K. Nomizu and L. Rodríguez
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UMBILICAL SUBMANIFOLDS AND MORSE FUNCTIONS

KATSUMI NOMIZU AND LUCIO RODRÍGUEZ

Let M^n be a differentiable manifold (of class C^∞). By a Morse function on M^n we mean a differentiable function whose critical points are all non-degenerate. If f is an immersion of M^n into a Euclidean space R^m, we may obtain Morse functions on M^n in the following way. Let p be a point of R^m and define a differentiable function L_p on M^n by

$$L_p(x) = d(p, f(x))^2 , \qquad x \in M^n$$

where d denotes the Euclidean distance in R^m. Then, for almost all $p \in R^m$, L_p is a Morse function on M^n (see [2], p. 36).

It is a well-known theorem of Reeb that if a compact differentiable manifold M^n admits a Morse function with exactly two critical points, then M^n is a topological sphere (see [2], p. 25). In the present note we shall prove the following results of a geometric nature (in contrast to a topological nature).

THEOREM A. *Let M^n be a connected compact differentiable manifold ($n \geqq 2$) immersed in a Euclidean space R^m. If every Morse function on M^n of the form $L_p, p \in R^m$, has exactly two critical points, then M^n is imbedded as a Euclidean n-sphere.*

Of course, a Euclidean n-sphere in R^m means a hypersphere in a Euclidean $(n+1)$-subspace R^{n+1} of R^m. As a matter of fact, Theorem A follows from the following more general result.

THEOREM B. *Let $M^n, n \geqq 2$, be a connected, complete Riemannian manifold isometrically immersed in a Euclidean space R^m. If every Morse function on M^n of the form $L_p, p \in R^m$, has index 0 or n at any of its critical points, then M^n is imbedded as a Euclidean n-subspace or a Euclidean n-sphere in R^m.*

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As another corollary, we obtain

THEOREM C. *Under the assumptions of Theorem* B, *if the index is always* 0, *then* M^n *is imbedded as a Euclidean n-subspace of* R^m.

1. Preliminaries.

It is necessary to recall certain concepts and results on focal points, which can be found in [2, pp. 32–38]. Although this reference treats submanifolds imbedded in a Euclidean space, the same results hold for immersed submanifolds.

Let f be an immersion of a differentiable manifold M^n into a Euclidean space R^m. A point of the normal bundle N of M^n is denoted by (x, ξ), where x is a point of M^n and ξ is a vector normal to $f(M^n)$ at $f(x)$. Let F be a differentiable mapping of N into R^m given by $F(x, \xi) = f(x) + \xi$. A point $p \in R^m$ is called a focal point of M if $p = F(x, \xi)$, where (x, ξ) is a point of N where the Jacobian F_* of F is degenerate. In this case, we also say that p is a focal point of (M, x). By virtue of Sard's theorem, the set of focal points of M has measure 0.

It is known that a point $p = F(x, \xi)$, where $(x, \xi) \in N$, is a focal point of (M, x) if and only if the endomorphism $I - A_\xi$ on the tangent space $T_x(M^n)$ is degenerate. Here I is the identity transformation of $T_x(M^n)$ and A_ξ is the symmetric endomorphism corresponding to the second fundamental form of M at x in the direction of ξ.

On the other hand, let $p \in R^m$ and consider the function $L_p(x) = d(f(x), p)^2$ on M^n. A point $x \in M^n$ is a critical point of L_p if and only if the vector ξ from $f(x)$ to p is normal to $f(M^n)$. In this case, the Hessian H of L_p at x, which is a bilinear symmetric function on $T_x(M) \times T_x(M)$, is given by

$$H(X, Y) = 2\langle I - A_\xi(X), Y\rangle , \qquad X, Y \in T_x(M^n) ,$$

where $\langle\ ,\ \rangle$ is the inner product on $T_x(M)$ induced from the Euclidean metric in R^m through the immersion f. Thus H is degenerate at x (i.e., x is a degenerate critical point of L_p) if and only if $I - A_\xi$ is degenerate (i.e., p is a focal point of (M, x)). If x is a nondegenerate critical point of L_p, the index at x is equal to the number of negative eigenvalues of $I - A_\xi$, counting multiplicities, in other words, the number of eigenvalues of A_ξ that are larger than 1, counting multiplicities.

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Finally, let $(x, \xi) \in N$, where ξ is a *unit* vector. For $t > 0$, let $p = F(x, t\xi)$. Then p is a focal point of (M, x) if and only if $1/t$ is an eigenvalue of A_ξ. Suppose $1/t$ is not an eigenvalue of A_ξ. Then the function L_p has x as a nondegenerate critical point and the index at x is equal to the number of positive eigenvalues (counting multiplicities) that are greater than $1/t$.

We now prove a lemma which is crucial in the proof of our results.

LEMMA. *Let $p \in R^m$ and assume that the function L_p has a nondegenerate critical point $x \in M^n$ of index k. Then there exists a point $q \in R^m$ such that L_q is a Morse function which has a critical point z of index k. (q and z may be chosen as close to p and x, respectively, as we want.)*

Proof. Let $p = F(x, \xi)$, where ξ is a normal vector at $f(x)$. By assumption, p is not a focal point of (M, x), that is, the Jacobian F_* is nondegenerate at (x, ξ). Thus there exists a neighborhood U of (x, ξ) in the normal bundle N such that F gives a diffeomorphism of U onto a neighborhood $V = F(U)$ of p in R^m. (Of course, U and V may be chosen as small as we like.) Now V has a point q such that L_q is a Morse function (i.e., q is not a focal point of M), because the set of focal points of M has measure 0. We have $q = F(z, \zeta)$ for some $(z, \zeta) \in U$. We show that the index of L_q at z is equal to k.

Consider a differentiable family of symmetric endomorphisms $I - A_\eta$ on $T_y(M^n)$, where (y, η) runs over U. If we denote the eigenvalues by

$$\lambda_1(y, \eta) \geqq \lambda_2(y, \eta) \geqq \cdots \geqq \lambda_n(y, \eta) ,$$

then it can be shown that each λ_i is a continuous function on U. Since F_* is nondegenerate at each point of U, none of these functions takes value 1 on U. The index of L_p at x being k by assumption, we have that $\lambda_1, \cdots, \lambda_k$ are greater than 1 at (x, ξ) and $\lambda_{k+1}, \cdots, \lambda_n$ are less than 1 at (x, ξ). It follows that the same arrangement holds at (z, ζ). This means that the index of L_q at z is equal to k. We have thus proved the lemma.

2. Proof of Theorem B.

Under the assumptions of Theorem B, we shall show the following fact. If $x \in M^n$ and if ξ is a unit vector normal to $f(M^n)$ at $f(x)$, then

$A_\xi = cI$ for some constant c, that is, A_ξ has only one eigenvalue (of multiplicity n). Suppose A_ξ has a non-zero eigenvalue, say, a. We may assume that $a > 0$, because if $a < 0$, then $A_{-\xi}$ has $-a > 0$ as eigenvalue; if we can show that $A_{-\xi} = (-a)I$, then we know that $A_\xi = -A_{-\xi} = aI$.

Assuming thus that a is the largest positive eigenvalue of A_ξ, take $t > 0$ such that $1/a < t < 1/b$, where b is the next largest positive eigenvalue if any (if a is the only positive eigenvalue, just consider $1/a < t$). Then $p = F(x, t\xi)$ is not a focal point of (M, x) and the function L_p has x as a nondegenerate critical point. The index at x is equal to the multiplicity, say, k, of the eigenvalue a. If L_p is a Morse function, the assumption in Theorem B implies $k = n$, since k cannot be 0. Now L_p may not be a Morse function (it can have a degenerate critical point elsewhere). By the lemma in Section 1, however, we know that there must exist a Morse function of the form L_q, $q \in R^m$, which has a critical point z of index k. Thus we may conclude that $k = n$. This means that a is an eigenvalue of A_ξ with multiplicity n so that $A_\xi = aI$.

What we have just shown implies that M^n is umbilical, that is, if η denotes the mean curvature vector field, then for any normal vector ξ at x we have

$$A_\xi = \langle \xi, \eta \rangle I .$$

Equivalently, every $X \in T_x(M^n)$ is a principal vector in the sense that there exists a 1-form ω on the normal space N_x such that

$$A_\xi(X) = \omega(\xi)X \quad \text{for all} \quad \xi \in N_x \quad \text{and} \quad X \in T_x(M) .$$

It is known (see [1, p. 231]) that a complete Riemannian manifold isometrically and umbilically immersed in R^m is actually imbedded as a Euclidean n-subspace or a Euclidean n-sphere. This completes the proof of Theorem B.

It is quite easy to derive Theorem A from Theorem B. If a Morse function L_p has exactly two critical points, then one is where L_p has a maximum (hence of index n) and the other is where L_p has a minimum (hence of index 0). Thus every Morse function L_p has index n or 0 at a critical point.

Suppose S^n is a Euclidean n-sphere in R^m and assume we have taken a rectangular coordinate system $x_1, \cdots, x_m$ in R^m so that

$$S^n = \left\{(x_1, \cdots, x_{n+1}, 0, \cdots, 0) ; \sum_{k=1}^{n+1} x_k^2 = r^2\right\} .$$

Then we can see that the set of focal points of S^n is the Euclidean $(m-(n+1))$-subspace defined by $x_1 = \cdots = x_{n+1} = 0$. If p is not a focal point, the Morse function L_p has exactly two critical points, one of index n and the other of index 0.

What we have just said is sufficient to derive Theorem C from Theorem B.

3. Remarks.

Our main results may be formulated without explicitly involving the notion of Morse functions and, indeed, under a weaker assumption. *Let D be a dense subset of R^m. In Theorems* A, B *and* C, *we may replace "every Morse function on M^n of the form $L_p, p \in R^m$" by "every function on M^n of the form $L_p, p \in D$".*

The proof of Theorem B under this weaker assumption remains almost the same as before except for a corresponding change in the lemma, namely, the conclusion of the lemma should be modified as follows: "*Then there exists a point $q \in D$ such that L_q has a critical point z of index k.*"

Finally, we note that if M^2 immersed in R^m is topologically a 2-sphere, then our original assumption in Theorem A is equivalent to the spherical two-piece property studied by T. F. Banchoff: *The spherical two-piece property and tight surfaces in spheres*, J. Differential Geometry 4 (1970), 193–205 (see, in particular, Theorem 3).

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Brown University