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ERRATUM/CORRIGENDUM, OCTOBER 2020, FOR 'A BOGOMOLOV UNOBSTRUCTEDNESS THEOREM FOR LOG-SYMPLECTIC MANIFOLDS IN GENERAL POSITION' (*J. INST. MATH. JUSSIEU* 19 (2018), 1509–1519)

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Abstract The general position hypothesis needs strengthening.

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I am grateful to Brent Pym (personal communication) for showing me by an example due jointly to him, Mykola Matviichuk and Travis Schedler – and related to their article in arxiv:math 201008692 – that the general position hypothesis in the paper is not strong enough. As their example shows, it is possible even with the 2-general position hypothesis for the inclusion of the log complex to the log-plus complex to not induce a surjection on local and even global cohomology, and for the Poisson structure to admit deformations where the Pfaffian divisor deforms non locally-trivially.

We are thus led to introduce a stronger general position condition that we call 'very general position' on a Poisson structure. Throughout this note, we keep the notations and assumptions of the original paper.

Definition 1. The Poisson pair (X,Π) is said to be in t-very general position if it is in t-general position and, for any index set $I\subset[m]$ with $|I|\leq t$, the standard basis vectors $e_1,\dots,e_m$ are linearly independent over $\mathbb{Q}$ modulo $\sum\limits_{i\in I}\mathcal{O}_X k_i$, where k_i is the ith column of the submatrix $(b_{ij}:i,j\in[m])$ of B.

Using the notations of §2.2 of the original paper, the significance of the t-very general property is this: for any $I\subset[m],|I|\leq t$, the local log 1-forms $\mathrm{dlog}(x_i),i=1,\dots,m$, are linearly independent over $\mathbb{Q}$ modulo $\sum_{i\in I} x_i\phi_i\mathcal{O}_X+\Omega^1_X$. In other words, for any set of integers $r_1,\dots,r_m$, not all zero, we have

$$\sum r_j \,\mathrm{dlog}(x_j) \notin \sum_{i\in I} x_i\phi_i\mathcal{O}_X+\Omega^1_X. \tag{1}$$

The following is a partial substitute for Lemma 4 of the original paper:

Lemma. *If* (X,Π) *is in* t*-very general position, then* $Q_I^\bullet$ *is exact in degrees* $<t$.

Proof. Continuing with the notations of the paper, let us further set

$$\mathrm{dlog}(x)^K=\bigwedge_{k\in K}\mathrm{dlog}(x_k), \qquad x^K=\prod_{k\in K}x_k.$$

The complex denoted Q_I is generated by subgroups

$$Q_I^{J,K}=x^I\phi_I\,\mathrm{dlog}(x)^J\,\mathrm{dlog}(x)^K\mathcal{O}_{D_I}, \quad J\subset I, K\cap I=\emptyset,$$

with the differential

$$\delta_I=\wedge\,\mathrm{dlog}\left(x^I\right)+d,$$

where the first summand acts on the $\mathrm{dlog}\left(x^I\right)$ factor where d acts solely on the $\mathcal{O}_{D_I}$ factor. Note that the two summands commute. Let $\hat{\mathcal{O}}_{D_I}$ be the formal completion of $\mathcal{O}_{D_I}$ at the maximal ideal $\mathfrak{m}$ of the origin. As usual, the flatness of $\hat{\mathcal{O}}_{D_I}$ over $\mathcal{O}_{D_I}$ yields that it will suffice to prove that the formal completion $Q_I\otimes\hat{\mathcal{O}}_{D_I}$ is exact in degrees $<t$ – that is, that the sequence

$$Q_I\otimes\left(\mathcal{O}_{D_I}/\mathfrak{m}^{k+1}\right)\to Q_I\otimes\left(\mathcal{O}_{D_I}/\mathfrak{m}^{k}\right)\to Q_I\otimes\left(\mathcal{O}_{D_I}/\mathfrak{m}^{k-1}\right)$$

is exact in the middle in degrees $<t$.

Now assume temporarily that the matrix $B=(b_{ij})$ is constant – that is, Φ has constant coefficients in the $\mathrm{dlog}(x_i)$, and hence so do the forms $x_i\phi_i=\sum b_{ij}\,\mathrm{dlog}\left(x_j\right)$. Then the complex admits an action by the torus $T=\mathbb{G}_m^m$, which acts trivially on any $\mathrm{dlog}(x_i)$ and hence acts only on the $\hat{\mathcal{O}}_{D_I}$ factor. Now $\mathcal{O}_{D_I}/\mathfrak{m}^k$ decomposes as a finite direct product of character sheaves $\mathcal{O}_{D_I}^M$ indexed by nonnegative M-tuples M with $M\cap I=\emptyset$ and $|M|<k$. This yields a similar decomposition of $Q_I^{J,K}$ and $Q_I^\bullet$ into subsheaves $Q_I^{J,K,M}$ and $Q_I^{\bullet M}$ as well. Moreover, on $Q_I^{\bullet M}$, δ_I is simply given by the wedge product, with

$$\eta:=\mathrm{dlog}\left(x^Ix^M\right).$$

Note that thanks to our t-very general hypothesis, η is a ‘primitive’ or cotorsion-free element of Q_I^1 – that is, it maps to a nonzero element of the fibre of Q_I^1 at the origin. As is well known and easy to prove (e.g., by making η part of a basis), the wedge product with such an element η yields an exact complex $\wedge^\bullet Q_I^{1,M}=Q_I^{\bullet M}$. Therefore Q_I is exact in degrees $<t$ when B is constant.

In the general case we use semicontinuity: let α_s be multiplication by $s\in\mathbb{C}$ defined locally near the origin, and set $\Phi_s=\alpha_s^*\Phi$. For $s=0$, Φ_s has constant coefficients, hence the corresponding complex is exact in degree $<t$. Then by semicontinuity, the corresponding

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complex remains exact in degrees $< t$ for all small enough s. But since the complexes are equivalent for all $s \neq 0$, it follows that the original complex is exact in degrees $< t$. □

As in the original paper, we deduce the following:

Theorem (Theorem 8 corrected). *The conclusions of Theorem 8 hold under the additional hypothesis that* (X,Π) *is in* 2*-very general position.*

Corollaries 9 and 10 in the paper should be similarly corrected to replace 2-general position by 2-very general position.

The following example, communicated by B. Pym, is due to M. Matviichuk, B. Pym and T. Schedler and is related to their paper in arxiv 2010.08692, though not explicitly contained in it.

Example 2 (Matviichuk, Pym and Schedler). Consider the matrix

$$B = (b_{ij}) = \begin{pmatrix} 0 & 1 & 2 & 4 \\ -1 & 0 & 3 & 5 \\ -2 & -3 & 0 & 6 \\ -4 & -5 & 1 & 0 \end{pmatrix} \tag{2}$$

and the corresponding log-symplectic form on $\mathbb{C}^4$, $\Phi = \sum_{i<j} b_{ij} \frac{dz_i}{z_i} \wedge \frac{dz_j}{z_j}$ and corresponding Poisson structure $\Pi = \Phi^{-1}$, both of which extend to $\mathbb{P}^4$ with Pfaffian divisor $D = (z_0 z_1 z_2 z_3 z_4)$, $z_0 =$ hyperplane at infinity. Then Π admits the first-order Poisson deformation with bivector $z_3 z_4 \partial_{z_1} \partial_{z_2}$, which in fact extends to a Poisson deformation of $(\mathbb{P}^4, \Pi)$ over the affine line $\mathbb{C}$, and the Pfaffian divisor deforms as $(z_3 z_4 z_0 (z_1 z_2 - t z_3 z_4))$, hence non locally-trivially. Correspondingly, the log-plus form $z_3 z_4 \phi_1 \phi_2$ is closed (and not exact). Then $d(z_3 z_4 \phi_1 \phi_2) = 0$ corresponds to the integral column relation

$$k_1 - k_2 + (e_1 + e_2) - (e_3 + e_4) = 0,$$

where the k_i and e_j are the columns of the B matrix and the identity, respectively, showing that Π, although 2-general is not 2-very general.

Matviichuk, Pym and Schedler (arxiv:math 201008692) consider deformations of the 'generalised geometry', in Hitchin's sense, associated to (X,Π) and give an analogue of the theorem in this note for the corresponding deformations (see their Remark 5.24).

References

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