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PLÜCKER COORDINATES FOR REGULAR CHAIN GROUPS

T. KAMBAYASHI

The theory of Plücker coordinates and Grassmann varieties is well-developed and well-known among the algebraic geometers. It gives a one-to-one correspondence between the set of all subspaces of a given dimension in the ambient projective space and the set of points on a certain projective algebraic variety called a Grassmann variety. The unacquainted can find the theory discussed in detail in Hodge-Pedoe [**1**, Chapters VII and XIV].

The purpose of the present paper is to show that the techniques of the Grassmann theory can be applied with almost equal force to the study of regular chain groups introduced by Tutte [**3**]. (See also Minty [**2**] for connections with graph theory.) It will be shown that the set of all regular chain groups of a given rank on a fixed set of vertices are in precise one-to-two correspondence with a certain set of ordered tuples of 0's and ± 1's (their Plücker coordinates) subject to defining conditions similar to those for a Grassman variety. This is done in § 2. Thus, the number of such regular chain groups is precisely one-half the number of such tuples admissible as Plücker coordinates (see (2.3)). In practice, however, the actual enumeration of regular chain groups is still a difficult problem, and in § 3 we offer what is hoped to be a first step in the direction of a complete solution.

The first section (§ 1) is mainly a quick review and a degree of reformulation of Tutte's theory, but it also contains a technical result (1.4) essential in § 2.

Throughout, $\mathbf{Z}$ denotes the additive group of all rational integers.

1. Regular chain groups according to Tutte (cf. [**3**]). For a natural number n, fixed throughout the present paper, denote by $[1\,n]$ the set $\{1, \ldots, n\}$ of integers between 1 and n, and define a *chain on n vertices* or, simply, a *chain* to be any mapping from $[1\,n]$ to the additive group $\mathbf{Z}$ of integers. If f and g are chains, their sum $f + g$ is defined by the formula: $(f + g)(i) = f(i) + g(i)$ for every $i \in [1\,n]$. A set of chains forming a group under this addition is called a *chain group*. *We may, and shall, identify a chain f with the element* $(f(1), \ldots, f(n))$ *of the direct sum* $\mathbf{Z}^n = \mathbf{Z} \oplus \ldots \oplus \mathbf{Z}$ *of n copies of* $\mathbf{Z}$. Let C be a chain group and take $f \in C$. *The support of* f is by definition the set $\{i \in [1\,n] : f(i) \neq 0\}$ and is denoted by $|f|$. Then, $|f|$ is empty if and only if f is the zero-chain, or $f = \bar{0} = (0, \ldots, 0)$. The chain $f \in C$ is said to be *elementary* (*in C*) if f is non-zero and no non-zero $g \in C$ exists whose

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support $|g|$ is a proper subset of $|f|$. If in addition $f(i) = 0$ or ± 1 for each $i \in [1\, n]$, we say f is *primitive* (*in* C). Finally, a chain group C is said to be *regular* if for each elementary $f \in C$ there exists a primitive $g \in C$ such that $|g| = |f|$. When that is so, clearly $f = \alpha g$ for some $\alpha \in \mathbf{Z}$. The definition implies that the zero chain group $\{\bar{0}\}$ is to be regarded as regular.

(1.1) *Every chain f belonging to a regular chain group C can be expressed as an integral linear combination, $f = \sum_i \alpha_i g_i$ $(\alpha_i \in \mathbf{Z})$, of primitive chains $g_i \in C$ such that $|g_i| \subseteq |f|$ for all i.*

Proof. Call k the size (cardinality) of the set $|f|$ and use induction: There exists a primitive chain $g \in C$ with $|g| \subseteq |f|$. For a suitable integer α we have then $|f - \alpha g| \subset |f|$ (proper inclusion), and the cardinality of $|f - \alpha g|$ is therefore less than k. Since $f - \alpha g$ is a linear combination of the desired type by the induction hypothesis, so is f.

(1.2) *A non-zero regular chain group is freely generated by a set of primitive elements.*

Proof. Let C be a regular chain group $\neq \{\bar{0}\}$, and for each $k \in [1\, n]$ let $C_k = \{f \in C : f(i) = 0 \text{ for all } i > k\}$. We have then a sequence $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n = C$ of regular chain groups. Clearly, C_1 is either $\{\bar{0}\}$ or freely generated by a primitive chain. Suppose that for every $j < k$ the group C_j either is $\{\bar{0}\}$ or satisfies our proposition. Let π_k be the restriction on C_k of the kth projection $\mathbf{Z}^n \rightarrow \mathbf{Z}$ which maps $(x_1, \ldots, x_k, \ldots, x_n)$ onto x_k. If $\pi_k : C_k \rightarrow \mathbf{Z}$ is a zero map, then $C_k = C_{k-1}$ and there is nothing to prove. If $\pi_k(C_k) \neq \{0\}$, then by (1.1) there is a primitive chain $h \in C_k$ with $\pi_k(h) = 1$. Then, for any chain $g \in C_k$ we have $g - \pi_k(g)h \in C_{k-1}$, whence $C_k = C_{k-1} + \mathbf{Z}h$. This last sum is evidently direct. This proves (1.2).

Let us introduce an *ad hoc* definition: a primitive chain will be called *k-primitive* if it is of the type $(*, \ldots, *, 1, 0, \ldots, 0)$ with the 1 at the kth spot and each asterisk representing 0 or ± 1. Then, the above proof of (1.2) shows that a regular chain group of rank r possesses a free base $\{g_1, \ldots, g_r\}$ where g_i is k_i-primitive for each $1 \leqq i \leqq r$ and $1 \leqq k_1 < \ldots < k_r \leqq n$ holds. Moreover, for each pair $i < j$, one can suppose that the k_ith coordinate spot of g_j has 0 as its entry. For, if not already so, g_j may be replaced by $g_j \pm g_i$ and then by a primitive chain having 1 as its k_jth coordinate entry and whose support is contained in $|g_j \pm g_i|$. That this replacement is possible is due to (1.1), and it is evident that the resulting base is again free. We have thus established the following:

(1.3) *A regular chain group C of rank r determines a set of r integers $k_1 < \ldots < k_r$ belonging to $[1\, n]$ and an $r \times n$ matrix*

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$$\begin{array}{c}
\begin{array}{ccccccccccccccc}
 & & & k_1 & & & & k_2 & & & & k_r & & &
\end{array} \\
\begin{bmatrix}
* & \dots & * & 1 & 0 & \dots & 0 & 0 & 0 & \cdots & & & 0 & \dots & 0 \\
* & \dots & * & 0 & * & \dots & * & 1 & 0 & \cdots & & & 0 & \dots & \cdot \\
* & \dots & * & 0 & * & \dots & * & 0 & * & \cdots & & & 0 & \dots & \cdot \\
\vdots & & & \vdots & & & & \vdots & & & & & \vdots & & \vdots \\
* & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * & 1 & 0 & \dots & 0
\end{bmatrix}
\end{array}$$

whose rows form a free base of primitive chains in C.

It is the next corollary of (1.3) that we actually need in § 2:

(1.4) *If C is a regular chain group, then the factor group $\mathbf{Z}^n/C$ is torsion-free.*

Proof. Suppose that a chain $f \in \mathbf{Z}^n$ satisfies $\alpha f \in C$ for some non-zero $\alpha \in \mathbf{Z}$. Then, $\alpha f = \sum_{i=1}^r \beta_i g_i$ with $\beta_i \in \mathbf{Z}$, where $g_1, \dots, g_r$ are the rows of the matrix in (1.3). Then, $\alpha f(k_i) = \sum_{i=1}^r \beta_i g_i(k_i) = \beta_i g_i(k_i) = \beta_i$ for all $1 \leqq i \leqq r$, whence α divides every β_i and $f = \sum_{i=1}^r f(k_i) g_i \in C$.

Observe that the converse of (1.4) is false, as seen by simple counter-examples such as the non-regular chain group $\{(2d, 3d) : d \in \mathbf{Z}\} \subseteq \mathbf{Z}^2$. *This example also shows that the regularity of a chain group is tied to the particular base of the ambient group $\mathbf{Z}^n$ and is not invariant under unimodular coordinate transformations on $\mathbf{Z}^n$.*

We conclude this section (§ 1) by citing the following theorem of Tutte [**3**, (4.5), p. 19] which plays a crucial role in § 2 below:

(1.5) *Let M be an $r \times n$ matrix of rank r with integer entries. Then, the rows of M span a regular chain group of rank r if and only if every $r \times r$ minor of M has determinant* 0 *or* ± 1 *and at least one such determinant is non-zero.*

See Tutte, *loc. cit.* for the proof.

2. Plücker coordinates of regular chain groups. Let $C \subseteq \mathbf{Z}^n$ be a regular chain group of rank r, and let $f_1, \dots, f_r$ be a $\mathbf{Z}$-free base of C. For every ordered r-tuple $\bar{\imath} = (i_1, \dots, i_r)$ of integers between 1 and n, define

$$P(\bar{\imath}) = P(i_1, \dots, i_r) = \det \begin{bmatrix} f_1(i_1) & f_1(i_2) & \dots & f_1(i_r) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ f_r(i_1) & f_r(i_2) & \dots & f_r(i_r) \end{bmatrix}.$$

By (1.5), we know $P(\bar{\imath}) = -1, 0$, or $+1$ for each $\bar{\imath}$, and not all $P(\bar{\imath})$'s are zero. It is clear that if $\bar{\imath}$ and $\bar{\jmath} = (j_1, \dots, j_r)$ differ from each other only by a

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permutation then $P(\bar{\imath})$ equals $P(\bar{\jmath})$ multiplied by the sign of that permutation. Let us fix, once and for all, one (arbitrary) ordering on the set of all those ordered r-tuples $\bar{\imath} = (i_1, \ldots, i_r)$ satisfying $i_1 < i_2 < \ldots < i_r$, and arrange the corresponding $P(\bar{\imath})$'s in that order; we thus obtain an ordered $\binom{n}{r}$-tuple $(\ldots, P(\bar{\imath}), \ldots)$ of integers 0, ± 1, not all of which are zero. We shall call the $\binom{n}{r}$-tuple *the Plücker coordinates of the regular chain group* C, and write $P(C) = \pm (\ldots, P(\bar{\imath}), \ldots)$. The appellation and the notation will presently be justified.

(2.1) *The Plücker coordinates are uniquely determined by the regular chain group* C, *up to a constant multiplicative factor of* ± 1.

Proof. If $\{g_1, \ldots, g_r\}$ is another $\mathbf{Z}$-free base of C, then there is a unimodular $r \times r$ matrix U with integer coefficients such that

$$\begin{bmatrix} g_1 \\ \cdot \\ \cdot \\ \cdot \\ g_r \end{bmatrix} = U \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_r \end{bmatrix}.$$

Consequently, for any ordered r-tuple $\bar{\imath} = (i_1, \ldots, i_r)$, we have

$$\begin{bmatrix} g_1(i_1) & \ldots & g_1(i_r) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ g_r(i_1) & \ldots & g_r(i_r) \end{bmatrix} = U \begin{bmatrix} f_1(i_1) & \ldots & f_1(i_r) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ f_r(i_1) & \ldots & f_r(i_r) \end{bmatrix}.$$

Since $\det(U) = +1$ or -1 independent of $\bar{\imath}$, the last matrix equality proves our assertion.

(2.2) *The correspondence* $C \to P(C) = \pm (\ldots, P(\bar{\imath}), \ldots)$ *is one-to-one* (= *injective*) *into the set of* $\binom{n}{r}$-*tuples of integers, up to the factor of* ± 1.

Proof. Let $u \in C$. Then the matrix

$$\begin{bmatrix} u(1) & u(2) & \ldots & u(n) \\ f_1(1) & f_1(2) & \ldots & f_1(n) \\ \cdot & & & \cdot \\ \cdot & & \ldots & \cdot \\ \cdot & & & \cdot \\ f_r(1) & \ldots & & f_r(n) \end{bmatrix}$$

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has rank r, whence all $(r+1) \times (r+1)$ minors have vanishing determinents: For all $i_0 < i_1 < \ldots < i_r$, we have

$$(*) \quad u(i_0)P(i_1, \ldots, i_r) - u(i_1)P(i_0, i_2, \ldots, i_r) + \ldots + (-1)^r u(i_r)P(i_0, i_1, \ldots, i_{r-1}) = 0.$$

Conversely, suppose that a chain $u \in \mathbf{Z}^n$ satisfies $\binom{n}{r+1}$ linear equations (*). Then, the $(r+1) \times n$ matrix above, viewed as a matrix with rational coefficients, has rank $\leqq r$ (and in fact $= r$), and hence $u = q_1 f_1 + \ldots + q_r f_r$ with rational coefficients q_i. After clearing all the denominators, one gets $au = b_1 f_1 + \ldots + b_r f_r$ with $a \in \mathbf{Z}$, $a \neq 0$, and all $b_i \in \mathbf{Z}$. By virtue of (1.4), it follows, then, that u itself is an integral linear combination of f_i's. Therefore $u \in C$. We have thus established that the homogeneous simultaneous equations (*) is a defining set of equations for C, and, therefore, that the Plücker coordinates $\pm(\ldots, P(\bar{\imath}), \ldots)$ up to the ± 1 factor determine C completely. This proves (2.2).

Next we shall determine the condition for an ordered $\binom{n}{r}$-tuple $(\ldots, U(\bar{\imath}), \ldots)$ of integers $0, \pm 1$ to be the Plücker coordinates of a regular chain group:

(2.3) THEOREM. *Let U be a function defined on the set of all ordered r-tuples $\bar{\imath} = (i_1, \ldots, i_r)$ of integers in $[1\, n]$ taking its values in $\{-1, 0, 1\}$. Then, in order that there exist a regular chain group C of rank r such that $P(C) = \pm(\ldots, U(\bar{\imath}), \ldots)$, it is necessary and sufficient that*

(A) *The $U(\bar{\imath})$'s are not all zero and skew-symmetric on the indices* (viz., *a transposition of i_p and i_q results in the change of sign of $U(\bar{\imath})$ only); and*

(B) *For every combination of $\bar{\imath} = (i_1, \ldots, i_r)$, $\bar{l} = (l_1, \ldots, l_r)$ and $s \in [1\, r]$, the relation*

$$U(i_1, \ldots, i_r)U(l_1, \ldots, l_r) = \sum_{p=1}^{r} U(i_1, \ldots, i_{s-1}, l_p, i_{s+1}, \ldots, i_r)U(l_1, \ldots, l_{p-1}, i_s, l_{p+1}, \ldots, l_r)$$

holds.

For the proof of (2.3) we shall need the following lemma, which is also of an independent interest:

(2.4) LEMMA. *Let C be a regular chain group of rank r and let $P(C) = \pm(\ldots, P(\bar{\imath}), \ldots)$ be its Plücker coordinates. For each $(r+1)$-tuple $(\bar{\imath}, j) = (i_1, \ldots, i_r, j)$ taken from $[1\, n]$, set*

$$F(\bar{\imath}, j; \bar{X}) = X(i_1)P(i_2, \ldots, i_r, j) - X(i_2)P(i_1, i_3, \ldots, i_r, j) + \ldots + (-1)^r X(j)P(i_1, \ldots, i_r).$$

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Suppose that $P(\bar{\imath}) \neq 0$ *for a particular* $\bar{\imath} = (i_1, \ldots, i_r)$. *Then, the set of* $n - r$ *equations* $F(\bar{\imath}, j; \bar{X}) = 0$, $j \neq i_p$ *for any* $p \in [1\, r]$, *is a defining set of equations for* C; *furthermore, every linear form with integral coefficients which vanish on* C *is an integral linear combination of the* $n - r$ *forms* $F(\bar{\imath}, j; \bar{X})$, $j \neq i_p$ *for* $p \in [1\, r]$.

Proof of (2.4). That a chain $\bar{x} = (x(1), \ldots, x(n)) \in C$ satisfies $F(\bar{\imath}, j; \bar{x}) = 0$ is already shown in (2.2). Conversely, let C' be the set of solution vectors (=chains) of the $n - r$ equations. $C' \subseteq \mathbf{Z}^n$ is a chain group of rank r as is evident from the shape of the forms, and contains C. But the rank of C is also r, and $\mathbf{Z}^n/C$ is torsion-free by (1.4). Thus, $C = C'$ follows, proving the first half of our assertion. Next, let $G(\bar{X}) = G(X(1), \ldots, X(n))$ be a linear form vanishing on C. Then, it is a linear combination of the $F(\bar{\imath}, j; \bar{X})$'s with rational coefficients; whence $bG(\bar{X}) = \sum_j a_j F(\bar{\imath}, j; \bar{X})$ with integers b and a_j's. But b must divide all a_j's because the coefficient of $X(j)$ on the right hand side of the equality is $\pm\, a_j$ for each $j \neq i_p$. The lemma is now proven.

Proof of (2.3). We first assume the existence of a regular chain group C with $P(C) = \pm\, (\ldots, U(\bar{\imath}), \ldots)$ and prove (B). (Notice that (A) requires no proof.) We fix an $\bar{\imath} = (i_1, \ldots, i_r)$ such that $U(\bar{\imath}) \neq 0$, and take an arbitrary $(r + 1)$-tuple $(l_0, l_1, \ldots, l_r)$ inside $[1\, n]$. Now let

$$\text{(1)} \qquad \Phi = \sum_{p=0}^{r} (-1)^p F(i_1, \ldots, i_r, l_p; \bar{X}) U(l_0, \ldots, \hat{l}_p, \ldots, l_r)$$

where $\hat{l}_p$ designates the absence of that index, and $F(\bar{\imath}, l_p; \bar{X})$ is the same as in (2.4) except now U's are substituted for P's. Then, Φ clearly vanishes on C. Moreover, a calculation shows that (1) leads to

$$\text{(2)} \quad \Phi = \sum_{q=1}^{r} (-1)^{q+1} \left[\sum_{p=0}^{r} (-1)^p U(i_1, \ldots, \hat{\imath}_q, \ldots, i_r, l_p) U(l_0, \ldots, \hat{l}_p, \ldots, l_r) \right] \times X(i_q) + (-1)^r U(i_1, \ldots, i_r) F(l_0, \ldots, l_r; \bar{X}).$$

Thus, Φ can be written as $\sum_{q=1}^{r} A_q X(i_q) + (-1)^r U(\bar{\imath}) F(\bar{l}; \bar{X})$ with $A_q \in \mathbf{Z}$, and the entire sum as well as its last term vanish on C. Consequently, $\sum_{q=1}^{r} A_q X(i_q)$ vanishes on C. It follows from (2.4) then that this last is a zero form: $A_q = 0$ for all $q \in [1\, r]$. Thus, from (2), we get

$$\text{(3)} \qquad \sum_{p=0}^{r} (-1)^p U(i_1, \ldots, i_{r-1}, l_p) U(l_0, \ldots, \hat{l}_p, \ldots, l_r) = 0$$

which holds valid for every $(r - 1)$-tuple $(i_1, \ldots, i_{r-1})$ for which there is a j such that $U(i_1, \ldots, i_{r-1}, j) \neq 0$. But, if $U(i_1, \ldots, i_{r-1}, j) = 0$ for all j, the equality (3) is trivially true. All in all, we have (3) for all $i_1, \ldots, i_{r-1}$ and all $l_0, \ldots, l_r$, taken from $[1\, n]$. We can now deduce (B) from (3) as follows.

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Firstly, write i_r in lieu of l_p in (3) to get

$$U(i_1, \ldots, i_r)U(l_1, \ldots, l_r) = \sum_{p=1}^{r} U(i_1, \ldots, i_{r-1}, l_p)U(l_1, \ldots, l_{p-1}, i_r, l_{p+1}, \ldots, l_r).$$

Secondly, pick any $s \in [1\ r]$ and perform the permutation

$$\begin{pmatrix} i_1 \ldots i_{s-1} i_s & i_{s+1} \ldots i_{r-1} i_r \\ i_1 \ldots i_{s-1} i_{s+1} i_{s+2} \ldots i_r & i_s \end{pmatrix}$$

on the last equality to get

$$U(i_1, \ldots, i_{s-1}, i_{s+1}, \ldots, i_r, i_s)U(l_1, \ldots, l_r) = \sum_{p=1}^{r} U(i_1, \ldots, i_{s-1}, i_{s+1}, \ldots, i_r, l_p)U(l_1, \ldots, l_{p-1}, i_s, l_{p+1}, \ldots, l_r).$$

By moving i_s on the left hand side and l_p on the right, one obtains (B) as desired.

We shall now prove the sufficiency of (A), (B). Thus, let $U(\bar{\imath})$'s be given satisfying (A) and (B). Without loss of generality, we may assume $U(1, 2, \ldots, r) \neq 0$, and in fact $U(1, 2, \ldots, r) = 1$. This done, we define

$$(4) \qquad f_p(q) = U(1, 2, \ldots, p-1, q, p+1, \ldots, r)$$

for all $p \in [1\ r]$, $q \in [1\ n]$. Let

$$(5) \qquad \begin{bmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_r \end{bmatrix} = \begin{bmatrix} f_1(1) \ldots f_1(r) & f_1(r+1) \ldots f_1(n) \\ \cdot \quad\quad \cdot & \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot & \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot & \cdot \quad\quad \cdot \\ f_r(1) \ldots f_r(r) & f_r(r+1) \ldots f_r(n) \end{bmatrix}$$

whose first $r \times r$ submatrix is actually the identity matrix.

Let

$$(6) \qquad P(i_1, \ldots, i_r) = \det \begin{bmatrix} f_1(i_1) \ldots f_1(i_r) \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ f_r(i_1) \ldots f_r(i_r) \end{bmatrix}$$

for each r-tuple $\bar{\imath} = (i_1, \ldots, i_r)$ from $[1\ n]$. We shall prove that $P(\bar{\imath}) = U(\bar{\imath})$ for all $\bar{\imath}$. (This will tell us that the rows $f_1, \ldots, f_r$ of (5) generate a regular chain group of rank r and the Plücker coordinates of that group are precisely the $U(\bar{\imath})$'s.) Toward that end, suppose that $\{i_1, \ldots, i_r\}$ contain as its subset precisely s integers $k_1, \ldots, k_s$ which are strictly larger than r, and write out

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$\{i_1, \ldots, i_r\} = \{1, \ldots, j_1 - 1, k_1, j_1 + 1, \ldots, j_s - 1, k_s, j_s + 1, \ldots, r\}$. Then

$$P(i_1, \ldots, i_r) = \det \begin{bmatrix} f(j_1; k_1) \ldots f(j_1; k_s) \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ f(j_s; k_1) \ldots f(j_s; k_s) \end{bmatrix} \tag{7}$$

is clear, where for typographical reasons we have written $f(p; q)$ in place of $f_p(q)$ if p has a suffix. If now $s = 0$ or $s = 1$, the desired result $P(\bar{\imath}) = U(\bar{\imath})$ is immediate. Therefore, let us assume $P(\bar{\imath}) = U(\bar{\imath})$ for all $\bar{\imath}$ for which $s < t$, and consider the case $s = t$:

By (B) with $(l_1, \ldots, l_r) = (1, \ldots, r)$ we have

$$U(\bar{\imath}) = \sum_{p=1}^{r} U(i_1, \ldots, i_{s-1}, p, i_{s+1}, \ldots, i_r) U(1, \ldots, p - 1, i_s, p + 1, \ldots, r) \tag{8}$$

and we may let $i_s = k_t =$ the last i not in $[1\ r]$. Note that

$$U(i_1, \ldots, i_{s-1}, p, i_{s+1}, \ldots, i_r) = 0$$

unless $p \in [1\ r]$ is one of $j_1, \ldots, j_t$. Now, by the hypothesis of our induction, we have

$$\begin{aligned} U(i_1, \ldots, i_{s-1}, p, i_{s+1}, \ldots, i_r) &= P(i_1, \ldots, i_{s-1}, p, i_{s+1}, \ldots, i_r) \\ &= \det \begin{bmatrix} f(j_1; k_1) & \ldots f(j_1; k_{t-1}) & f(j_1; p) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ f(j_{t-1}; k_1) & \ldots f(j_{t-1}; k_{t-1}) & f(j_{t-1}; p) \\ f(j_t; k_1) & \ldots f(j_t; k_{t-1}) & f(j_t; p) \end{bmatrix} \\ &= (-1)^{t+\lambda} \det \begin{bmatrix} f(j_1; k_1) & \ldots f(j_1; k_{t-1}) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ f(j_{\lambda-1}; k_1) & \ldots f(j_{\lambda-1}; k_{t-1}) \\ f(j_{\lambda+1}; k_1) & \ldots f(j_{\lambda+1}; k_{t-1}) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ f(j_{t-1}; k_1) & \ldots f(j_{t-1}; k_{t-1}) \end{bmatrix} \end{aligned} \tag{9}$$

where $p = j_\lambda$ and the last equality is the outcome of the expansion by the last column. Finally, then, by (8)

$$U(i_1, \ldots, i_r) = \sum_{\lambda=1}^{t} (-1)^{t+\lambda} \det(\cdot) f(j_\lambda; k_t)$$

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where $\cdot$ is the matrix on the right hand side of (9), and consequently $U(\bar{\imath}) = \det[f(j_\alpha; k_\beta)] = P(\bar{\imath})$.

3. Toward enumeration of regular chain groups. The results in the foregoing section (§ 2) showed that there exists a fundamental parallel between the regular chain groups in a given $\mathbf{Z}^n$ and the linear subspaces in a given finite dimensional vector space. Possessing the main results (2.1), (2.2), and (2.3), one can now freely talk of Plücker coordinates for regular chain groups, and can pursue their analogy with the classical Plücker coordinates further. We shall briefly discuss a few more results parallel to known theorems in the classical Grassmann theory. These may one day prove useful in enumerating the regular chain groups of a given rank in a given $\mathbf{Z}^n$. Because our results in this direction are incomplete (and also because it should be by now clear how readily classical Grassmann theorems are translated in our situation), we shall not go into details of the proofs.

(3.1) *Let $P(C) = \pm (\ldots, P(\bar{\imath}), \ldots)$ be the Plücker coordinates of a regular chain group C of rank r, and suppose $P(1, \ldots, r) = 1$. Then, every $P(\bar{\imath})$ can be expressed as a polynomial with integer coefficients in the $r(n - r)$ 'variables' $P(j, 2, \ldots, r), \ldots, P(1, \ldots, p - 1, j, p + 1, \ldots, r), \ldots, P(1, \ldots, r - 1, j)$ for $j \in [r + 1, n]$.*

Indeed, since the $P(\bar{\imath})$'s satisfy (B) of (2.3), we apply the formula with $(l_1, \ldots, l_r) = (1, \ldots, r)$ and any one $i_s \notin [1\,r]$ to obtain

$$P(\bar{\imath}) = \sum_{p=1}^{r} P(i_1, \ldots, i_{s-1}, p, i_{s+1}, \ldots, i_r) P(1, \ldots, p - 1, i_s, p + 1, \ldots, r).$$

Now there is one less index $\notin [1, r]$ among $i_1, \ldots, i_{s-1}, p, i_{s+1}, \ldots, i_r$. Repeat the process until all indices larger than r are eliminated.

(3.2) *With the notations and the assumptions of* (3.1), *the $r(n - r)$ quantities $P(1, \ldots, p - 1, j, p + 1, \ldots, r)$ for $j \in [r + 1\,n]$, $p \in [1\,r]$ can freely vary within $\{-1, 0, 1\}$.*

Indeed, the relation (B) of (2.3) among this type of $P(\bar{\imath})$'s can only be trivial.

These facts suggest that one should first count the Plücker coordinates of a given rank r with $P(1, \ldots, r) = 1$. By (3.2), there are $3^{r(n-r)}$ possible combinations of values for the free variables $P(1, \ldots, p - 1, j, p + 1, \ldots, r)$ to take. All other $P(\bar{\imath})$'s are well determined by (3.1) for each combination. If values other than 0, ± 1 are obtained for any one of these other $P(\bar{\imath})$'s, the particular combination must be discarded. After this count of the regular chain groups with $P(1, \ldots, r) = 1$ is done, one must count those with $P(1, \ldots, r) = 0$. In this regard, we have

(3.3) *Let $C \subseteq \mathbf{Z}^n$ be a regular chain group of rank r, and*

$$P(C) = \pm (\ldots, P(\bar{\imath}), \ldots).$$

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Then, $P(1, 2, \ldots, r) = 0$ *if and only if* C *contains a non-zero chain of the form* $(0, \ldots, 0, a_{r+1}, \ldots, a_n)$.

Indeed, let $f_1 = (f_1(1), \ldots, f_1(n)), \ldots, f_r = (f_r(1), \ldots, f_r(n))$ be a $\mathbf{Z}$-free base of C. Then, $P(1, \ldots, r) = 0$ means the rank of $[f_i(j)]$ $(i \in [1\ r]$, $j \in [1\ r])$ is less than r, or $\alpha_1 f_1(j) + \ldots + \alpha_r f_r(j) = 0$ for all $j \in [1\ r]$ by a suitable choice of constants α_i's in $\mathbf{Z}$, not all zero. The converse is clear.

We conclude the paper by an example.

Example. The set of all regular chain groups of rank 2 contained in $\mathbf{Z}^4$ is parametrized by the Plücker coordinates $\pm(\ldots, P(ij), \ldots)$ with 6 essential entries. The $P(ij)$'s are 0 or ± 1 but not all zero, $P(ji) = -P(ij)$, and they satisfy the only nontrivial identity

$$P(34)P(12) = P(14)P(32) - P(13)P(42).$$

The number of those C's with $P(12) = 1$ are 3^4 less the number of possible combinations of x, y, z, w ranging over $\{-1, 0, 1\}$ such that $xw - yz = \pm 2$. This last number is clearly 8. Thus there are $3^4 - 8 = 73$ regular chain groups of the given dimension with $P(12) = 1$. As for those with $P(12) = 0$, their number is, by (1.3) and (3.3), equal to 65, which could be derived from the above identity, too. Thus, the total number of regular chain group of rank 2 in $\mathbf{Z}^4$ is 138.

Added in proof. The added items of references 4, 5 and 6 below deal with materials related to the subject matter discussed above. The author thanks Professors G. J. Minty and W. T. Tutte who brought the items to his attention (February 20, 1973).

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Northern Illinois University,
DeKalb, Illinois

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