instead of increasing the bet exponentially the bet may be increased arithmetically; that is, the bet is increased by one unit after a loss and reduced by one unit after a win. Such a system is called a *d'Alembert* system; sometimes any stake variation system is included under that title. Reducing by one unit after a loss and increasing by one unit after a win is called a reverse or contra-d'Alembert system. A similar system, truncated after two, three or four bets, is known as a Paroli system (see, for example, Squire [14], page 83). The advantage of the arithmetic d'Alembert

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system is that stakes do not increase so rapidly as they do for the martingale system so that the upper limit is reached more slowly; on the other hand the sequences of bets which end in a gain are also longer.

Another form of d'Alembert type strategy is the Labouchère system, sometimes called cross-out, cancellation or top-and-bottom system. It has also been called the Neapolitan martingale (Figgis, [8]) or, in one form, the split martingale. This system gives rise to some unsolved mathematical problems, and is of interest in other respects, and will be discussed in more detail later.

4. The stake spreading system

Suppose that the tableau has been divided up into k mutually exclusive and exhaustive groups of numbers $(k = 2, 3, 6, 12, 18 \text{ or } 36)$. A typical bet will consist of stakes s_{ri} on r of these groups $(i = 1, 2, \ldots, r;\ r = 1, 2, \ldots, k)$. Since an additional group of numbers is bet upon after every loss (apart from losses to the zero, when the stakes are simply replaced), the number of groups on which bets are made will depend upon the current loss position. The index i will determine the order in which the groups of numbers were added to the bet. Thus s_{r1} is staked on the first losing group in the sequence, s_{r2} the losing group which followed it and so on. s_{11} is the stake (which without loss of generality could be assumed to be unity) that the gambler makes after a win, and is made on the group that has just won.

The behaviour of the staking process on a "fair" wheel is of little interest. Suppose however that one of the groups of numbers has a bias of size δ; that is, the probability of it winning is $36/37k + \delta$. We will suppose for convenience that the bias on each of the remaining $k - 1$ groups is $-\delta/(k-1)$, with the zero unbiased. With random betting the advantage to the bank is therefore assumed to be unchanged. This bias assumption gives two bias patterns, according as δ is positive or negative.

The staking pattern may then easily be formulated as a Markov chain with $\left(k^2+(3k-2)\right)/2$ states. There are $k - 1$ states where there are bets on r groups of numbers $(r = 1, 2, \ldots, k-1)$, but none is made on the

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"favoured" group and $k(k+1)/2$ states where a bet is made on the "favoured" group and $r-1$ other groups. For example, the Markov chain for a stake spreading system based on columns $(k=3)$ has 8 states as follows:

Stake	
01	no bet on favoured column; s_{11} on unfavoured column;
02	no bet on favoured column; s_{21}, s_{22} on unfavoured columns;
11	s_{11} on favoured column;
21	s_{21} on favoured column; s_{22} on unfavoured column;
22	s_{22} on favoured column; s_{21} on unfavoured column;
31	s_{31} on favoured column; s_{32}, s_{33} on unfavoured columns;
32	s_{32} on favoured column; s_{31}, s_{33} on unfavoured columns;
33	s_{33} on favoured column; s_{31}, s_{32} on unfavoured columns.

The transition matrix for the bias pattern proposed is given in Table 2. Matrices with a similar structure may easily be constructed for the other values of k.

TABLE 2

Transition matrix for column stake spreading system

		States							
		01	02	11	21	22	31	32	33
States	01	$\frac{13}{37}-\frac{\delta}{2}$	$\frac{12}{37}-\frac{\delta}{2}$	0	0	$\frac{12}{37}+\delta$	0	0	0
	02	$\frac{24}{37}-\frac{\delta}{2}$	$\frac{1}{37}$	0	0	0	0	0	$\frac{12}{37}+\delta$
	11	0	0	$\frac{13}{37}+\delta$	$\frac{24}{37}-\delta$	0	0	0	0
	21	$\frac{12}{37}-\frac{\delta}{2}$	0	$\frac{12}{37}+\delta$	$\frac{1}{37}$	0	$\frac{12}{37}-\frac{\delta}{2}$	0	0
	22	$\frac{12}{37}-\frac{\delta}{2}$	0	$\frac{12}{37}+\delta$	0	$\frac{1}{37}$	0	$\frac{12}{37}-\frac{\delta}{2}$	0
	31	$\frac{24}{37}-\delta$	0	$\frac{12}{37}+\delta$	0	0	$\frac{1}{37}$	0	0
	32	$\frac{24}{37}-\delta$	0	$\frac{12}{37}+\delta$	0	0	0	$\frac{1}{37}$	0
	33	$\frac{24}{37}-\delta$	0	$\frac{12}{37}+\delta$	0	0	0	0	$\frac{1}{37}$

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Although, in general, other criteria may be appropriate to the individual gambler, we will concentrate here on his expected gain per unit stake when the chain is in equilibrium. To determine this we first need to determine the equilibrium solution to the chain, the vector $\mathbf{u}$ which satisfies the equation $\mathbf{u}^T = \mathbf{u}^T\mathbf{P}$, with the elements of $\mathbf{u}$ adding to unity, where $\mathbf{P}$ is the Markov transition matrix. This expected gain per unit stake to the player in equilibrium may be shown to be of the form (assuming a single zero wheel)

$$E(G) = -\frac{1}{37} + k\delta\left(\sum_{r=1}^{k}\sum_{i=1}^{r} b_{ri}s_{ri}\right) \Big/ \left(\sum_{r=1}^{k}\sum_{i=1}^{r} a_{ri}s_{ri}\right), \tag{4.1}$$

unless $k = 2$, when $-1/37$ is replaced by $-1/74$. The coefficients a_{ri}, b_{ri} depend on δ through the equilibrium solution $\mathbf{u}$ and the denominator and numerator in the above expression represent, respectively, the expected stake and the expected gain due to bias for the selected staking pattern when the chain is in equilibrium. These coefficients may be obtained from the vector $\mathbf{u}$ by matrix equations of the type $\mathbf{a} = \mathbf{A}\mathbf{u}$ and $\mathbf{b} = \mathbf{B}\mathbf{u}$; Table 3 gives the matrices $\mathbf{A}$ and $\mathbf{B}$ for the column stake spreading system $(k = 3)$. Similar matrices may easily be obtained for the other values of k.

For the six-line stake spreading system an extensive examination of the effect of different staking systems on expected gain per unit stake has been conducted. It emerges that the effect of bias is the same for any staking pattern, in which $s_{rr} = s_1$, say, and $s_{ri} = s_2$, $r \neq i$. This includes not only the fixed bet size system $(s_1 = s_2)$ but also the simple "follow-the-winner" system $(s_2 = 0)$, where the gambler bets on the most recently winning group; also a "bet-on-losers" system $(s_1 = 0)$, although such a system would involve a hypothetical bet on a winner to start each sequence of losing bets. It is conjectured that this may be a general property of the stake spreading systems for the other values of k, with the type of bias considered.

It is interesting that introducing a d'Alembert type bet variation system with, say, $s_{1i} = 1$, $s_{2i} = 2$, $s_{3i} = 3$, and so on, does worse than the fixed bet size system. A comparison is given in Table 4. It may

TABLE 3

Matrices relating coefficients in the expected gain to the states for biased wheel in column stake spreading system

Coefficients	States 01	02	11	21	22	31	32	33
a_{11}	1	0	1	0	0	0	0	0
a_{21}	0	1	0	1	1	0	0	0
a_{22}	0	1	0	1	1	0	0	0
a_{31}	0	0	0	0	0	1	1	1
a_{32}	0	0	0	0	0	1	1	1
a_{33}	0	0	0	0	0	1	1	1

Coefficients	States 01	02	11	21	22	31	32	33
b_{11}	$-\frac{1}{2}$	0	1	0	0	0	0	0
b_{21}	0	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$	0	0	0
b_{22}	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	0	0
b_{31}	0	0	0	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$
b_{32}	0	0	0	0	0	$-\frac{1}{2}$	1	$-\frac{1}{2}$
b_{33}	0	0	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	1

TABLE 4

Expected gain per unit stake in two six-line stake spreading systems

Bias	Fixed bet size	d'Alembert bets	Bias	Fixed bet size	d'Alembert bets
-0.10	0.0470	0.0380	0.01	-0.0263	-0.0265
-0.09	0.0329	0.0247	0.02	-0.0241	-0.0248
-0.08	0.0203	0.0132	0.03	-0.0204	-0.0221
-0.07	0.0092	0.0033	0.04	-0.0152	-0.0183
-0.06	-0.0004	-0.0051	0.05	-0.0085	-0.0134
-0.05	-0.0085	-0.0120	0.06	-0.0004	-0.0076
-0.04	-0.0152	-0.0175	0.07	0.0092	-0.0007
-0.03	-0.0204	-0.0217	0.08	0.0203	0.0071
-0.02	-0.0241	-0.0247	0.09	0.0329	0.0159
-0.01	-0.0263	-0.0265	0.10	0.0470	0.0257
0	-0.0270	-0.0270	-	-	-

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be noted that the direction of the bias is irrelevant for fixed size bets, but not for d'Alembert type variation of bets; also that a fairly substantial bias is required before the expected gain per unit stake becomes positive.

5. Optimum stakespreading system (repeat betting)

Since the stakes, s_{ri}, in the stake spreading system can be varied, it is natural to ask what values they should be given in order to maximise the expected gain per unit stake. I am indebted to Dr C.K. Wright for the following solution.

For a positive bias $(\delta > 0)$ we wish to choose $\mathbf{s}$ in order to maximise g, say, subject to the condition

$$\mathbf{b}^T\mathbf{s} - g\mathbf{a}^T\mathbf{s} = 0 , \tag{5.1}$$

where $\mathbf{a}, \mathbf{b}$ are the vectors of the coefficients in (4.1) and $\mathbf{s}$ is the vector of stakes. The elements of both $\mathbf{a}$ and $\mathbf{s}$ are necessarily non-negative.

Now (5.1) has no solution for a given g if and only if *all* the elements of $\mathbf{b}^T - g\mathbf{a}^T$ are either positive or negative. Thus (5.1) has no solution if and only if either

$$g < \min_{r,i} \frac{b_{ri}}{a_{ri}} \tag{5.2a}$$

or

$$g > \max_{r,i} \frac{b_{ri}}{a_{ri}} \tag{5.2b}$$

since all $a_{ri} > 0$.

Hence the maximum value obtainable for g is given by $\max_{r,i} \frac{b_{ri}}{a_{ri}} = \frac{b_{\alpha\beta}}{a_{\alpha\beta}}$, say. This implies that the maximum gain per unit stake is obtained by taking $s_{\alpha\beta} = 1$ and $s_{ri} = 0$, $r \neq \alpha$ and $i \neq \beta$. A similar result may be obtained if δ is assumed to be negative.

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For the six-line stake spreading system it turns out that over the range of δ considered the maximum coefficient ratio is that given by b_{11}/a_{11} . It is conjectured that not only is this true for all obtainable values of δ , but that it is also true for stake spreading systems based on other groups of numbers.

In practice this means that the optimum would be given by taking $s_{11} = 1$ and $s_{ri} = 0$, $r \neq i \neq 1$. Thus a bet is only made on the spin immediately after a "repeat" is observed and the bet is made on the group of numbers that is repeated. Thus the gambler should observe the table until one of the groups is repeated; he should then bet on that group. If it wins he bets on it again. If it loses he should wait until either that group again appears or one of the other groups appears twice, whichever is the sooner, and then bet on the repeated group. The expected gains per unit stake (in equilibrium) for repeat betting on various groups and different values of bias are given in Table 5. These may be compared with the non-optimal results of Table 4 for the six-line system.

TABLE 5

Expected gains per unit stake for repeat betting

	Number groups					
Bias δ	Even Chances (k = 2)	Dozens or columns (k = 3)	Six-lines (k = 6)	Streets (k = 12)	Splits (k = 18)	Straights (k = 36)
-0.05	0.0068	-0.0050	0.0048	0.0198	0.0256	-
-0.04	-0.0005	-0.0128	-0.0059	0.0057	0.0126	-
-0.03	-0.0061	-0.0189	-0.0148	-0.0071	-0.0014	-
-0.02	-0.0102	-0.0234	-0.0214	-0.0175	-0.0142	-0.0078
-0.01	-0.0127	-0.0261	-0.0256	-0.0245	-0.0235	-0.0208
0	-0.0135	-0.0270	-0.0270	-0.0270	-0.0270	-0.0270
0.01	-0.0127	-0.0261	-0.0255	-0.0242	-0.0228	-0.0181
0.02	-0.0102	-0.0233	-0.0208	-0.0151	-0.0088	0.0137
0.03	-0.0061	-0.0185	-0.0128	0.0009	0.0166	0.0748
0.04	-0.0005	-0.0119	-0.0013	0.0245	0.0548	0.1700
0.05	0.0068	-0.0033	0.0138	0.0562	0.1068	0.3021
Odds	1:1	2:1	5:1	11:1	17:1	35:1
Expected interval between bets (δ = 0)	1.50	1.89	2.77	4.04	5.01	7.20

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Essentially the repeat system makes use of hypothetical bets. Using the full stake spreading system, bets of zero are made whenever the system would give bets on more than one group of numbers, and unit bet is made whenever the system would have resulted in a bet on a single group. The pattern of bets with the resulting gains (or losses) is illustrated for a sequence of six-line results (numbering the six-lines from 1 to 6) in Table 6. For convenience the betting pattern has been started with a hypothetical bet on the first six-line to appear. Note that when the zero appears both real and hypothetical bets are replaced.

TABLE 6

Example of the operation of the six-line repeat system

Spin number	Result	Six-line bets						Total gain/loss
		1	2	3	4	5	6	
1	2	-	-	-	-	-	-	
2	2		0					
3	1		1					-1
4	2	0	0					
5	3		1					-2
6	6		0	0				
7	1		0	0			0	
8	0	0	0	0			0	
9	5	0	0	0			0	
10	1	0	0	0		0	0	
11	2	1						-3
12	2	0	0					
13	1		1					-4
14	2	0	0					
15	2		1					+1
16	1		1					0
17	3	0	0					
18	4	0	0	0				
19	4	0	0	0	0			
20	1				1			-1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

0 ≡ Hypothetical bet. 1 ≡ Real bet

A mathematical (and practical) problem of interest is the distribution of the intervals between bets. Consider a repeat system based on a division of the tableau into k mutually exclusive and exhaustive groups of numbers for a true wheel with, in addition, a single zero. Suppose that a real bet has just been made; if the result is a zero or the group on which the bet has been placed another real bet is made immediately. Other-

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wise hypothetical bets are made until a repeat occurs.

Let $p_{n,r}$ be the probability that after n spins of the wheel r of the groups (including the one immediately preceding this sequence) have occurred without there having been a real bet. It may be seen that $p_{n,1} = 0$, $n \geq 1$; $p_{0,1} = 1$; $p_{0,r} = 0$, $r \neq 1$, and

$$(5.3) \qquad p_{n,r} = \binom{n-1}{r-1} \frac{(k-1)!}{(k-r)!} \left(\frac{36}{k}\right)^{r-1} \frac{1}{36^n} , \quad n \geq r-1 , \quad r \geq 2 .$$

If N_k is the number of spins from one bet to the next then

$$(5.4) \qquad P(N_k > n) = \sum_{r=2}^{\min(k,n+1)} p_{n,r} ,$$

from which the distribution of N_k may be obtained. In particular

$$(5.5) \quad E(N_k) = \sum_{n=0}^{\infty} P(N_k > n) = \sum_{r=1}^{n} \frac{(k-1)!}{(k-r)!k^{r-1}} ; \quad k = 2, 3, 6, 12, 18, 36 .$$

Values of this expected value for the different values of k are given in Table 5. Note that the expected number of spins to a repeat is 37/36 times this value; in this case another real bet is made either when a repeat occurs or when a zero immediately follows a real bet. A bias in the wheel would tend to reduce this expected value; the analysis involving semi-Markov chains will not be included here.

To obtain the results for this type of system on a real wheel its 6-line version was tried on data given by O'Neil-Dunne ([12]). These date purport to be the results for 31 days' play in the Casino de Macao. The wheel was used from 6.00 a.m. to 5.55 a.m. on each of these days, the five minute interval being used to oil and level the wheel. The results for the 6-line repeat system are given in Table 7. During the 31 days 20,080 spins were made and 7169 bets would have been made giving an average interval between bets of 2.80 compared with the theoretical result for a true wheel of 2.77. The total loss per unit stake over the period would have been 3.33 per cent compared with 2.70 per cent for a true wheel. There is no evidence for any bias on the wheel, but the pattern of results for each of the days will repay study; in particular, it may be noticed that the gambler would have shown a profit on 14 days, shown a loss on 16 days and

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TABLE 7

Effect of 6-line repeat system on Macao data

Day	Number of spins	Number of bets	Average interval	Units won	Rate	Max gain	Max loss
1	632	231	2.74	+9	+0.0390	26	20
2	716	251	2.86	+7	+0.0280	24	36
3	720	254	2.83	-56	-0.2205	17	55
4	642	232	2.77	+20	+0.0862	37	8
5	684	238	2.87	+2	+0.0084	16	21
6	648	223	2.91	+5	+0.0224	26	9
7	637	219	2.91	-27	-0.1233	4	46
8	689	243	2.84	+3	+0.0123	24	24
9	634	241	2.63	+77	+0.3195	98	4
10	576	210	2.74	0	0	18	17
11	720	277	2.60	+17	+0.0614	22	7
12	720	259	2.78	+11	+0.0425	11	34
13	720	266	2.71	-2	-0.0075	46	5
14	640	216	2.96	-48	-0.2222	0	49
15	648	236	2.75	-14	-0.0593	5	34
16	557	207	2.69	+9	+0.0435	22	15
17	502	175	2.87	-43	-0.2457	0	48
18	720	264	2.73	-12	-0.0455	8	40
19	648	225	2.88	-45	-0.0200	5	49
20	648	224	2.89	-20	-0.0893	3	21
21	720	243	2.96	-33	-0.1358	5	66
22	576	208	2.77	-58	-0.2788	11	65
23	680	237	2.87	-81	-0.3418	0	87
24	485	174	2.78	+42	+0.2414	45	4
25	648	229	2.83	-1	-0.0044	8	31
26	648	231	2.81	+15	+0.0649	65	2
27	664	244	2.72	-16	-0.0656	0	46
28	648	238	2.72	+26	+0.1092	47	0
29	696	244	2.85	+34	+0.1393	8	36
30	576	196	2.94	-10	-0.0510	24	13
31	638	234	2.73	+18	+0.0769	22	18
Total	20 080	7 169	2.80	-239	-0.0333	114	285

broken exactly even on the remaining day, results which are consistent with random betting on the 6-lines.

6. Labouchère systems

A basic principle of successful betting is for the gambler to bet a small amount (zero if possible) when the "edge" is against him and to bet as much as possible when the "edge" is in his favour in order to maximise his absolute expected gain. If the aim is to maximise the expected gain per unit stake this policy is slightly modified (Downton, [3]). An

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alternative approach to exploiting bias in a roulette wheel is to make use of a stake variation system to attempt to operate such a policy. The parallel Labouchère system described in Leigh ([11]) is one such system, although it is doubtful if the system could be as successful as the author claimed.

The Labouchère system is a bet variation system for the *chances simples*. In its standard form it involves writing down a line of integers. The size of the first bet (in suitable units) is the sum of the two extreme integers in the line. If the bet wins then these two numbers are deleted; if it loses the number corresponding to the amount lost is added to the right of the line. The next bet is then the new sum of the extremes, and this process continues until all the numbers, both the original ones and those added, have been deleted. Completion of such a sequence results in a win equal to the total of the original k numbers. The system is named after Henry du Pre Labouchère (1831-1912), an English journalist and Member of Parliament. While the system is commonly used with the four numbers 1, 2, 3, 4 as starting values, Labouchère himself used the five numbers 3, 4, 5, 6, 7 and attributed the system to Condorcet (Thorold, [15], pages 60-61, quotes from an article in Labouchère's paper *Truth* published on 15th February 1877). The system in itself suffers from the same flaw as the martingale (or doubling-up) system, in that although the sequence of bets is certain to terminate sooner or later it may require a bet which exceeds either the house limit or the gambler's capital. The gambler then suffers a loss out-weighing earlier gains. The reverse Labouchère system adds the amount won to the line of numbers after a win and deletes the two extreme numbers after a loss, on the fallacious argument that if the Labouchère system gives a large loss counterbalancing a series of small wins, then its mirror image will give an occasional large win to counterbalance a series of small losses.

A parallel Labouchère system involves operating separate Labouchère processes (either both standard or both reverse) on a pair of opposing *chances simples*. This results in differential betting on the opposing chances, with, it is hoped, more being bet on the favoured chance than on the unfavoured one in the event of there being bias present.

There has been little analysis of even single Labouchère processes. Downton ([5]) used a system of difference equations to compute the probability distributions of the length of completed standard and reverse Labouchère sequences starting with four numbers and unrestricted by house or capital limits, but these distributions do not provide information about the way in which bet sizes build up, which remains an unsolved problem. In fact, explicit expressions for the moment of these distributions may be obtained.

The standard Labouchère system deletes two numbers after a win (with probability p, say) and adds a number after a loss (with probability $1 - p$). It may therefore be regarded as a random walk on the positive integers, where that walk has an absorbing barrier at zero and has probability p of taking two steps to the left and $1 - p$ of taking one step to the right. If X_n is the number of steps taken to reach the origin from n and $g(X_n)$ is some function of X_n, then

(6.1) $$E\{g(X_n-1)\} = pE\{g(X_{n-2})\} + (1-p)E\{g(X_{n+1})\} .$$

In particular

(6.2) $$E(X_n) = 1 + pE(X_{n-2}) + (1-p)E(X_{n+1}) ,$$

and

(6.3) $$E\left(X_n^2\right) = 2E(X_n) - 1 + pE\left(X_{n-2}^2\right) + (1-p)E\left(X_{n+1}^2\right) .$$

From (6.2) it may be shown that

(6.4) $$E(X_n) = [1+(1+\alpha)n-(-\alpha)^{-n}]/(3p-1)(1+\alpha)$$

where

$$\alpha = \sqrt{\{(1/p)-(3/4)\}} + \tfrac{1}{2} .$$

(6.4) may be used with (6.3) to obtain an explicit expression (in terms of powers of α) for $E\left(X_n^2\right)$ and hence $\operatorname{var}(X_n)$. These expressions are complicated and will not be given here. When n is large we have

(6.5) $$E(X_n)/n \sim 1/(3p-1) \quad \text{and} \quad \operatorname{var}(X_n)/n \sim 9p(1-p)/(3p-1)^2 .$$

Values of $E(X_n)/n$ and $\operatorname{var}(X_n)/n$ are given in Table 8 for standard and reverse Labouchère sequences on single and double zero wheels and on a

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TABLE 8

$E(X_n)/n$ and $\mathrm{var}(X_n)/n$ for X_n, the length of a completed (unrestricted) Labouchère series of games

Initial number of integers	Standard Labouchère		Fair wheel	Reverse Labouchère	
	Double zero wheel	Single zero wheel		Single zero wheel	Double zero wheel
1	3.800, 43.65	3.501, 33.56	3.236, 25.97	3.010, 20.47	2.826, 16.57
2	2.660, 34.76	2.436, 26.90	2.236, 20.97	2.066, 16.65	1.927, 13.58
3	2.736, 33.39	2.512, 25.71	2.315, 19.92	2.146, 15.72	2.009, 12.74
4	2.569, 32.80	2.354, 25.34	2.163, 19.72	2.001, 15.64	1.868, 12.73
5	2.567, 31.99	2.355, 24.65	2.167, 19.12	2.006, 15.11	1.876, 12.26
6	2.517, 31.87	2.307, 24.61	2.120, 19.13	1.962, 15.15	1.832, 12.32
7	2.506, 31.47	2.298, 24.26	2.113, 18.83	1.956, 14.89	1.827, 12.09
8	2.484, 31.39	2.277, 24.22	2.093, 18.82	1.937, 14.90	1.809, 12.11
9	2.475, 31.18	2.269, 24.04	2.986, 18.67	1.930, 14.76	1.803, 11.99
10	2.464, 31.11	2.258, 24.00	2.076, 18.64	1.921, 14.75	1.794, 11.99
11	2.456, 30.99	2.252, 23.90	2.070, 18.56	1.915, 14.68	1.789, 11.92
12	2.449, 30.93	2.245, 23.85	2.063, 18.53	1.909, 14.66	1.783, 11.91
13	2.444, 30.85	2.240, 23.79	2.059, 18.48	1.905, 14.61	1.779, 11.87
14	2.439, 30.80	2.235, 23.75	2.055, 18.45	1.901, 14.59	1.775, 11.86
15	2.434, 30.75	2.231, 23.71	2.051, 18.42	1.898, 14.57	1.772, 11.83
16	2.431, 30.70	2.228, 23.68	2.048, 18.39	1.895, 14.55	1.769, 11.82
17	2.427, 30.67	2.225, 23.65	2.045, 18.37	1.892, 14.53	1.767, 11.80
18	2.424, 30.63	2.222, 23.63	2.042, 18.35	1.890, 14.51	1.765, 11.79
19	2.422, 30.60	2.220, 23.60	2.040, 18.33	1.888, 14.50	1.763, 11.78
20	2.420, 30.58	2.218, 23.58	2.038, 18.31	1.886, 14.48	1.761, 11.77
∞	2.375, 30.06	2.176, 23.18	2.000, 18.00	1.850, 14.24	1.727, 11.56

"fair" wheel (without a zero). It will be seen from that table that the mean and variance of X_n are relatively sensitive to changes in p; this is a property which is required of a parallel bet variation system designed to exploit bias.

Although no theory for such parallel systems is known, it is possible to see the type of criteria which would make such a system effective. Martingale type systems produce sequences of bets which end with a fixed positive gain. Therefore the shorter such sequences are the greater will be the rate at which these gains are acquired. On the other hand if one is aiming at maximising the expected gain per unit stake the actual amount bet during a sequence of bets should not be allowed to grow too large. Therefore an effective parallel bet variation system should have sequence length sensitive to bias, but at the same time bet size should be relatively insensitive to sequence length. Limited trials of parallel Labouchère

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systems and parallel martingales suggest that the former is better in both these respects than the latter; however, not only are there no effective theoretical results to assess the effectiveness of the parallel Labouchère system, but also it remains a completely open question as to whether it might be improved by some other bet variation system.

7. Conclusion

This review of roulette betting strategies in the game of roulette, has suggested that not all of the systems proposed are entirely without merit. The exploitation of bias in roulette wheels may no longer be possible on the scale of that achieved by Jaggers (see, for example, Kingston, [*10*]), but nevertheless bias on modern wheels does still seem to exist (see, for example, Wilson, [*17*]). Whether that bias is of the form assumed in the present paper, or consists of correlations as Pearson ([*13*]) seems to imply is uncertain.

One may however look at these systems in a different way. The output from a roulette wheel consists of a stream of (perhaps random) integers. The betting strategy consists of a scoring system which is trying to detect bias. In this it is analogous to a significance test for deviations from randomness, although instead of attempting to maximise power as in the conventional significance test we are trying to optimise a different criterion, the expected gain per unit stake. Just as a significance test which is optimal against a specific alternative may still be powerful against other alternatives, so a betting strategy which is optimal for a particular type of bias may still be good if the bias takes a different form. More generally it is suggested that statisticians wishing to detect deviations from randomness might find useful ideas among roulette betting strategies, and, equally, gamblers seeking betting strategies might find useful ideas among the significance tests that have been proposed to detect deviations from randomness.

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