Can. J. Math., Vol. XXXV, No. 6, 1983, pp. 1075-1109

SOME RESULTS ON L-INDISTINGUISHABILITY FOR SL(r)

FREYDOON SHAHIDI

Introduction. Fix a positive integer r. Let $\mathbf{A}_F$ be the ring of adeles of a number field F. For a parabolic subgroup P of SL_r, we fix a Levi decomposition $P = MN$, and we let

$$\mathfrak{a}_M = \mathrm{Hom}\,(X(M)_{\mathbf{Q}}, \mathbf{R}).$$

Let $W(M) = W(\mathfrak{a}_M)$ be the Weyl group of $\mathfrak{a}_M$. It follows from a recent work of James Arthur [**1**, **2**] (also cf. [**3**]) that, among the terms appearing in the trace formula for $SL_r(\mathbf{A}_F)$, coming from the Eisenstein series, are those which are a constant multiple (depending only on M and w) of

(1) $\quad$ trace $(M_{P|P}(w, 0)I(\sigma, P, f))$

where σ is a cusp form on $M(\mathbf{A}_F)$ satisfying $w\sigma \cong \sigma$,

$$f \in C_c^\infty(SL_r(\mathbf{A}_F)),$$

and

$$w \in W(M)_{\mathrm{reg}}$$

$(W^G(\mathfrak{a}_M)_{\mathrm{reg}}$ in the notation of [**2**, **3**]). Here $W(M)_{\mathrm{reg}}$ is the set of all $w \in W(\mathfrak{a}_M)$ for which $\mathfrak{a}_G$ is the space of fixed vectors. $M_{P|P}(w, 0)$ is the Langlands' M-function [**12**], and the operator $I(\sigma, P, f)$ is given by

$$I(\sigma, P, f) = \int_{SL_r(A_F)} I(\sigma, P)(g)f(g)dg,$$

where $I(\sigma, P)$ is the representation of $SL_r(\mathbf{A}_F)$ induced from σ.

To have $W(M)_{\mathrm{reg}}$ non-empty, P is forced to be the intersection of a parabolic subgroup $\widetilde{P}$ of GL_r with SL_r whose Levi factors are isomorphic to m copies of GL_p, where p and m are positive integers with $r = mp$. Then $W(M)$ is isomorphic to the symmetric group in m letters and $W(M)_{\mathrm{reg}}$ is equal to the Coxeter conjugacy class in $W(M)$ which contains the permutation $\widetilde{w}_c$ sending $1 \to m \to m - 1 \to \ldots \to 2 \to 1$.

Received December 1, 1982 and in revised form April 3, 1983. This research was partially supported by NSF grant MCS 81-01600.

https://doi.org/10.4153/CJM-1983-060-9 Published online by Cambridge University Press

These terms are of particular interest (cf. [**3**]) and are not expected to cancel with other terms coming from the Eisenstein series.

On the other hand, there are cusp forms on a certain endoscopic group (cf. [**13**, **17**, **18**]) of $SL_r(\mathbf{A}_F)$, namely the group of elements in $GL_p(\mathbf{A}_E)$ whose determinants have norm one, where E is a cyclic extension of F of degree m, which are not expected to lift to cusp forms on $SL_r(\mathbf{A}_F)$. Consequently, there must be no contribution from their traces to the final form of the trace formula for $SL_r(\mathbf{A}_F)$. These are the terms which are expected to cancel off the terms of form (1). Local components of these traces are linear combinations of characters, each taken over an L-packet, of representations of local components of $SL_r(\mathbf{A}_F)$, whose coefficients are the values of the pairings between the corresponding S-groups and the representations in the corresponding L-packets (cf. [**13**]). In fact, using the conjectured transfer of orbital integrals (cf. [**13**]), these traces on the group of elements of norm one in $GL_p(\mathbf{A}_E)$, when stabilized, lift to such linear combinations. Similar linear combinations must then be expected for local components of (1) (cf. [**11**] for SL_2), and the purpose of this paper is to establish them over non-archimedean places.

More precisely, let k be a non-archimedean local field. Let $G = SL_r(k)$, $r = mp$, and denote by P the parabolic subgroup of G whose Levi factors are isomorphic to the intersection of G with $\widetilde{M} \subset GL_r(k)$ which is a product of m copies of $GL_p(k)$. We assume that P is standard in the sense that it contains the subgroup of upper triangulars. Write $P = MN$ with the Levi factor $M = \widetilde{M} \cap G$. Let σ be an irreducible tempered representation of M. Then σ defines an L-packet $\{\sigma\}$ of tempered representations of M (cf. Section 1). Let

$$\tau = \bigoplus_{\sigma \in \{\sigma\}} \sigma.$$

Set

$$I(\sigma, P) = \operatorname*{Ind}_{P\uparrow G} \sigma \quad \text{and} \quad I(\tau, P) = \operatorname*{Ind}_{P\uparrow G} \tau.$$

Let

$$W(\sigma) = \{w \in W(M), w\sigma \cong \sigma\}.$$

Then in fact $W(\sigma) = W(\tau)$ (cf. [**6**]). Inspired by [**5**, **6**], to every $w \in G$ representing an element of $W(\sigma)$, we attach a pair of complex functions ω_w and $\omega_{w,0}$, the first one on $\widetilde{M}$, which depends upon the realization of τ with $w\tau$ (except when $\{\sigma\}$ is a singleton in which case $\omega_w = \omega_{w,0}$); and the second, a character of $\widetilde{M}_0$, the stabilizer of any $\sigma \subset \tau$, which does not (cf. Lemma 1.2); and in Section 1 of this paper we show that the set of all $\omega_{w,0}$

($w \in W(\sigma)$) is a group which when σ is in the discrete series is isomorphic to the R-group $R(\sigma)$ defined in general by A. W. Knapp and G. Zuckerman **[10]** (cf. Proposition 2.4 of the present paper).

Next, we observe that the local components of $M_{P|P}(w, 0)$ are the standard intertwining operators [**14**, **15**, **19**], defined by

$$A(\sigma, w)f(g) = \int_{N_w} f(w^{-1}ng)dn \quad (g \in G),$$

where w is a representative of a permutation matrix $\widetilde{w} \in W(\sigma)$ in G (cf. Section 2), and $N_w \subset N$ is defined as in Section 2. Here $f \in V(\sigma, P)$, the space of $I(\sigma, P)$. Similarly, we define $A(\tau, w)f$ for f in the space of $I(\tau, P)$. We observe that $A(\tau, w)$ intertwines the irreducible constituents of $I(\tau, P)$, sending those in $I(\sigma, P)$ to themselves.

Now, we consider a particular (and significantly important) normalization of these operators. The normalizing factors are defined in terms of certain local Langlands' root numbers and L-functions (cf. [**7**, **9**, **15**, **16**]). We refer to Section 2 of the present paper for their exact definitions. For each $w \in W(\sigma)$, let $R(\sigma, w)$ (as well as $R(\tau, w)$) be this normalized operator.

Finally, using the results of [**6**], we observe that there exists a finite set $A \subset k^*$ such that if we fix a base representation $\pi_1 \subset I(\tau, P)$, then every other representation in $I(\tau, P)$ is of the form $\widetilde{a} \cdot \pi_1$ for some unique $1 \neq a \in A$, where

$$\widetilde{a} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL_r(k),$$

and $\widetilde{a} \cdot \pi_1$ is defined by

$$(\widetilde{a} \cdot \pi_1)(g) = \pi_1(\widetilde{a}^{-1}g\widetilde{a}) \quad (g \in G).$$

Fixing $\pi_1 \subset I(\tau, P)$, write $\pi_a = \widetilde{a} \cdot \pi_1$, and let

$$A_\sigma = \{a \in A | \pi_a \subset I(\sigma, P)\}.$$

Finally, for every $\pi \subset I(\tau, P)$, let χ_π be its character. Then our Theorem 3.1 asserts that: there exist a realization of τ with $w\tau$ and a base representation $\pi_1 \subset I(\tau, P)$ such that for every $f \in C_c^\infty(G)$:

$$\text{trace } (R(\sigma, w)I(\sigma, P, f)) = \sum_{a \in A_\sigma} \omega_w(\widetilde{a})\chi_{\pi_a}(f),$$

and furthermore if π_1 is changed, the coefficients $\omega_w(\widetilde{a})$ remain proportional.

Moreover, Corollary 3.7 implies that with a suitable change of the above realization and introduction of a new set $A' \subset A$ of indices with representatives $\tilde{a}$ in $\tilde{M}_0$, together with a bijection $\alpha_\sigma : A' \cong A_\sigma$, defined for every $\sigma \subset \tau$, we have

$$\text{trace}\,(R(\sigma, w)I(\sigma, P, f)) = \sum_{A'} \omega_{w,0}(a)\, \chi_{\pi_{\alpha_\sigma(a)}}(f).$$

Now, suppose σ is in the discrete series and assume that there exists $\phi: W_k \to {}^L M$, generating $\{\sigma\}$ (cf. [**6**]). Then

$$R(\sigma) \cong S(\phi)/S_M(\phi)$$

(Proposition 1.9) and, pulling back an element $\varkappa \in S(\phi)$ to $\omega_{w,0} \in R(\sigma)$ and accepting the previous conjectural discussion, $\omega_{w,0}(a)$ is the pairing $\langle \varkappa, \pi_{\alpha_\sigma}(a) \rangle$ conjectured by Langlands [**13**] in this case.

One consequence of the proof of Theorem 3.1 is Corollary 3.6 which for every $\tilde{w} \in W(\tau)$, computes the effect of $A(\tau, w)$ on the subspaces of $I(\tau, P)$.

The functions ω_w and R-groups are studied in Section 1, most of which is based on the results of [**6**]. Intertwining operators and their normalization are explained in Section 2. Section 3 is devoted to the proof of Theorem 3.1.

The results are expected to generalize to any reductive group, but the proofs are based on two facts which are particular to the group SL_r. The first that every irreducible admissible representation of M is a subrepresentation of the restriction of one of $\tilde{M}$ to M, and that there exists a simply transitive $\tilde{M}$-action on this restriction, and the second that the tempered representations of $GL_r(k)$ are non-degenerate and therefore possess Whittaker models [**8**]. The first result is due to S. S. Gelbart and A. W. Knapp [**6**]. We must remark that our Theorem 3.1 does not make use of their working hypotheses.

When $m = r = 2$, P is minimal and Theorem 3.1 becomes Lemma 3.6 of [**11**] whose proof, even though short, is fairly involved. In the present paper, the results of [**16**] play a very important role in providing us with a method which applies to the general case.

Many thanks are due to James Arthur. First for inviting me to the University of Toronto where I obtained these results. Second for his warm hospitality which made my stay very pleasant and useful. Third, but not last, for generously sharing his insights and ideas with me. I would also like to thank the Department of Mathematics at the University of Toronto for their hospitality.

1. Preliminaries and R-groups. Let p and r be two positive integers. Assume that p divides r. Let m be such that $r = mp$. For a non-archimedean local field k, let $G = SL_r(k)$ and $\widetilde{G} = GL_r(k)$. Let $\widetilde{P}$ be the standard parabolic subgroup of $\widetilde{G}$ attached to the partition $p + \ldots + p = r$ of r. Set $P = \widetilde{P} \cap G$. It is a standard parabolic subgroup of G. For global reasons (cf. [**3**]), explained in the introduction, these are the only parabolic subgroups of G in which we are interested. Write $\widetilde{P} = \widetilde{M}N$ and $P = MN$ with standard Levi factors $\widetilde{M}$ and M. Then $M = \widetilde{M} \cap G$.

Let σ be an irreducible tempered representation of M. We start with the following lemma.

LEMMA 1.1. *There exists an irreducible tempered representation $\widetilde{\sigma}$ of $\widetilde{M}$ such that σ is a constituent of $\widetilde{\sigma}|M$. Furthermore, suppose $\widetilde{\sigma}_1$ is another such representation of $\widetilde{M}$; then there exists a character $\nu \in \hat{k}^*$ of k^* such that $\widetilde{\sigma}_1 \cong \widetilde{\sigma} \otimes \nu \cdot \det$.*

Proof. Let $Z_{\widetilde{G}}$ denote the center of $\widetilde{G}$. Then $M \cdot Z_{\widetilde{G}}$ is an open normal subgroup of $\widetilde{M}$ such that $\widetilde{M}/M \cdot Z_{\widetilde{G}}$ is finite abelian. Now the existence of $\widetilde{\sigma}$ follows from Lemma 2.3 of [**6**]. To prove the second assertion, we first observe that using an argument similar to Lemma 2.6 of [**11**], one can conclude that

$$\operatorname*{Ind}_{P\uparrow G}(\widetilde{\sigma}|M) = \operatorname*{Ind}_{\widetilde{P}\uparrow \widetilde{G}} \widetilde{\sigma}|G$$

is multiplicity free. Consequently, so is $\widetilde{\sigma}|M$. Then by Lemma 2.4 of [**6**], there exists a character of $\widetilde{M}$, which can easily be shown is of the form $\nu \cdot \det$, $\nu \in \hat{k}^*$, such that

$$\widetilde{\sigma}_1 \cong \widetilde{\sigma} \otimes \nu \cdot \det.$$

Remark. From now on, whenever we use a character $\nu \in \hat{k}^*$ as a character of $\widetilde{M}$ or $\widetilde{G}$, we shall in fact mean the character $\nu \cdot \det$.

Let W_k be the Weil group of $\bar{k}$ over k, and let ϕ be an admissible homomorphism from W_k into ${}^LM$, the L-group of M which may be considered to be the quotient of

$$\overbrace{GL_p(\mathbf{C}) \times \ldots \times GL_p(\mathbf{C})}^{m}$$

by $\mathbf{C}^*$, embedded diagonally there. Clearly ϕ is an admissible homomorphism from W_k into ${}^LG \cong PGL_r(\mathbf{C})$. Then, as in [**6**], ϕ lifts to an admissible homomorphism $\widetilde{\phi}$ from W_k into ${}^L\widetilde{G} \cong GL_r(\mathbf{C})$. Furthermore, the image of W_k under $\widetilde{\phi}$ lies inside

https://doi.org/10.4153/CJM-1983-060-9 Published online by Cambridge University Press

$$ {}^{L}\widetilde{M} \cong \overbrace{GL_p(\mathbf{C}) \times \ldots \times GL_p(\mathbf{C})}^{m}. $$

Write $\widetilde{\phi}(w) = (\widetilde{\phi}_1(w), \ldots, \widetilde{\phi}_m(w))$, where each $\widetilde{\phi}_i$ is an admissible homomorphism from W_k into $GL_p(\mathbf{C})$. By Langlands' conjectured reciprocity (cf. [**6**]), let $\widetilde{\sigma}_i$ be the representation of $GL_p(k)$ attached to $\widetilde{\phi}_i$, $1 \leqq i \leqq m$. Then similar arguments as in Theorems 4.1, 4.2, and 4.3 of [**6**] show that, if $\widetilde{\sigma} = \otimes \widetilde{\sigma}_i$, then the irreducible constituents of $\widetilde{\sigma}|M$ provide the L-packet of M attached to ϕ. For this reason, for every $\widetilde{\sigma}$ in Lemma 1.1 containing σ, we use $\{\sigma\}$ to denote the irreducible constituents of $\widetilde{\sigma}|M$ and we call them the L-packet attached to σ.

Now, let

$$ I(\widetilde{\sigma}, \widetilde{P}) = \operatorname*{Ind}_{\widetilde{P}\uparrow\widetilde{G}} \widetilde{\sigma}. $$

Observe that since $\widetilde{\sigma}$ is tempered, $I(\widetilde{\sigma}, \widetilde{P})$ is irreducible [**8**]. Set $\tau = \widetilde{\sigma}|M$. Then, if

$$ I(\sigma, P) = \operatorname*{Ind}_{P\uparrow G} \sigma \quad \text{and} \quad I(\tau, P) = \operatorname*{Ind}_{P\uparrow G} \tau, $$

we have

$$ I(\sigma, P) \subset I(\tau, P) \cong I(\widetilde{\sigma}, \widetilde{P})|G. $$

From the results of [**6**], it is clear that irreducible constituents of $I(\tau, P)$ constitute a tempered L-packet for G. For an irreducible admissible representation ρ of $\widetilde{G}$, let

$$ X(\rho) = \{\mu \in k^*|\rho \cong \rho \otimes \mu \cdot \det\}. $$

Incidentally, we would like to remark that, when r is a prime, $X(\rho)$ is a r-group. A similar notion is defined for an irreducible admissible representation of $\widetilde{M}$.

Now, from Lemma 2.1 of [**6**], it follows that the number of irreducible constituents of $I(\tau, P)$, i.e., the number of elements in the L-packet of G provided by $I(\widetilde{\sigma}, \widetilde{P})$, is equal to the cardinality n of $X(I(\widetilde{\sigma}, \widetilde{P}))$. As we mentioned in the proof of Lemma 1.1, the fact that $I(\widetilde{\sigma}, \widetilde{P})|G$ is multiplicity free can be proved by an argument similar to Lemma 2.6 of [**11**].

Write

$$ I(\tau, P) = \bigoplus_{i=1}^{n} \pi_i. $$

Let

$$\widetilde{G}_{\pi_i} = \{g \in \widetilde{G} | g\pi_i \cong \pi_i\},$$

where $g\pi_i$ is the representation $h \mapsto \pi_i(g^{-1}hg)$ of G. Now, Corollary 2.2 of [**6**] implies that $\widetilde{G}_{\pi_i}$ is independent of i and is in fact equal to the subgroup

$$\widetilde{G}_0 = \{g \in \widetilde{G} | \nu(\det g) = 1 \text{ for all } \nu \text{ in } X(I(\widetilde{\sigma}, \widetilde{P}))\}$$

of $\widetilde{G}$.

Again by Lemma 2.1 of [**6**], $\widetilde{G}/\widetilde{G}_0$ acts simply transitively on the set of π_i's. Since $\widetilde{G}_0 \supset G$, we may choose the representatives of $\widetilde{G}/\widetilde{G}_0$ to be of the form

$$\widetilde{a} = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL_r(k)$$

with $a \in k^*$.

Now, consider the map $a \mapsto \widetilde{G}_0 \cdot \widetilde{a}$ from k^* onto $\widetilde{G}/\widetilde{G}_0$. From the definition of $\widetilde{G}_0$, it follows that the kernel of this map is equal to

$$N_{\widetilde{\sigma}} = \bigcap_{\nu \in X(I(\widetilde{\sigma}, \widetilde{P}))} \operatorname{Ker} \nu,$$

and therefore we have an isomorphism

$$k^*/N_{\widetilde{\sigma}} \cong \widetilde{G}/\widetilde{G}_0.$$

If we now fix a representation π_1 in the L-packet, we can index other representations in the L-packet by a fixed set

$$A = \{a_i | 1 \leqq i \leqq n, a_1 = 1\}$$

of representatives of $k^*/N_{\widetilde{\sigma}}$. More precisely, we let π_{a_i} denote $\widetilde{a}_i\pi_1$, $1 \leqq i \leqq n$. Similar remarks can be applied to the pair $(\widetilde{M}, M)$.

Now, let $W(M)$ denote the Weyl group of A in G. It is isomorphic to S_m, the symmetric group in m letters. Again for global reasons (cf. [**3**]), we are particularly interested in those elements of $W(M)$ which lie in the Coxeter conjugacy class. More precisely, the conjugacy class which contains the elements $\widetilde{w}_c$, defined by the permutation $1 \to m \to m - 1 \to \dots \to 2 \to 1$. As in [**2**, **3**], we denote this conjugacy class by $W(M)_{\text{reg}}$. We observe that, modulo the center of G, only the trivial element of the center of M can be fixed by an element of $W(M)_{\text{reg}}$.

The following lemma is crucial. It is basically Lemma 2.4 of [**6**] and we have only rearranged its proof.

LEMMA 1.2. a) *Let $w \in G$, representing the element of $W(M)$, be such that $w\sigma \cong \sigma$. Choose $\tilde{\sigma}$ such that $\sigma \subset \tilde{\sigma}$. Fix an intertwining operator $\phi:\tilde{\sigma}|M \cong w\tilde{\sigma}|M$. Then there exists a non-zero complex valued function $\omega_{w,\sigma}$ on $\tilde{M}$, which depends on ϕ (except when $\tilde{\sigma}|M = \sigma$), but is independent of $\tilde{\sigma}$ and the choice of σ in its class, such that for all $m \in \tilde{M}$, $\tilde{\sigma}(m^{-1}) \cdot w\tilde{\sigma}(m)$ acts as $\omega_{w,\sigma}(m)$ on the space of σ (here σ runs over its L-packet). Furthermore, the restriction $\omega_{w,0}$ of $\omega_{w,\sigma}$ to $\tilde{M}_0$, the stabilizer of any representation in the L-packet of σ, is a character of $\tilde{M}_0$ which is independent of ϕ and the choice of σ in its L-packet. Moreover it is trivial on M, and therefore factors through the determinant. In particular, when $X(\tilde{\sigma})$ is trivial, $\omega_{w,\sigma}$ is a character of $\tilde{M}$.*

b) *Let $\omega_{w,0} = \omega_{w,\sigma}|\tilde{M}_0$. Extend $\omega_{w,0}$ in any way to a character of $\tilde{M}$ which we still denote by $\omega_{w,0}$. Then $w\tilde{\sigma} \cong \tilde{\sigma} \otimes \omega_{w,0}$. In particular, $\omega_{w,0}$ is unique up to an element of $X(\tilde{\sigma})$, and thus, modulo $X(\tilde{\sigma})$, $\omega_{w,0}$ determines a unique class.*

c) *Suppose w is changed to $w' = ws$ with $s \in M$. Define $\phi':\tau \cong w'\tau$ by $\phi' = \tau(s^{-1})\phi$. Then for each σ, $\omega_{w',\sigma} = \omega_{w,\sigma}$, and $\omega_{w',0} = \omega_{w,0}$. In particular, $\omega_{w,0}$ is independent of the choice of w in G.*

Proof. We first realize $\tau = \tilde{\sigma}|M$, and $w\tau$, which are equivalent on the same space, i.e., let $w\tilde{\sigma}$ denote $\phi^{-1} \cdot w\tilde{\sigma} \cdot \phi$. To prove the existence of $\omega_{w,\sigma}$, we only need check that for every $m \in \tilde{M}$ and $h \in M$

$$(1.2.1) \quad \tilde{\sigma}^{-1}(m) \cdot w\tilde{\sigma}(m) \cdot \tilde{\sigma}(h) = \tilde{\sigma}(h) \cdot \tilde{\sigma}^{-1}(m) \cdot w\tilde{\sigma}(m).$$

Then the existence of $\omega_{w,\sigma}$ would immediately follow from Schur's Lemma. Fix $m \in \tilde{M}$. Then for every $h \in M$, $mhm^{-1} \in M$, and therefore by our realization

$$\tau(mhm^{-1}) = w\tau(mhm^{-1})$$

or

$$w\tilde{\sigma}(mhm^{-1}) = \tilde{\sigma}(mhm^{-1})$$

which implies (1.2.1).

The fact that $\omega_{w,\sigma}$ is independent of $\tilde{\sigma}$ is now a consequence of Lemma 1.1. Clearly, it also depends only on the class of σ.

Next, for $m_1 \in \tilde{M}_0$, write

$$\tilde{\sigma}(m_1^{-1}) \cdot w\tilde{\sigma}(m_1) = \omega_{w,\sigma,0}(m_1)I$$

on the space V_σ of σ, where $\omega_{w,\sigma,0} = \omega_{w,\sigma}|\tilde{M}_0$ (we have not yet proved that

$\omega_{w,\sigma,0}$ is independent of the choice of σ in its L-packet). Multiplying both sides on the right by $w\widetilde{\sigma}(m_2)$ and on the left side by $\widetilde{\sigma}(m_2^{-1})$, $m_2 \in \widetilde{M}_0$, then implies

$$\omega_{w,\sigma,0}(m_1) \cdot \omega_{w,\sigma,0}(m_2)I = \omega_{w,\sigma,0}(m_1m_2)I \quad (\text{on } V_\sigma),$$

proving that $\omega_{w,\sigma,0}$ is a character of $\widetilde{M}_0$. The fact that $\omega_{w,\sigma,0}$ is independent of the choice of σ in its L-packet will be proved as a corollary of part b). The rest of part a) follows from definitions.

To prove the second assertion, write

$$\widetilde{\sigma}|M = \bigoplus_{i=1}^{N} \sigma_i$$

with $\sigma_1 = \sigma$.

Choose $m \in \widetilde{M}$, such that $m\sigma_1 \cong \sigma_i$. Then $m(w\sigma_1) \cong w\sigma_i$, and

$$\widetilde{\sigma}(m) \cdot w\widetilde{\sigma}(m^{-1}) \cdot \omega_{w,\sigma,0}(m)$$

acts on the space of σ_i as an scalar $B_{w,\sigma_i}(m)$. Observe that for $m \in \widetilde{M}_0$, $B_{w,\sigma}(m) = 1$. Now, if $m \in \widetilde{M}$ and $m_0 \in \widetilde{M}_0$, then on the space V_{σ_i} of σ_i

$$\begin{aligned} B_{w,\sigma_i}(mm_0) &= \widetilde{\sigma}(m)\omega_{w,\sigma,0}^{-1}(m_0)w\widetilde{\sigma}(m^{-1})\omega_{w,\sigma,0}(mm_0) \\ &= B_{w,\sigma_i}(m). \end{aligned}$$

Define B_w on $V = \bigoplus_{i=1}^{N} V_{\sigma_i}$ to be $B_{w,\sigma_i}(m)$ on V_{σ_i} for any $m \in \widetilde{M}$ such that $m\sigma_1 \cong \sigma_i$. Then an argument similar to the one given at the end of the proof of Lemma 2.4 of [**6**] implies

$$B \cdot w\widetilde{\sigma}(m) = \widetilde{\sigma}(m)\omega_{w,\sigma,0}(m) \cdot B.$$

Now, suppose σ' is another element in the L-packet. Then from the equivalence

$$\widetilde{\sigma} \otimes \omega_{w,\sigma,0} \cong \widetilde{\sigma} \otimes \omega_{w,\sigma',0}$$

it follows that

$$\omega_{w,\sigma,0}\, \omega_{w,\sigma',0}^{-1} \in X(\widetilde{\sigma}).$$

Restricting to $\widetilde{M}_0$, we then have

$$\omega_{w,\sigma,0}\, \omega_{w,\sigma',0}^{-1}(m) = 1, \forall m \in \widetilde{M}_0.$$

This proves the independence of $\omega_{w,\sigma,0}|M_0 = \omega_{w,0}$ from the choice of σ in its L-packet.

From now on, we shall use $\overline{\omega}_{w,0}$ to denote the unique class of $\omega_{w,0}$ modulo $X(\widetilde{\sigma})$. Now, let w be as in Lemma 1.2.

COROLLARY 1.3. *The functions* $\{\omega_{w,\sigma}\}_\sigma$ *satisfy the following cocycle relation.*

$$\omega_{w,\sigma}(mm') = \omega_{w,m'0}(m)\omega_{w,\sigma}(m') \quad (m, m' \in \widetilde{M}).$$

In particular

$$\omega_{w,a^{-1}\sigma}(\widetilde{a}) = \omega_{w,\sigma}^{-1}(\widetilde{a}^{-1}) \quad (a \in A).$$

Proof. Fix σ in τ. Let I be the identity on V_σ. Then

$$\widetilde{\sigma}^{-1}(m)w\widetilde{\sigma}(m) = \omega_{w,\sigma}(m)I.$$

But for $m' \in \widetilde{M}$,

$$\widetilde{\sigma}(m')V_\sigma = V_{m'\sigma} \quad \text{and} \quad w\widetilde{\sigma}(m')V_\sigma = V_{m'\sigma}.$$

Consequently

$$\begin{aligned}\widetilde{\sigma}^{-1}(m)w\widetilde{\sigma}(m)V_\sigma &= \widetilde{\sigma}^{-1}(m)w\widetilde{\sigma}(m)w\widetilde{\sigma}(m'^{-1})V_{m'\sigma}\\ &= \omega_{w,\sigma}(m)w\widetilde{\sigma}(m'^{-1})V_{m'\sigma}.\end{aligned}$$

Multiplying both sides on the left by $\widetilde{\sigma}(m')$, we get

$$\widetilde{\sigma}^{-1}(mm'^{-1})w\widetilde{\sigma}(mm'^{-1})V_{m'\sigma} = \omega_{w,\sigma}(m)\widetilde{\sigma}(m')w\widetilde{\sigma}(m'^{-1})V_{m'\sigma}$$

or

$$\omega_{w,m'\sigma}(mm'^{-1}) = \omega_{w,\sigma}(m)\omega_{w,m'\sigma}(m'^{-1}).$$

Changing σ to $m'^{-1}\sigma$ now completes the corollary.

Now, fix $\pi_1 \subset I(\tau, P)$, irreducible, and A as before, and let $A_\sigma \subset A$ be the set of all $a \in A$ such that $\widetilde{a} \cdot \pi_1$ is in $I(\sigma, P)$. Let

$$b_{w,\sigma} = \omega_{w,\sigma} \cdot \omega_{w,0}^{-1}.$$

Then $b_{w,\sigma}(m) = 1$ for all $m \in \widetilde{M}_0$. Furthermore, it is easy to check that

$$b_{w,\sigma}(mm_0) = b_{w,\sigma}(m)$$

for all $m \in \widetilde{M}$ and $m_0 \in \widetilde{M}_0$. Again, let

$$N_{\widetilde{\sigma}} = \bigcap_{\nu \in X(I(\widetilde{\sigma},\widetilde{P}))} \operatorname{Ker} \nu$$

which is independent of the choice of $\widetilde{\sigma}$. We then have

LEMMA 1.4. *The functions* $b_{w,\sigma}$ *and* $\omega_{w,0}$ *are both functions on* $k^*/N_{\widetilde{\sigma}}$. *Consequently for each* $a \in A_\sigma$, *the value of* $\omega_{w,\sigma}(\widetilde{a})$ *is independent of the choices of the representatives of* $k^*/N_{\widetilde{\sigma}}$.

Proof. Using simple standard intertwining operators and Lemma 1.2, the following equivalences are clear.

$$\begin{aligned} I(\widetilde{\sigma}, \widetilde{P}) \otimes \omega_{w,0} &\cong I(\widetilde{\sigma} \otimes \omega_{w,0}, \widetilde{P}) \\ &\cong I(w\widetilde{\sigma}, \widetilde{P}) \\ &\cong I(\widetilde{\sigma}, \widetilde{P}). \end{aligned}$$

Here we have realized $\omega_{w,0}$ with $\omega_{w,0} \cdot \det$ for some unique $\omega_{w,0} \in \hat{k}^*$. Thus

$$\omega_{w,0} \in X(I(\widetilde{\sigma}, \widetilde{P})) \quad \text{and} \quad \omega_{w,0}(N_{\widetilde{\sigma}}) = 1.$$

For $b_{w,\sigma}$, suppose $a = na'$, where $n \in N_{\widetilde{\sigma}}$ and $a' \in k^*$. Then $\nu(n) = 1$ for all $\nu \in X(I(\widetilde{\sigma}, \widetilde{P}))$ and in particular for all $\nu \in X(\widetilde{\sigma})$. But then by Corollary 2.2 of [**6**], $\widetilde{n} \in \widetilde{M}_0$. Now

$$b_{w,\sigma}(\widetilde{a}) = b_{w,\sigma}(\widetilde{n}\,\widetilde{a}') = b_{w,\sigma}(\widetilde{a}')$$

which completes the lemma.

Since throughout this paper M is fixed, we use $W(\sigma)$ to denote the following subgroup of $W(M)$

$$W(\sigma) = \{w \in W(M) | w\sigma \cong \sigma\}.$$

For every $w \in W(\sigma)$, Lemma 1.2 attaches a family of functions $\{\omega_{w,\sigma}\}$, where σ_i runs over the L-packet of σ.

Fix w as before. Choose i, uniquely, such that $\pi_1 \subset I(\sigma_i, P)$. Let ω_w be the function on $\widetilde{M}$ defined by

$$\omega_m(m) = \omega_{w,\sigma_i}(m).$$

It is clearly well defined, but its values depend on the choice of π_1. It can also be explained in terms of where $m\pi_1$ appears. In fact we have

LEMMA 1.5. *Choose l, uniquely, such that $m\pi_1 \subset I(\sigma_l, P)$. Then*

$$\omega_{\mathrm{w}}(m) = \omega_{w,\sigma_l}^{-1}(m^{-1}).$$

Proof. For each $m \in \widetilde{M}$, the mapping $f \mapsto f_1$, from $m \cdot I(\sigma_i, P)$ onto $I(m\sigma_i, P)$ defined by $f_1(g) = f(m^{-1}gm)$ defines an isomorphism which commutes with the action of G. Consequently,

$$m\pi_1 \subset I(m\sigma_i, P),$$

which implies $\sigma_l = m\sigma_i$. Now by Corollary 1.3, we have

$$\begin{aligned}\omega_w(m) &= \omega_{w,\sigma_i}(m)\\ &= \omega^{-1}_{w,m\sigma_i}(m^{-1})\\ &= \omega^{-1}_{w,\sigma_l}(m^{-1}),\end{aligned}$$

which completes the lemma.

Now, if we change our base representation π_1 to another representation $\pi_1' \subset I(\sigma_j, P)$ for some unique j, then the function ω_w would certainly change. The following lemma describes this change.

LEMMA 1.6. *Let $\sigma_j = m_j\sigma_i$ for some $m_j \in \widetilde{M}$. Then*

$$\omega_{w,\sigma_j}(m) = \omega^{-1}_{w,\sigma_i}(m_j)\omega_{w,\sigma_i}(m)\omega_{w,m\sigma_i}(mm_jm^{-1})$$

for all $m \in \widetilde{M}$.

Proof. By Corollary 1.3

$$\begin{aligned}\omega_{w,m\sigma_i}(mm_jm^{-1}) &= \omega_{w,m\sigma_i}(m^{-1})\omega_{w,\sigma_i}(mm_j)\\ &= \omega^{-1}_{w,\sigma_i}(m)\omega_{w,\sigma_i}(mm_j)\\ &= \omega^{-1}_{w,\sigma_i}(m)\omega_{w,\sigma_i}(m_j)\omega_{w,\sigma_j}(m)\end{aligned}$$

which proves the lemma.

Now, given $w \in W(\sigma)$, let, as before,

$$\omega_{w,0} = \omega_{w,\sigma}|\widetilde{M}_0.$$

Define

$$R(\sigma) = \{\omega_{w,0}|w \in W(\sigma)\};$$

we have:

LEMMA 1.7. *$R(\sigma)$ is a group under the multiplication*

$$\omega_{w_1,0}\cdot\omega_{w_2,0} = \omega_{w_1w_2,0}$$

and the inverse operation

$$\omega^{-1}_{w,0} = \omega_{w^{-1},0}.$$

Proof. Fix $m \in \widetilde{M}_0$, and choose $\sigma \subset \tau$. Then on V_σ

$$\widetilde{\sigma}^{-1}(m)w_1w_2\widetilde{\sigma}(m) = \omega_{w_1w_2,0}(m)I$$

which, using Lemma 1.1 and $w_2\sigma \cong \sigma$, can be written as (again on V_σ)

$$\widetilde{\sigma}^{-1}(m)w_2\widetilde{\sigma}(m)w_2\widetilde{\sigma}(m^{-1})w_1w_2\widetilde{\sigma}(m) = \omega_{w_2,0}(m)\omega_{w_1,0}(m)I.$$

Thus

$$\omega_{w_1,0} \cdot \omega_{w_2,0} = \omega_{w_1w_2,0}$$

and the lemma follows.

Set

$$W'(\sigma) = \{w \in W(\sigma) \mid \widetilde{\sigma} \cong w\widetilde{\sigma}\}.$$

We now have

PROPOSITION 1.8. *The group $R(\sigma)$ is isomorphic to the quotient $W(\sigma)/W'(\sigma)$.*

Proof. Let Θ be the map from $W(\sigma)$ onto $R(\sigma)$ defined by $\Theta(w) = \omega_{w,0}$. We only need to show $W'(\sigma) = \text{Ker}(\Theta)$. First suppose $\omega_{w,0} = 1$. Extend $\omega_{w,0}$ to $\widetilde{M}$ again to be equal to the trivial character of $\widetilde{M}$. Then by part b) of Lemma 1.2, $w\widetilde{\sigma} \cong \widetilde{\sigma}$. Thus $w \in W'(\sigma)$. Conversely, choose $w \in W'(\sigma)$. Then again by part b) of Lemma 1.2,

$$w\widetilde{\sigma} \cong \widetilde{\sigma} \otimes \widetilde{\omega}_{w,0} \cong \widetilde{\sigma},$$

where $\widetilde{\omega}_{w,0}$ denotes an extension of $\omega_{w,0}$. But then $\widetilde{\omega}_{w,0} \in X(\widetilde{\sigma})$ which implies

$$\omega_{w,0} = \widetilde{\omega}_{w,0}|\widetilde{M}_0 = 1,$$

completing the proposition.

Remark. In Proposition 2.4 of the next section we shall prove that when σ is in the discrete series, those $w \in W(\sigma)$ which belong to $W'(\sigma)$ are exactly the ones which make the Plancherel measure zero, and therefore our R-group is in complete agreement with its definition given in general in **[10]**.

Finally for $w \in W(\sigma)/W'(\sigma)$, let $\omega_{w,0} \in \hat{k}^*$ be the character of k^* fixed in Lemma 1.2. Then by part b) of the same lemma,

$$w\widetilde{\sigma} \cong \widetilde{\sigma} \otimes \omega_{w,0} \cdot \det,$$

and consequently

$$\begin{aligned} I(\widetilde{\sigma}, \widetilde{P}) &\cong I(w\widetilde{\sigma}, \widetilde{P}) \\ &\cong I(\widetilde{\sigma}, \widetilde{P}) \otimes \omega_{w,0} \end{aligned}$$

and therefore $\omega_{w,0} \in X(I(\widetilde{\sigma}, \widetilde{P}))$. Furthermore, the map $w \mapsto \omega_{w,0}$ is well defined if we consider the image $\bar{\omega}_{w,0}$ of $\omega_{w,0}$ inside $X(I(\widetilde{\sigma}, \widetilde{P}))/X(\widetilde{\sigma})$, and moreover it establishes an injection from $R(\widetilde{\sigma})$ into $X(I(\widetilde{\sigma}, \widetilde{P}))/X(\widetilde{\sigma})$.

Now, assume $\widetilde{\sigma}$ is in the discrete series. Take $\omega \in X(I(\widetilde{\sigma}, \widetilde{P}))$. Then

$$I(\widetilde{\sigma}, \widetilde{P}) \cong I(\widetilde{\sigma} \otimes \omega, \widetilde{P}),$$

and from Theorem 5.4.4.1 of [**19**] it follows that $w\widetilde{\sigma} \cong \widetilde{\sigma} \otimes \omega$ for some $w \in W(M)$. Then $w\sigma \cong \sigma$ for every σ in $\widetilde{\sigma}|M$, and the map $w \mapsto \bar{\omega}_{w,0}$ is onto $X(I(\widetilde{\sigma}, \widetilde{P}))/X(\widetilde{\sigma})$, establishing an isomorphism

$$R(\sigma) \cong X(I(\widetilde{\sigma}, \widetilde{P}))/X(\widetilde{\sigma}).$$

Now, let ϕ be the homomorphism of W_k into ${}^LM$ to which the L-packet $\widetilde{\sigma}|M$ is attached. Let $S(\phi)$ be the S-group of ϕ when ϕ is being considered as a map into ${}^LG$. More precisely, it is the quotient of the centralizer of the image of ϕ inside ${}^LG$ by its connected component. Similarly define the S-group $S_M(\phi)$ for the original ϕ, i.e., as a map into ${}^LM$. Then by theorem 4.3 of [**6**],

$$X(I(\widetilde{\sigma}, \widetilde{P})) = S(\phi) \text{ and } X(\widetilde{\sigma}) \cong S_M(\phi).$$

Therefore, we have:

PROPOSITION 1.9. a) *Suppose $\widetilde{\sigma}$ is in the discrete series. Then the mapping $w \mapsto \omega_{w,0}$ establishes an isomorphism between $R(\sigma)$ and $S(\phi)/S_M(\phi)$, where ϕ is the homomorphism from W_k into ${}^LM$ to which the L-packet of σ is attached.*

b) *Under the same assumption, suppose that the L-packet of $\phi: W_k \to {}^LM$ is a singleton. Then $R(\sigma) \cong S(\phi)$. In particular assume P is minimal. Then $R(\sigma) \cong S(\phi)$.*

The following two lemmas are important.

LEMMA 1.10. *Fix $\pi_1 \subset I(\sigma_i, P)$ and $\pi_1' \subset I(\sigma_j, P)$ and let A_σ and $A_\sigma' \subset A$ be the sets of all $a \in A$ and $a' \in A$ such that $\widetilde{a} \cdot \pi_1 \subset I(\sigma, P)$ and $\widetilde{a}' \cdot \pi_1' \subset I(\sigma, P)$, respectively. Choose $a_j \in A$, uniquely, such that $\pi_1' = \widetilde{a}_j\pi_1$. Assume $a \in A_\sigma$ and $a' \in A_\sigma'$ are so that $\widetilde{a}\pi_1 = \widetilde{a}'\pi_1'$. Then as a runs in A, the values of $\omega_{w,\sigma_i}(\widetilde{a})$ and $\omega_{w,\sigma_j}(\widetilde{a}')$ are proportional. More precisely*

$$\omega_{w,\sigma_i}(\widetilde{a}) = \omega_{w,\sigma_i}(\widetilde{a}_j) \cdot \omega_{w,\sigma_j}(\widetilde{a}').$$

Proof. Since $\widetilde{a}\pi_1 = \widetilde{a}_j \cdot \widetilde{a}'\pi_1$ we have $\widetilde{a} = \widetilde{a}_j\widetilde{a}'$. But now by Corollary 1.3

$$\begin{aligned}\omega_{w,\sigma_i}(\widetilde{a}) &= \omega_{w,\sigma_i}(\widetilde{a}'\,\widetilde{a}_j)\\ &= \omega_{w,\widetilde{a}_j\sigma_i}(\widetilde{a}')\omega_{w,\sigma_i}(\widetilde{a}_j)\\ &= \omega_{w,\sigma_i}(\widetilde{a}_j)\omega_{w,\sigma_j}(\widetilde{a}').\end{aligned}$$

LEMMA 1.11. *Choose* σ_1 *and* σ_2 *in the L-packet of* σ. *Fix* $\pi_1 \subset I(\sigma_i, P)$. *Let* A_{σ_1} *and* A_{σ_2} *be as above. If* $a_l\pi_1 \subset I(\sigma_1, P)$ *and* $a'_l \pi_1 \subset I(\sigma_2, P)$, $l = 1, \ldots n_1$, *then with, possibly, a rearrangement of* $\{a'_l\}$, *the values of* $\omega_w(\widetilde{a}_l)$ *and* $\omega_w(\widetilde{a}'_l)$ *are proportional.*

Proof. Set

$$N_{\widetilde{\sigma}} = \bigcap_{\nu \in X(I(\widetilde{\sigma}, \widetilde{P}))} \operatorname{Ker} \nu$$

and

$$N'_{\widetilde{\sigma}} = \bigcap_{\nu \in X(\widetilde{\sigma})} \operatorname{Ker} \nu.$$

Then

$$k^*/N_{\widetilde{\sigma}}/N'_{\widetilde{\sigma}}/N_{\widetilde{\sigma}} \cong k^*/N'_{\widetilde{\sigma}},$$

and n_1 is the cardinality of $N'_{\widetilde{\sigma}}/N_{\widetilde{\sigma}}$, and therefore the same for both σ_1 and σ_2. Let $b_1, \ldots, b_{n_1}$ be the set of representatives of $N'_{\widetilde{\sigma}}/N_{\widetilde{\sigma}}$ in A. Fix $a_1 \in A_{\sigma_1}$ and $a'_1 \in A_{\sigma_2}$. Then

$$A_{\sigma_1} = \{a_1 b_i | i = 1, \ldots, n_1\}$$

and

$$A_{\sigma_2} = \{a'_1 b_i | i = 1, \ldots, n_1\}.$$

But now

$$\omega_{w,\sigma_i}(\widetilde{a}_1 \widetilde{b}_i) = \omega_{w,\sigma_i}(\widetilde{a}_1)\omega_{w,\sigma_i}(\widetilde{b}_i)$$

and

$$\omega_{w,\sigma_i}(\widetilde{a}'_1 \widetilde{b}_i) = \omega_{w,\sigma_i}(\widetilde{a}'_1)\omega_{w,\sigma_i}(\widetilde{b}_i),$$

using a simple argument as in Lemma 1.2 since $\widetilde{b}_i \in \widetilde{M}_0$. This proves the lemma.

2. Intertwining operators and normalizing factors. Fix a complex number s, and let $V(s, \widetilde{\sigma}, \widetilde{P})$ be the space of all the smooth functions $\widetilde{f}$ from $\widetilde{G}$ into the space of $\widetilde{\sigma}$ such that

$$\widetilde{f}(mng) = \delta_{\widetilde{P}}^{s+1/2}(m)\widetilde{\sigma}(m)\widetilde{f}(g),$$

for all $m \in \widetilde{M}$, $n \in N$, and $g \in \widetilde{G}$. Here

$$\delta_{\widetilde{P}}(m) = |\det \operatorname{Ad}_{\mathfrak{n}}(m)|,$$

where $\mathfrak{n}$ is the Lie algebra of N, $\operatorname{Ad}_{\mathfrak{n}}$ denotes the adjoint representation of

https://doi.org/10.4153/CJM-1983-060-9 Published online by Cambridge University Press

$\widetilde{M}$ on $\mathfrak{n}$, and the absolute value is that of k. Now, let $I(s, \widetilde{\sigma}, \widetilde{P})$ be the representation of $\widetilde{G}$ obtained by right translations. In other words

$$I(s, \widetilde{\sigma}, \widetilde{P}) = \operatorname*{Ind}_{\widetilde{P}\uparrow\widetilde{G}} \widetilde{\sigma} \otimes \delta_{\widetilde{P}}^{s}.$$

Observe that $I(0, \widetilde{\sigma}, \widetilde{P}) = I(\widetilde{\sigma}, \widetilde{P})$ was defined in Section 1.

For a permutation $\widetilde{w} \in W(\widetilde{M}) \cong W(M)$, we choose a representative in G which we denote by w. We will be more specific about these representatives in a moment. Now, let N^- denote the transpose of N, and set

$$N_w = N \cap \widetilde{w}N^-\widetilde{w}^{-1}.$$

Given $\widetilde{f} \in V(s, \widetilde{\sigma}, \widetilde{P})$, define

$$A(s, \widetilde{\sigma}, w)\widetilde{f}(g) = \int_{N_w} \widetilde{f}(w^{-1}ng)dn \quad (g \in \widetilde{G}).$$

It is proved in [**19**] (also see [**14, 15**]) that for Re $(s) \gg 0$, the integral is absolutely convergent, and furthermore, as a function of s, has a meromorphic continuation to the whole complex plane. We still use $A(s, \widetilde{\sigma}, w)\widetilde{f}(g)$ to denote this continuation. Let $A(s, \widetilde{\sigma}, w)$ be the corresponding operator. Observe that $A(0, \widetilde{\sigma}, w)$ sends $V(0, \widetilde{\sigma}, \widetilde{P})$ into $V(0, w\widetilde{\sigma}, \widetilde{P})$, where as before $w\widetilde{\sigma}$ is defined by

$$w\widetilde{\sigma}(m) = \widetilde{\sigma}(w^{-1}mw).$$

We use $A(\widetilde{\sigma}, w)$ to denote $A(0, \widetilde{\sigma}, w)$. We finally remark that since $A(\widetilde{\sigma}, w)$ is non-zero and $I(\widetilde{\sigma}, \widetilde{P})$ is irreducible, the map $A(\widetilde{\sigma}, w)$ is in fact onto, or more precisely a bijection.

Now, let $V(\widetilde{\sigma}, \widetilde{P})$ be the space of $I(\widetilde{\sigma}, \widetilde{P})$. Given $\widetilde{f} \in V(\widetilde{\sigma}, \widetilde{P})$ let $f = \widetilde{f}|G$. Then the mapping $\widetilde{f} \mapsto f$ from $V(\widetilde{\sigma}, \widetilde{P})$ into $V(\tau, P)$, $\tau = \widetilde{\sigma}|M$, induces an isomorphism between $I(\widetilde{\sigma}, \widetilde{P})\,|G$ and $I(\tau, P)$. In particular

$$A(\widetilde{\sigma}, w)\widetilde{f} = A(\tau, w)f,$$

where $A(\tau, w)$ is defined to be the value of the analytic continuation of the integral

$$\int_{N_w} f(w^{-1}ng)dn$$

at $s = 0$, with $f \in V(s, \tau, P)$ and $g \in G$. $A(s, \widetilde{\sigma}, w)$ may have a pole at $s = 0$ and therefore $A(\tau, w)$ may not be defined, but we are only interested in a normalization of it which is in fact unitary and therefore well defined.

We shall now explain our normalizing factors.

As in [**15**], write $\widetilde{w}^{-1} = \widetilde{w}_1 \cdot \widetilde{w}_2 \ldots . \widetilde{w}_{l(\widetilde{w})}$, where each $\widetilde{w}_i$ is a simple reflection in $W(\widetilde{M})$ and furthermore there is no other decomposition of $\widetilde{w}^{-1}$ whose number of factors is less than $l(\widetilde{w})$, in other words a reduced decomposition of $\widetilde{w}^{-1}$. The number $l(\widetilde{w})$ is unique while the decomposition itself is not. For each i, $1 \leqq i \leqq l(\widetilde{w})$, define a matrix ϵ_i as follows. Suppose $\widetilde{w}_i$ interchanges j^{th} and $j + 1^{\text{th}}$ blocks. Then ϵ_i is a diagonal matrix in $\widetilde{G}$ which is equal to one everywhere along its diagonal except at $j + 1$ block, where it is equal to the scalar matrix $(-1)^p$. Observe that the matrices $\epsilon_i w_i$ and consequently the representative w, defined by

$$w^{-1} = \epsilon_1 \widetilde{w}_1 \ldots . \epsilon_{l(\widetilde{w})} \cdot \widetilde{w}_{l(\widetilde{w})}$$

are all in $SL_r(k) = G$. It follows from a standard lemma on reduced decompositions that w^{-1} is independent of the decomposition of $\widetilde{w}^{-1}$. For every $\widetilde{w}$, this is the representative which we would like to fix throughout this paper.

Now, for each i, let $\sigma_{i,1}$ and $\sigma_{i,2}$ be two adjacent representations of $\widetilde{w}_{i-1} \ldots . \widetilde{w}_1 \widetilde{\sigma}$ which are interchanged by $\widetilde{w}_i$.

To proceed, we let ψ be an additive character of k. Assume that the largest ideal on which ψ is trivial is the ring of integers 0 of k. Fix a complex number s, and for a pair of tempered representations ρ_1 and ρ_2 of $GL_p(k)$, let $\epsilon(s, \rho_1 \times \rho_2, \psi)$ and $L(s, \rho_1 \times \rho_2)$ be the local Langlands root number and L-function attached to the pair (ρ_1, ρ_2) by H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika in [**9**] (see also [**7**]). Their definitions are fairly involved but we only remark that $\epsilon(s_1, \rho_1 \times \rho_2, \psi)$ and $L(s, \rho_1 \times \rho_2)^{-1}$ are, respectively, a monomial and a polynomial in q^{-s}, and furthermore they are the local factors appearing in the functional equation satisfied by global forms, if any, for which ρ_1 and ρ_2 are local components (cf. [**7**]). Moreover, we observe that by Proposition 9.4 of [**9**], $L(s, \rho_1 \times \rho_2)$ is holomorphic for Re $(s) > 0$. To conclude this discussion, we give an explicit formula for $L(s, \rho_1 \times \rho_2)$, when either ρ_1 or ρ_2 is supercuspidal. For each i, $i = 1, 2$, let ω_i be the central character of ρ_i. Denote by $\overline{\omega}$ a uniformizing parameter for 0. This is an element satisfying $|\overline{\omega}| = q^{-1}$. Then it follows almost immediately from the definitions that $L(s, \rho_1 \times \rho_2) = 1$, if either there is no $s_0 \in \mathbf{C}$ such that $\rho_1 \cong \rho_2 \otimes |\det|^{s_0}$, or $\omega_1\omega_2$ is a ramified character, and

$$L(s, \rho_1 \times \rho_2) = (1 - \omega_1\omega_2(\overline{\omega})q^{-ps})^{-1}$$

otherwise (cf. Section 2.3 of [**14**]).

Let $w'_{0,r}$ and $w'_{0,p}$ be, respectively, permutation matrices in $GL_r(k)$ and $GL_p(k)$ whose only non-zero (in fact 1) entries are along their second diagonals. Then

$$w''_{0,p} = \overbrace{(w'_{0,p}, \ldots, w'_{0,p})}^{m} \in \widetilde{M}$$

is a matrix in $GL_r(k)$. Set $w'_0 = w''_{0,p} \cdot w'_{0,r}$. We now define the following diagonal matrix

$$\epsilon_0 = \begin{cases} I \in GL_r(k) & p \text{ even or } m \equiv 0, 1(4) \\ -I \in GL_r(k) & p \text{ odd and } m \equiv 3(4) \\ \begin{pmatrix} -I_{m/2} & 0 \\ 0 & I_{m/2} \end{pmatrix} \in GL_r(k) & p \text{ odd and } m \equiv 2(4). \end{cases}$$

At all cases, set $w_0 = \epsilon_0 w'_0$. Then $w_0 \in SL_r(k) = G$. We now write

$$\widetilde{\sigma} = \bigotimes_{j=1}^{m} \sigma_{1,j}$$

and for each j let $\omega_{1,j}$ be the central character of $\sigma_{1,j}$. We also define the following factor

$$r_0(\widetilde{\sigma}, \widetilde{w}) = \begin{cases} \prod_{j=1}^{m/2} \omega_{1,j}\omega_{1,\widetilde{w}^{-1}(j)}(-1) & p \text{ odd and } m \equiv 2(4) \\ 1 & \text{otherwise.} \end{cases}$$

The significance of this factor will be explained later.

Given i, $1 \leqq i \leqq l(\widetilde{w})$, again let $\sigma_{i,1}$ and $\sigma_{i,2}$ be two adjacent representations of $\widetilde{w}_{i-1} \ldots \widetilde{w}_1\sigma$ which are interchanged by $\widetilde{w}_i$ ($\widetilde{w}_i$'s are permutations).

LEMMA 2.1. *Each of the two factors* $r_0(\widetilde{\sigma}, \widetilde{w})$ *and*

$$\prod_{i=1}^{l(\widetilde{w})} \epsilon(0, \sigma_{i,1} \times \widetilde{\sigma}_{i,2}, \psi)L(1, \sigma_{i,1} \times \widetilde{\sigma}_{i,2})/L(0, \sigma_{i,1} \times \widetilde{\sigma}_{i,2}),$$

and (*consequently*) *their product, are independent of the decomposition of* $\widetilde{w}$; *nor do they depend on the choice of* $\widetilde{\sigma}$. *Here* $\widetilde{\sigma}_{i,2}$ *denotes the contragradient of the representation* $\sigma_{i,2}$, $i = 1, \ldots, l(\widetilde{w})$.

Proof. The first assertion follows from the corollary of Lemma 2.1.2 of [**14**]. To prove the second assertion, we observe that any other choice of $\widetilde{\sigma}$ is of the form $\widetilde{\sigma} \otimes \nu$ for some $\nu \in \hat{k}^*$. Then every $\sigma_{1,j}$ will change to $\sigma_{1,j} \otimes \nu$ which results in a change of every $\omega_{1,j}$ to $\omega_{1,j}\nu^p$, $1 \leqq j \leqq m$. The fact that the root number and the L-function remain the same is now trivial. The same is true for $r_0(\widetilde{\sigma}, \widetilde{w})$ which completes the lemma.

Now, we set

$$r(\tau, w) = r_0(\widetilde{\sigma}, \widetilde{w}) \cdot \prod_{i=1}^{l(\widetilde{w})} \epsilon(0, \sigma_{i,1} \times \widetilde{\sigma}_{i,2}, \psi)$$
$$\times L(1, \sigma_{i,1} \times \widetilde{\sigma}_{i,2})/L(0, \sigma_{i,1} \times \widetilde{\sigma}_{i,2}).$$

This is the normalizing factor in which we are interested. Observe that when σ and $\widetilde{\sigma}$ are components of global forms, the coefficient $r_0(\widetilde{\sigma}, \widetilde{w})$ is equal to 1 and our normalizing factor reduces to

$$\prod_{i=1}^{l(\widetilde{w})} \epsilon(0, \sigma_{i,1} \times \widetilde{\sigma}_{i,2}, \psi)L(1, \sigma_{i,1} \times \widetilde{\sigma}_{i,2})/L(0, \sigma_{i,1} \times \widetilde{\sigma}_{i,2}).$$

This has a particularly important global significance (cf. **[12]**).

If for every element $\widetilde{w} \in W(\widetilde{M}) = W(M)$, we choose a representative w in G as before, and let $A(\tau, w)$ be the value of the analytic continuation of

$$\int_{N_w} f(w^{-1}ng)dn$$

at 0, then our normalized operator $R(\tau, w)$ is defined by

$$R(\tau, w) = r(\tau, w)A(\tau, w).$$

Now, let $\widetilde{w}$ be equal to the full Coxeter element (or rather its inverse) $\widetilde{w}_c$ in $W(\widetilde{M})$. More precisely, $\widetilde{w}_c$ is the permutation $1 \to m \to m - 1 \to \dots \to 2 \to 1$. We then have

LEMMA 2.2. *Suppose $w = w_c$. Assume $w_c\sigma \cong \sigma$. Then*

a) *There exists an irreducible tempered representation π of $GL_p(k)$ and $\omega \in \hat{k}^*$ with $\omega^m \in X(\pi)$ such that*

$$\widetilde{\sigma} = \bigotimes_{j=1}^{m} (\pi \otimes \omega^j).$$

Any other π is of the form $\pi \otimes \nu \cdot \det$ for some $\nu \in \hat{k}^*$ and therefore the L-packet generated by π is unique. Furthermore, for a given π, ω is unique modulo the elements of $X(\pi)$.

b) *The normalizing factor $r(\tau, w_c)$ is given by*

$$r(\tau, w_c) = r_0 \prod_{j=1}^{m-1} \epsilon(0, \pi \times (\widetilde{\pi} \otimes \omega^j), \psi)$$
$$\times L(1, \widetilde{\pi} \times (\pi \otimes \omega^{-j}))/L(0, \pi \times (\widetilde{\pi} \otimes \omega^j)),$$

https://doi.org/10.4153/CJM-1983-060-9 Published online by Cambridge University Press

where $r_0 = \omega(-1)$ *if* p *is odd and* $m \equiv 2(4)$, *and* $r_0 = 1$ *otherwise.*

Proof. a) Write

$$\widetilde{\sigma} = \bigotimes_{j=1}^{m} \sigma_{1,j}.$$

Let ω denote $\omega_{w,0}$ as a character of k^*. Then by part b) of Lemma 1.2 $w_c\widetilde{\sigma} \cong \widetilde{\sigma} \otimes \omega$. Now part a) follows immediately if we let $\pi = \sigma_{1,m}$.

b) First observe that $\widetilde{w}_c^{-1} = \widetilde{w}_{12} \ldots\ldots \widetilde{w}_{m-1,m}$ is a reduced decomposition of $\widetilde{w}_c^{-1}$ and therefore

$$\sigma_{i,1} = \pi \otimes \omega \quad \text{and} \quad \sigma_{i,2} = \pi \otimes \omega^{i+1}, \, i = 1, \ldots, m-1.$$

Now, straightforward calculations show that if $\widetilde{w} = \widetilde{w}_c$, then $r_0(\widetilde{\sigma}, \widetilde{w}) = r_0$. We only need remark that r_0 is well defined. In fact, any other choice of ω is of the form $\omega\omega_0$ with $\omega_0 \in X(\pi)$. Then for odd p

$$\begin{aligned} \omega\omega_0(-1) &= \omega\omega_0^p(-1) \\ &= \omega(-1). \end{aligned}$$

It remains to show that

$$\prod_{j=1}^{m-1} L(1, \widetilde{\pi} \times (\pi \otimes \omega^{-j})) = \prod_{j=1}^{m-1} L(1, \pi \times (\widetilde{\pi} \otimes \omega^{j})).$$

But then by properties of L-functions we have

$$\begin{aligned} \prod_{j=1}^{m-1} L(1, \widetilde{\pi} \times (\pi \otimes \omega^{-j})) &= \prod_{j=1}^{m-1} L(1, \pi \times (\widetilde{\pi} \otimes \omega^{-j})) \\ &= \prod_{j=1}^{m-1} L(1, \pi \times (\widetilde{\pi} \otimes \omega^{m-j})) \\ &= \prod_{j=1}^{m-1} L(1, \pi \times (\widetilde{\pi} \otimes \omega^{j})), \end{aligned}$$

which completes the lemma.

COROLLARY 2.3. *Let* m' *be a positive integer with* $m' \leqq m$. *Denote by* $S_{m'}$ *the symmetric group in* m' *letters. Let* w *be the permutation in* $S_{m'}$, *defined by* $1 \to m' \to m' - 1 \to \ldots 2 \to 1$. *Fix an embedding* ρ *of* $S_{m'}$ *into* S_m. *Suppose* $\rho(w)\tau \cong \tau$. *Choose* $\widetilde{\sigma}$ *such that* $\widetilde{\sigma}|M = \tau$. *Then with notation as before*

$$\prod_{i=1}^{l(\rho(w))} L(1, \sigma_{i,1} \times \widetilde{\sigma}_{i,2}) = \prod_{i=1}^{l(\rho(w))} L(1, \widetilde{\sigma}_{i,1} \times \sigma_{i,2}).$$

Proof. This can easily be proved if one uses the corollary after Lemma 2.1.2 of [**14**] together with Lemma 2.2 of the present paper.

Now, let $\widetilde{w} \in W(\widetilde{M})$. Then $\widetilde{w}$ is conjugate to a product of elements of type $\rho(w)$ for different pairs (m', ρ). Furthermore this element (product) is unique. In fact, every conjugacy class in $w(\widetilde{M})$ has a unique permutation of this type. From now on, for every conjugacy class C in $W(\widetilde{M})$, let $\widetilde{w}_C$ be this unique permutation, and denote by w_C its representative in G as before. The full Coxeter element w_c would now be only an example for which the conjugacy class is the Coxeter conjugacy class which we denote by $c = W(M)_{\text{reg}}$.

Finally for the sake of completeness we prove:

PROPOSITION 2.4. *For every positive root α generating N, let w_α be the corresponding transposition, and let $W''(\sigma)$ be the subgroup of $W(\sigma)$ generated by those transpositions $\widetilde{w}_\alpha$ for which the Plancherel measure $\mu(\sigma, w_\alpha)$, defined by*

$$A(\sigma, w_\alpha)A(\sigma, w_\alpha^{-1}) = \mu(\sigma, w_\alpha)^{-1},$$

is zero. Suppose σ is in the discrete series. Then $W'(\sigma) = W''(\sigma)$.

Proof. Given two positive integers l and k with $l > k$, let $\alpha = (l, k)$ be the corresponding positive root. Denote by $\widetilde{w}_\alpha$ the corresponding transposition. Then as in Theorem 1.1 of [**6**], every $1 \neq \widetilde{w} \in W'(\sigma)$ is a product of transpositions, each of which is again in $W'(\sigma)$. Consequently $W'(\sigma)$ is generated by transpositions. Thus to prove $W'(\sigma) = (W''\sigma)$, we only need to show that they are generated by the same transpositions. Therefore first assume $\widetilde{w}_\alpha \in W'(\sigma)$, $\alpha = (l, k)$. Write

$$\widetilde{\sigma} = \bigotimes_{j=1}^{m} \sigma_{1,j}.$$

Then $\sigma_{1,l} \cong \sigma_{1,k}$. By Theorem 6.1 of [**16**], up to a positive constant multiple, $\mu(\sigma, w_\alpha)$ is equal to

$$|L(1, \sigma_{1,l} \times \widetilde{\sigma}_{1,k})|^2/|L(0, \sigma_{1,l} \times \widetilde{\sigma}_{1,k})|^2.$$

It follows from the properties of these L-functions (cf. [**9**]) that since $\sigma_{1,l}$ and $\sigma_{1,k}$ are both unitary, the numerator which is never zero will also have no poles, and therefore the zeros of $\mu(\sigma, w_\alpha)$ are all given by the poles of

$$L(0, \sigma_{1,l} \times \tilde{\sigma}_{1,k}).$$

But then from Propositions 9.1 and 9.2 of [**9**], it follows that $L(s, \sigma_{1,l} \times \tilde{\sigma}_{1,l})$ has a pole at $s = 0$ which implies $\tilde{w}_\alpha \in W''(\sigma)$ (in this direction being tempered is enough). Conversely, suppose $\tilde{w}_\alpha \in W''(\sigma)$, $\alpha = (l, k)$. Then $L(s, \sigma_{1,l} \times \tilde{\sigma}_{1,k})$ has a pole at $s = 0$. As in Proposition 9.2 of [**9**], write $p = at$ with a and t positive integers, and let π_0 be an irreducible unitary supercuspidal representation of $GL_a(k)$ such that

$$\sigma_{1,l} = \sigma(\pi_1, \pi_2, \ldots, \pi_t)$$

is the unique discrete series component of

$$\mathrm{Ind}(GL_p(k), Q, \pi_1, \pi_2, \ldots, \pi_t),$$

where $\pi_i = \pi_0 \otimes \alpha^{(t+1)/2-i}$, $1 \leqq i \leqq t$, $\alpha(g) = |\det g|$, and Q is the obvious standard parabolic subgroup of $GL_p(k)$. Similarly, choose positive integers b and q with $p = bq$, and an irreducible unitary supercuspidal representation of σ_0 of $GL_b(k)$ such that

$$\sigma_{1,k} = \sigma(\sigma_1, \sigma_2, \ldots, \sigma_q),$$

where $\sigma_i = \sigma_0 \otimes \sigma^{(q+1)/2-i}$, $1 \leqq i \leqq q$. Then by Proposition 9.2 of [**9**],

$$L(s, \sigma_{1,l} \times \tilde{\sigma}_{1,k}) = \prod_{i=1}^{q} L(s, \pi_1 \times \tilde{\sigma}_i),$$

and consequently there exists an i, $1 \leqq i \leqq q$, such that $L(s, \rho_1 \times \tilde{\sigma}_i)$ has a pole at $s = 0$. Now from our previous discussion of L-functions for supercuspidal representations, we conclude that $\pi_1 \cong \sigma_i$ which immediately implies $a = b$ and therefore $t = q$. Furthermore $\pi_0 \cong \sigma_0 \otimes \alpha^{1-i}$. But since i is an integer and σ_0 and π_0 are both unitary, this implies that $i = 1$ and $\pi_0 \cong \sigma_0$. Consequently $\sigma_{1,l} \cong \sigma_{1,k}$ which implies $\tilde{w}_\alpha \in W''(\sigma)$. This completes the proposition.

3. The theorem. Let ψ be as in the previous section. Denote by U the subgroup of upper triangular matrices in $\tilde{G}$ with ones on their diagonals. Set

$$\chi(u) = \psi(u_{12} + u_{23} + \ldots \ldots u_{r-1,r})$$

for every $u \in U$. Then χ is a character of U. Since $\tilde{\sigma}$ is tempered, it is non-degenerate (cf. [**8**]). Consequently $I(\tilde{\sigma}, \tilde{P})$ possesses a non-zero χ-Whittaker functional $\tilde{\lambda}_\chi$ which is unique up to a complex multiple. More precisely, $\tilde{\lambda}_\chi$ is a functional on the space $V(\tilde{\sigma}, \tilde{P})$ of $I(\tilde{\sigma}, \tilde{P})$ satisfying

$$\tilde{\lambda}_\chi(I(\tilde{\sigma}, \tilde{P})(u)\tilde{f}) = \chi(u)\tilde{\lambda}_\chi(\tilde{f})$$

for all $u \in U$ and $\tilde{f} \in V(\tilde{\sigma}, \tilde{P})$. It is then, by restriction, a functional on $I(\tau, P)$.

Choose $\pi_1 \subset I(\tau, P)$ so that $\tilde{\lambda}_\chi | V_1 \not\equiv 0$, where V_1 is the space of π_1. It follows from Lemma 3.6 below that π_1 is unique. We normalize the measure dn, defining $A(\tau, w)$, so that the measures defining the corresponding rank one operators in a decomposition of $A(\tau, w)$ (according to a reduced decomposition of w^{-1}, cf. **[15]**) are all normalized as in Theorem 5.1 of **[16]**. Fix $\sigma \subset \tau$. Let $C_c^\infty(G)$ be the space of all the locally constant functions of compact support on G, and for $f \in C_c^\infty(G)$, let

$$I(\sigma, P, f) = \int_G f(g) I(\sigma, P)(g) dg$$

be its Fourier transform. Finally, for $\pi \subset I(\tau, P)$, let χ_π be the character of π. Fix $w \in G$ as in Section 2 such that $w\sigma \cong \sigma$. Then

THEOREM 3.1. *Let $\pi_1 \subset I(\tau, P)$ be (uniquely) so that $\lambda_\chi | \pi_1 \not\equiv 0$, and let $A_\sigma \subset A$ be as before the set of all $a \in A$ such that $\tilde{a}\pi_1 \subset I(\sigma, P)$. Then there exists $\Phi: \tau \cong w\tau$ such that*

a) *Fix $f \in C_c^\infty(G)$. Then*

$$\operatorname{trace}\,(R(\sigma, w) I(\sigma, P, f)) = \sum_{a \in A_\sigma} \omega_{w'}(\tilde{a}) \chi_{\pi_a}(f).$$

We refer to Lemma 1.2 *for the definition of $\omega_{w'}$.*

b) *Suppose π_1 is changed to $\pi_1' \subset I(\tau, P)$. With $\pi_1 \subset I(\sigma_i, P)$ and $\pi_1' \subset I(\sigma_j, P)$. Choose $a_j \in A$, uniquely, such that $\pi_1' \cong \tilde{a}_j \cdot \pi_1$. Let A_σ' be the new set of indices which sends π_1' to the constituents of $I(\sigma, P)$. Then*

$$\operatorname{trace}\,(R(\sigma, w) I(\sigma, P, f)) = \omega_{w,\sigma_i}(\tilde{a}_j) \sum_{a \in A_\sigma'} \omega_{w'}(\tilde{a}) \chi_{\pi_a}(f).$$

Consequently, the coefficients appearing in the above linear combination of characters χ_{π_a} remain proportional, when the base representation π_1 is changed.

c) *Suppose $\sigma = \sigma_1$ is changed to $\sigma_2 \subset \tau$. Then there exists a one-one correspondence ρ from A_{σ_1} onto A_{σ_2} such that the coefficients of χ_{π_a}, $a \in A_{\sigma_1}$, in the expansion of*

$$\operatorname{trace}\,(R(\sigma_1, w) I(\sigma_1, P, f))$$

are proportional to the coefficients of $\chi_{\pi_{\rho(a)}}$, $a \in A_{\sigma_1}$, in the expansion of

$$\operatorname{trace}\,(R(\sigma_2, w) I(\sigma_2, P, f)).$$

d) *Suppose that the L-packet of* σ *is a singleton (in particular when P is minimal). Then for every* $w \in S(\phi)$, *there exists a character* ω_w *of* k^*, *such that*

$$\operatorname{trace}(R(\sigma, w)I(\sigma, P, f)) = \sum_{a \in A} \omega_w(a)\chi_{\pi_a}(f) \quad (f \in C_c^\infty(G))$$

In this case Φ *is unique.*

Proof. We first observe that applying Lemmas 1.10, 1.11, and 1.2 to part a) of Theorem 3.1 implies, respectively, parts b), c), and d) of the theorem. To prove part a) we need some preparation.

Write

$$\widetilde{\sigma} = \bigotimes_{j=1}^{m} \sigma_{1,j},$$

and for each j, let $V_{1,j}$ denote the space of $\sigma_{1,j}$. Since $\widetilde{\sigma}$ is tempered, so is every $\sigma_{1,j}$ and therefore they are all non-degenerate. For each j, let λ_j be a χ-Whittaker functional for $V_{1,j}$. More precisely, it satisfies

$$\lambda_j(\sigma_{1,j}(u)v) = \chi(u)\lambda_j(v)$$

for all $u \in GL_p(k) \cap U$ and all $v \in V_{1,j}$. Here χ also denotes its restriction to $GL_p(k) \cap U$. For $\widetilde{f} \in V(\widetilde{\sigma}, \widetilde{P})$, the following principal value integral is convergent (cf. [**4**])

$$(3.1.1) \qquad \widetilde{\lambda}_\chi(\widetilde{f}) = \int_N \langle \widetilde{f}(w_0'n), \lambda_1 \otimes \lambda_2 \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)} dn,$$

and defines a non-zero χ-Whittaker functional for $V(\widetilde{\sigma}, \widetilde{P})$. Here $\langle \ , \ \rangle$ denotes the pairing between $\otimes_{j=1}^m V_j$ and its dual.

Next for $\widetilde{w} \in W(\widetilde{M})$, let $A(\widetilde{\sigma}, \widetilde{w})$ be the intertwining operator between $I(\widetilde{\sigma}, \widetilde{P})$ and $I(\widetilde{w}\widetilde{\sigma}, \widetilde{P})$. For $\widetilde{f} \in V(\widetilde{\sigma}, \widetilde{P})$, define

$$(3.1.2) \qquad \widetilde{\lambda}_\chi'(\widetilde{f}) = \int_N \langle (A(\widetilde{\sigma}, \widetilde{w})\widetilde{f})(w_0' n), \lambda_1 \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)} dn.$$

Then as is well explained in [**14**], it follows from the uniqueness of Whittaker models for $I(\widetilde{\sigma}, \widetilde{P})$, that there exists a non-zero complex number

$C_\chi(\tilde{\sigma}, \tilde{w})$ such that

$$(3.1.3)\qquad C_\chi(\tilde{\sigma}, \tilde{w})\tilde{\lambda}'_\chi(\tilde{f}) = \tilde{\lambda}_\chi(\tilde{f})$$

for all $\tilde{f} \in V(\tilde{\sigma}, \tilde{P})$.

Now, let $f = \tilde{f}|G$ be in $V(\tau, P)$. Define

$$(3.1.4)\qquad \lambda_\chi(f) = \int_N \langle f(w_0 n), \lambda_1, \otimes \lambda_2 \otimes \ldots . \otimes \lambda_m \rangle \overline{\chi(n)} dn.$$

Since $w\tau \cong \tau$, then using again f to denote its image under $V(\tau, P) \cong V(w\tau, P)$, we have

$$(3.1.5)\qquad r_0(\tilde{\sigma}, \tilde{w}) r_1(\tilde{\sigma}, \tilde{w}) C_\chi(\tilde{\sigma}, \tilde{w}) \lambda_\chi(A(\tau, w)f) = \lambda_\chi(f)$$

for all $f \in V(\tau, P)$, where

$$r_1(\tilde{\sigma}, \tilde{w}) = \prod_{i=1}^{l(\tilde{w})} \omega_{i,2}^p(-1).$$

The two factors $r_0(\tilde{\sigma}, \tilde{w})$ and $r_1(\tilde{\sigma}, \tilde{w})$ are introduced to compensate the effect of changing w'_0 and $\tilde{w}$ to w_0 and w in (3.1.3), respectively.

Now, let

$$r'(\tilde{\sigma}, \tilde{w}) = C_\chi(\tilde{\sigma}, \tilde{w}) \prod_{i=1}^{l(\tilde{w})} L(1, \sigma_{i,1} \times \tilde{\sigma}_{i,2}) / L(1, \tilde{\sigma}_{i,1} \times \sigma_{i,2}).$$

Then from Theorem 5.1 of [**16**], it follows that

$$(3.1.6)\qquad r'(\tilde{\sigma}, \tilde{w}) = r_1(\tilde{\sigma}, \tilde{w}) \cdot \prod_{i=1}^{l(\tilde{w})} \epsilon(0, \sigma_{i,1} \times \tilde{\sigma}_{i,2}, \psi) L(1, \sigma_{i,1} \times \tilde{\sigma}_{i,2}) / L(0, \sigma_{i,1} \times \tilde{\sigma}_{i,2}).$$

Furthermore, if we let

$$R'(\tilde{\sigma}, \tilde{w}) = r'(\tilde{\sigma}, \tilde{w}) A(\tilde{\sigma}, \tilde{w}),$$

then, using the results in [**15**], it follows that for every two permutation matrices $\tilde{w}_1$ and $\tilde{w}_2$ in $W(\tilde{M})$

$$(3.1.7)\qquad R'(\tilde{\sigma}, \tilde{w}_1\tilde{w}_2) = R'(\tilde{w}_2\tilde{\sigma}, \tilde{w}_1) R'(\tilde{\sigma}, \tilde{w}_2).$$

Also observe that

$$(3.1.8)\qquad R'(\tilde{\sigma}, \tilde{w}) \,|I(\tau, P) = r_0(\tilde{\sigma}, \tilde{w}) R(\tau, w).$$

To prove part a) of the theorem, one only needs to compute the effect of $R(\tau, w)$ on every irreducible constituent of $I(\tau, P)$.

Fix $\widetilde{w} \in W(\widetilde{M})$ for which w is a representative in G. Let C be the conjugacy class of $\widetilde{w}$. Choose $\widetilde{w}_1$ such that

$$\widetilde{w} = \widetilde{w}_1^{-1}\widetilde{w}_C\widetilde{w}_1.$$

Suppose $w\tau \cong \tau$. Then $w_C(w_1\tau) \cong w_1\tau$, and

$$R(w_1\tau, w_1^{-1}):I(w_1\tau, P) \to I(\tau, P)$$

establishes a one-one correspondence between irreducible constituents of $I(w_1\tau, P)$ and $I(\tau, P)$. Furthermore, for each σ_i, it sends irreducible constituents of $I(w_1\tau, P)$ and $I(\tau, P)$. Furthermore, for each σ_i, it sends irreducible constituents of $I(w_1\sigma_i, P)$ to those of $I(\sigma_i, P)$. Under this correspondence, for each i, we identify the irreducible constituents of $I(w_1\sigma_i, P)$ with their images in $I(\sigma_i, P)$. In particular, we shall identify our base representation $\pi_1 \subset I(\sigma_i, P)$ with its preimage in $I(w_1\tau, P)$.

Bearing in mind the above discussion, the following proposition reduces the proof of Theorem 3.1 to the case when $w = w_C$.

PROPOSITION 3.2. *Let w, w', and w_1 represent elements of $W(M)$ in G as before, and let $\widetilde{w}$, $\widetilde{w}'$, and $\widetilde{w}_1$ be the corresponding permutation matrices. Suppose $\widetilde{w} = \widetilde{w}_1^{-1}\widetilde{w}'\widetilde{w}_1$. Let $\tau = \widetilde{\sigma}|M$, and suppose $w\tau \cong \tau$. Then $w'(w_1\tau) \cong w_1\tau$, and*

a) *the following diagram is commutative*

$$\begin{tikzcd}
I(w_1\tau, P) \arrow[r, "R(w_1\tau, w_1^{-1})"] \arrow[d, "R(w_1\tau, w')"'] & I(\tau, P) \arrow[d, "{R(\tau, w),}"] \\
I(w_1\tau, P) \arrow[r, "R(w_1\tau, w_1^{-1})"'] & I(\tau, P)
\end{tikzcd}$$

and

b) *for all $m \in \widetilde{M}$ and $\sigma \subset \tau$*

$$\omega_{w', w_1\sigma}(w_1 m w_1^{-1}) = \omega_{w,\sigma}(m).$$

In particular

$$\omega_{w'}(\widetilde{a}) = \omega_w(\widetilde{a})$$

for all $a \in \mathfrak{a}$.

We need the following lemma.

https://doi.org/10.4153/CJM-1983-060-9 Published online by Cambridge University Press

LEMMA. 3.3. *Let w, w', and w_1 be so that $\tilde{w} = \tilde{w}_1^{-1}\tilde{w}'\tilde{w}_1$. Then the following diagram is commutative.*

$$\begin{tikzcd}
I(\tilde{w}_1\tilde{\sigma}, \tilde{P}) \arrow[r, "R'(\tilde{w}\tilde{\sigma}, \tilde{w}_1^{-1})"] \arrow[d, "R'(\tilde{w}\tilde{\sigma}, \tilde{w}')"'] & I(\tilde{\sigma}, \tilde{P}) \arrow[d, "R'(\tilde{\sigma}, \tilde{w})"] \\
I(\tilde{w}'\tilde{w}_1\tilde{\sigma}, \tilde{P}) \arrow[r, "R'(\tilde{w}'\tilde{w}_1\tilde{\sigma}, \tilde{w}_1^{-1})"'] & I(\tilde{w}\tilde{\sigma}, \tilde{P})
\end{tikzcd}$$

Proof. This is an immediate consequence of relation (3.1.7).

Proof of Proposition 3.2. a) From Lemma 3.3 and relation (3.1.8), it follows that to prove part a) we only need to show

(3.2.1) $$r_0(\tilde{\sigma}, \tilde{w})r_0(\tilde{w}_1\tilde{\sigma}, \tilde{w}_1^{-1}) = r_0(\tilde{w}'\tilde{w}_1\tilde{\sigma}, \tilde{w}_1^{-1})r_0(\tilde{w}_1\tilde{\sigma}, \tilde{w}').$$

Let

$$R''(\tilde{\sigma}, \tilde{w}) = C_\chi(\tilde{\sigma}, \tilde{w})A(\tilde{\sigma}, \tilde{w}).$$

Then again (cf. Proposition 3.1.4 of [**14**])

(3.2.2) $$R''(\tilde{\sigma}, \tilde{w}_1\tilde{w}_2) = R''(\tilde{w}_2\tilde{\sigma}, \tilde{w}_1)R''(\tilde{\sigma}, \tilde{w}_2).$$

By relation (3.1.5)

(3.2.3) $$\lambda_\chi(R''(\tilde{w}_1\tilde{\sigma}, \tilde{w}\tilde{w}_1^{-1})f) = r_0(\tilde{w}_1\tilde{\sigma}, \tilde{w}\tilde{w}_1^{-1})\lambda_\chi(f)$$

for $f \in V(w_1\tau, P)$.

Next by relation (3.2.2), we have the following two decompositions of $R''(\tilde{w}_1\tilde{\sigma}, \tilde{w}\tilde{w}_1^{-1})$.

$$\begin{aligned} R''(\tilde{w}_1\tilde{\sigma}, \tilde{w}\tilde{w}_1^{-1}) &= R''(\tilde{\sigma}, \tilde{w})R''(\tilde{w}_1\tilde{\sigma}, w_1^{-1}) \\ &= R''(\tilde{w}'\tilde{w}_1\tilde{\sigma}, \tilde{w}_1^{-1})R''(\tilde{w}_1\tilde{\sigma}, \tilde{w}'). \end{aligned}$$

Using the first decomposition, the left hand side of (3.2.3) is equal to

$$r_0(\tilde{\sigma}, \tilde{w})r_0(\tilde{w}_1\tilde{\sigma}, \tilde{w}_1^{-1})\lambda_\chi(f)$$

which implies

(3.2.4) $$r_0(\tilde{\sigma}, \tilde{w})r_0(\tilde{w}_1\tilde{\sigma}, \tilde{w}_1^{-1}) = r_0(\tilde{w}_1\tilde{\sigma}, \tilde{w}\tilde{w}_1^{-1}).$$

Similarly the second decomposition implies

(3.2.5) $$r_0(\tilde{w}'\tilde{w}_1\tilde{\sigma}, \tilde{w}_1^{-1})r_0(\tilde{w}_1\tilde{\sigma}, \tilde{w}') = r_0(\tilde{w}_1\tilde{\sigma}, \tilde{w}\tilde{w}_1^{-1}).$$

Now, (3.2.1) follows from comparing (3.2.4) with (3.2.5).

b) Fix $\sigma \subset \tau$, and let V_σ be the space of σ. Then

$$\omega_{w',w_1\sigma}(m)V_\sigma = w_1\widetilde{\sigma}(m^{-1})w'w_1\widetilde{\sigma}(m)V_\sigma.$$

Changing m to $w_1 m w_1^{-1}$, we then get

$$\omega_{w',w_1\sigma}(w_1 m w_1^{-1}) = \omega_{w,\sigma}(m),$$

completing the proposition.

We now prove the theorem for $w = w_C$.

For $a \in k^*$, we let $\widetilde{a}_0$ be the matrix

$$\widetilde{a}_0 = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

in $GL_p(k)$. As before fix $\pi_1 \subset I(\tau, P)$ such that $\lambda_\chi | V_1 \not\equiv 0$. Let $a \in A$, and choose $v \in V_a$, the space of π_a. Then, realizing $\widetilde{a} \cdot \pi_1$ on V_1 as π_a on V_a, we have

$$\begin{aligned}\lambda_\chi(\pi_a(u)v) &= \lambda_\chi(\pi_1(\widetilde{\alpha}^{-1}u\widetilde{a})v) \\ &= \chi(\widetilde{a}^{-1}u\widetilde{a})\lambda_\chi(v) \\ &= \chi_a(u)\lambda_\chi(v),\end{aligned}$$

where $\chi_a(u) = \chi(\widetilde{a}^{-1}u\widetilde{a})$, $\chi_1 = \chi$, is another character of U. Therefore π_a has a non-zero χ_a-Whittaker functional.

Fix an isomorphism $\phi:\tau \cong w\tau$ and let $\widetilde{\lambda} = \lambda_1 \otimes \ldots \otimes \lambda_m$. By the uniqueness of Whittaker models for $\widetilde{\sigma}$, there exists a non-zero complex number b_ϕ such that

$$b_\phi\langle\phi(v), \widetilde{\lambda}\rangle = \langle v, \widetilde{\lambda}\rangle \quad \text{for every } v \in \bigotimes_{j=1}^{m} V_j.$$

Now, let $\Phi = b_\phi \cdot \phi$. Then

$$\langle\Phi(v), \widetilde{\lambda}\rangle = \langle v, \widetilde{\lambda}\rangle.$$

For each j, $1 \leqq j \leqq m$, let $\sigma'_{1,j}$ denote the dual of $\sigma_{1,j}$. More precisely

$$\langle v, \sigma'_{1,j}(m)\lambda\rangle = \langle\sigma_{1,j}^{-1}(m)v, \lambda\rangle,$$

where $v \in V_{1,j}$ and λ is in the full dual of $V_{1,j}$.

LEMMA 3.4. *Given $\widetilde{f} \in I(\widetilde{\sigma}, \widetilde{P})$, let $f \in I(\tau, P)$ be its restriction to G. Let $l = \widetilde{w}_C^{-1}(m)$. Then*

(3.1.9) $$\tilde{\lambda}_{\chi_a}(\tilde{f}) = \int_N \langle \Phi(f(w_0 n)), \lambda_1 \otimes \ldots \otimes \lambda_{l-1} \otimes \sigma'_{1,l}(\tilde{a}_0)\lambda_l \otimes \lambda_{l+1} \ldots \otimes \lambda_m \rangle \overline{\chi(n)} dn$$

is a χ_a*-Whittaker functional for* $I(\tilde{\sigma}, \tilde{P})$.

Proof. Take $u \in U$. Write $u = u_0 n_0$ with $n_0 \in N$ and $u_0 \in U \cap M$. Then

$$\begin{aligned}\Phi(f(w_0 nu)) &= \Phi(\tau(w_0 u_0 w_0^{-1}) f(w_0 n n_0)) \\ &= w_C \tau(w_0 u_0 w_0^{-1}) \Phi(f(w_0 n n_0)) \\ &= w_C \tilde{\sigma}(w_0 u_0 w_0^{-1}) \Phi(f(w_0 n n_0)).\end{aligned}$$

Then

$$\begin{aligned}\tilde{\lambda}_{\chi_a}(\tilde{I}(\tilde{\sigma}, \tilde{P})(u)\tilde{f}) &= \int_N \langle \Phi(f(w_0 nu)), \lambda_1 \otimes \ldots . \otimes \sigma'_{1,l}(\tilde{a}_0)\lambda_l \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)} dn \\ &= \chi_a(u) \tilde{\lambda}_{\chi_a}(\tilde{f}),\end{aligned}$$

proving the lemma.

Again

(3.1.10) $$\tilde{\lambda}'_{\chi_a}(\tilde{f}) = \int_N \langle (A(\tilde{\sigma}, \tilde{w}_C)\tilde{f})(w_0 n), \lambda_1 \otimes \ldots \otimes \sigma'_{1,l}(\tilde{a}_0)\lambda_l \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)} dn$$

is related to $\tilde{\lambda}_{\chi_a}$ by

(3.1.11) $$\tilde{\lambda}'_{\chi_a}(r_0(\tilde{\sigma}, \tilde{w}_C)\, C_{\chi_a}(\tilde{\sigma}, \tilde{w}_C)\tilde{f}) = \tilde{\lambda}_{\chi_a}(\tilde{f}),$$

where $C_{\chi_a}(\tilde{\sigma}, \tilde{w}_C)$ exists, as before, by the uniqueness of χ_a-Whittaker functionals for $I(\tilde{\sigma}, \tilde{P})$. We have multiplied $C_{\chi_a}(\tilde{\sigma}, \tilde{w}_C)$ by $r_0(\tilde{\sigma}, \tilde{w}_C)$ to compensate the effect of changing w'_0 to w_0. Observe that

$$C_{\chi_1}(\tilde{\sigma}, \tilde{w}_C) = C_{\chi}(\tilde{\sigma}, \tilde{w}_C)$$

(by the definition of Φ).

Now, if $f = \tilde{f}|G$, then using

$$A(\tilde{\sigma}, \tilde{w}_C)\,|I(\tau, P) = r_1(\tilde{\sigma}, \tilde{w}_C) A(\tau, w_C),$$

relation (3.1.11) changes to

(3.1.12) $$r_0(\tilde{\sigma}, \tilde{w}_C) r_1(\tilde{\sigma}, \tilde{w}_C) C_{\chi_a}(\tilde{\sigma}, \tilde{w}_C) \lambda_{\chi_a}(A(\tau, w_C)f) = \lambda_{\chi_a}(f).$$

Here $\lambda_{\chi_a}(f)$ is defined by (3.1.9) for every $f \in V(\tau, P)$.

Let

$$r_a(\tau, w_C) = r_0(\tilde{\sigma}, \tilde{w}_C)r_1(\tilde{\sigma}, \tilde{w}_C)C_{\chi_a}(\tilde{\sigma}, \tilde{w}_C),$$

and set

$$R_a(\tau, w_C) = r_a(\tau, w_C)A(\tau, w_C).$$

Observe that since the operators are normalized, the product formula, Corollary 2.3 of the present paper, and Theorem 5.1 of [**16**] (cf. relation (3.1.6) of the present paper) now imply

$$R_1(\tau, w_C) = R(\tau, w_C).$$

Now, relation (3.1.12) can be written as

$$\lambda_{\chi_a}(R_a(\tau, w_C)f - f) = 0.$$

Since π_a appears with multiplicity one in $I(\tau, P)$, from Schur's Lemma it follows that $R_a(\tau, w_C)$ acts as a scalar on V_a. In Lemma 3.6 below, we shall prove that $\lambda_{\chi_a}|V_a \not\equiv 0$. These two observations then imply that for every $a \in A$, and every f in the space V_a of π_a

$$R_a(\tau, w_C)f = f.$$

LEMMA 3.5. *For every $a \in A$, choose f in the space of π_a. Then*

$$R(\tau, w_C)f = \gamma_a f,$$

where

$$\gamma_a = C_\chi(\tilde{\sigma}, \tilde{w}_C)/C_{\chi_a}(\tilde{\sigma}, \tilde{w}_C).$$

To complete the theorem it remains to calculate γ_a. To do this we prove:

LEMMA 3.6. *Fix $a \in A$. Let V_a be the space of $\pi_a = \tilde{a}\pi_1$. Then*

a) *There exists a function $f \in V_a$ such that $\lambda_{\chi_a}(f) \neq 0$.*

b) $\gamma_a = \omega_{w_C}(\tilde{a})$.

c) *V_a is the unique subspace of $I(\tau, P)$ which has a χ_a-Whittaker functional.*

Proof. We first assume $p \geqq 2$. Suppose $\pi_1 \subset I(\sigma_i, P)$ for some unique $\sigma_i \subset \tau$. Let $\sigma = \tilde{a} \cdot \sigma_i$. Then $\pi_a \subset I(\sigma, P)$. Take $f \in V_a$. Now, using $\Phi:\tau \cong w_C\tau$, (3.1.9) can be written as (for simplicity we use $\tilde{I}$ to denote $I(\tilde{\sigma}, \tilde{P})$)

$$(3.5.1)\qquad \tilde{\lambda}_{\chi_a}(\tilde{f}) = \int \langle \Phi(f(w_0 n)),\ \lambda_1 \otimes \ldots \otimes \sigma'_{1,l}(\tilde{\alpha}_0)\lambda_l \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)}dn$$

$$= \int \langle w_C \tilde{\sigma}(w_0 \tilde{a}^{-1} w_0^{-1})\Phi(f(w_0 n)),\ \lambda_1 \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)}dn$$

$$= \omega_{w_C,\sigma}(\tilde{a}^{-1}) \int \langle \tilde{\sigma}(w_0 \tilde{a} w_0^{-1}) f(w_0 n),\ \lambda_1 \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)}dn$$

$$= \omega_{w_C,\tilde{a}\sigma_i}(\tilde{a}^{-1}) \int \langle \tilde{\sigma}(w_0 \tilde{a}^{-1} w_0^{-1}) \tilde{f}(w_0 n),\ \lambda_1 \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)}dn$$

$$= |a|^{r-p} \omega^{-1}_{w_C,\sigma_i}(\tilde{a}) \tilde{\lambda}_{\chi_1}(\tilde{I}(\tilde{a}^{-1})\tilde{f})$$

$$= |a|^{r-p}\ \omega^{-1}_{w_C}(\tilde{a}) \tilde{\lambda}_{\chi_1}(\tilde{I}(\tilde{a}^{-1})\tilde{f}),$$

using Corollary 1.3. Here $\tilde{f}$ is defined by $\tilde{f}|G = f$. Now, observe that $I(\tilde{a}^{-1})\tilde{f} \in V_1$ and therefore

$$\tilde{\lambda}_{\chi_1}(\tilde{I}(\tilde{a}^{-1})\tilde{f}) \neq 0 \quad \text{for some } f \in V_a,$$

proving the first assertion.

Next, consider relation (3.1.10) and write

$$(3.5.2)\qquad \tilde{\lambda}'_{\chi_a}(\tilde{f}) = \int \langle w_C \tilde{\sigma}(w_0 \tilde{a}^{-1} w_0^{-1})(A(\tilde{\sigma}, \tilde{w}_C)\tilde{f})(w_0 n),\ \lambda_1 \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)}dn$$

$$= |a|^{r-p} \int \langle (A(\tilde{\sigma}, \tilde{w}_C)\tilde{I}(a^{-1})\tilde{f})(w_0 n),\ \lambda_1 \otimes \ldots \otimes \lambda_m \rangle \overline{\chi(n)}dn$$

$$= |a|^{r-p}\ \tilde{\lambda}'_{\chi_1}\ (\tilde{I}(\tilde{a}^{-1})\tilde{f}).$$

We recall that

$$\tilde{\lambda}_\chi / \tilde{\lambda}'_\chi = r_0(\tilde{\sigma}, \tilde{w})\ \tilde{\lambda}_{\chi_1} / \tilde{\lambda}'_{\chi_1}.$$

Now the lemma in this case follows immediately if we compare (3.5.1) with (3.5.2), and use (3.1.3) and (3.1.11). Finally part c) can be proved using an argument similar to the one given in Corollary 2.7 of [**11**].

We remark that the isomorphism $\Phi:\tau \cong w_C\tau$ is crucial in obtaining this result.

Now suppose $p = 1$. Then $m = r$ and the induced representations are in the principal series. P and $\tilde{P}$ are now the corresponding subgroups of upper triangulars. Finally, the subgroups M and $\tilde{M}$ are the corresponding subgroups of diagonals. We only prove the theorem in this case when $w_C = w_c$, the full Coxeter element. The proof for a general w_C is similar.

By part a) of Lemma 2.2, there exist characters π and ω of k^* with $\omega^m = 1$ such that

$$\widetilde{\sigma} = \bigotimes_{j=1}^{m} \pi\omega^j.$$

Furthermore, for $a \in k^*$

$$\tau\left(\begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & a^{-1} \end{pmatrix}\right) = \omega(a)$$

which uniquely determines ω.

An easy calculation now shows that $\omega_{w_c} = \omega \cdot \det$. Again for each $a \in A$, the corresponding Whittaker functional is given by the principal value integral

$$\widetilde{\lambda}_{\chi_a}(\widetilde{f}) = \int_N \widetilde{f}(w_0 n)\overline{\chi}(\widetilde{a}^{-1} n \widetilde{a})dn.$$

We then have

$$\begin{aligned}\pi^{-1}(a)\widetilde{\lambda}_{\chi_a}(\widetilde{f}) &= \widetilde{\sigma}(w_0\widetilde{a}^{-1}w_0^{-1})\,\widetilde{\lambda}_{\chi_a}(\widetilde{f})\\ &= \int \widetilde{f}(w_0\widetilde{a}^{-1}n\widetilde{a}\,\widetilde{a}^{-1})\overline{\chi}(\widetilde{a}^{-1}n\widetilde{a})dn\\ &= |a|^{r-1}\widetilde{\lambda}_{\chi_1}(\widetilde{I}(\widetilde{a}^{-1})\widetilde{f}).\end{aligned}$$

Similarly

$$\pi\omega(a^{-1})\widetilde{\lambda}'_{\chi_a}(\widetilde{f}) = w_c\widetilde{\sigma}(w_0\widetilde{a}^{-1}w_0^{-1}) \int A(\widetilde{\sigma}, \widetilde{w}_c)\widetilde{f}(w_0 n)\overline{\chi}(\widetilde{a}^{-1}n\widetilde{a})dn$$

$$\begin{aligned}&= \int A(\widetilde{\sigma}, \widetilde{w}_c)\widetilde{f}(w_0\widetilde{a}^{-1}n\widetilde{a}\cdot\widetilde{a}^{-1})\overline{\chi}(\widetilde{a}^{-1}n\widetilde{a})dn\\ &= |a|^{r-1}\,\widetilde{\lambda}'_{\chi_1}(\widetilde{I}(\widetilde{a}^{-1})\widetilde{f}).\end{aligned}$$

Again the lemma follows from definitions.

COROLLARY 3.6. *Suppose $\widetilde{w} \in W(\tau)$. Then, given $a \in A$, the intertwining operator $A(\tau, w)$ acts on the space of $\pi_a \subset I(\tau, P)$, as the scalar*

$$r(\tau, w)^{-1} \cdot \omega_w(\widetilde{a}).$$

Now, given $\sigma \subset \tau$, choose $a_\sigma, a_1, \ldots, a_{n_1} \in A$ with $a_1, \ldots, a_{n_1}$ representatives of $N'_{\widetilde{\sigma}}/N_{\widetilde{\sigma}}$, where as before (Lemma 1.11)

$$N'_{\tilde{\sigma}} = \bigcap_{\nu \in X(\tilde{\sigma})} \operatorname{Ker} \nu,$$

such that $\pi_{a_\sigma a_j} \subset I(\sigma, P), j = 1, \ldots, n_1$.

Next, suppose σ is in the discrete series and let ϕ be the homomorphism of W_k into ${}^LM$, if any, which generates the L-packet of σ (cf. Proposition 1.9). It is easy to see (using Theorem 4.2 of [**6**] and Theorem 5.4.4.1 of [**19**], cf. Proposition 1.9) that the S-group $S(\phi)$ is the set of all the extensions of the characters $\omega_{w,0} = \omega_{w,\sigma}|\widetilde{M}_0$, $\sigma \subset \tau$, from $\widetilde{M}_0$ to $\widetilde{M}$, $\forall\, w \in W(\sigma)$. We observe that when σ is only tempered we have

$$R(\sigma) \subset S(\phi)/S_M(\phi).$$

We now prove the following important corollary of Theorem 3.1. Again we fix A and $\pi_1 \subset I(\sigma_i, P)$ with $\lambda_\chi|\pi_1 \not\equiv 0$ (then $a_{\sigma_i} = 1$).

COROLLARY 3.7. a) *Up to a constant multiple (which is $\omega_w(a_\sigma)$), the coefficients of the expansion of*

$$\operatorname{trace}\,(R(\sigma, w)I(\sigma, P, f))$$

are equal to $\omega_{w,0}(a_j)$, i.e., the pairing between $\omega_{w,0} \in S(\phi)$ and $\pi_{a_\sigma a_j} \subset I(\sigma, P)$.

b) *Let $A' = \{a_1, \ldots, a_{n_1}\}$ and for each $\sigma \subset \tau$, let $\alpha_\sigma : A' \to A_\sigma$ be defined by $\alpha_\sigma(a_j) = a_\sigma a_j$. Observe that $A' = A_{\sigma_i}$. Then there exists a unique intertwining operator $\Phi_0 : \tau \cong w\tau$ such that for every $\sigma \subset \tau$ and $f \in C_c^\infty(G)$*

$$\operatorname{trace}\,(R(\sigma, w)I(\sigma, P, f)) = \sum_{a \in A'} \omega_{w,0}(a)\chi_{\pi_{\alpha_\sigma(a)}}(f).$$

c) *Suppose σ is in the discrete series and $\phi : W_k \to {}^LM$ generates the L-packet of σ. Let*

$$\rho : R(\sigma) \cong S(\phi)/S_M(\phi).$$

Given $\ast \in S(\phi)$, choose $\omega_{w,0} \in R(\sigma)$ such that $\rho(\omega_{w,0}) = \ast\, S_M(\phi)$. Also, given $\sigma \subset \tau$ and $a \in A'_\sigma$, let $\pi_{\alpha_\sigma(a)}$ be as in part b). *Set*

$$\langle \ast, \pi_{\alpha_\sigma a)} \rangle = \omega_{w,0}(a)$$

Let $\Phi_0 : \tau \cong w\tau$ be as in part b). *Then*

$$\operatorname{trace}\,(R(\sigma, w)I(\sigma, P, f)) = \sum_{a \in A'} \langle \ast, \pi_{\alpha_\sigma(a)} \rangle \chi_{\pi_{\alpha_\sigma(a)}}(f).$$

Proof. We only need to prove part b). Let $\Phi : \tau \cong w\tau$ be an isomorphism as in Theorem 3.1. If $\Phi' : \tau \cong w\tau$ is another one, let ω'_w be the corresponding function. For $\sigma \subset \tau$, let V_σ be the common space for σ and $w\sigma$. Then there exists a complex number $c_\sigma \neq 0$ such that

$$\Phi'|V_\sigma = c_\sigma \Phi|V_\sigma.$$

Moreover, if $\pi_1 \subset I(\sigma_i, P)$, then $c_{\sigma_i} = 1$. In fact, since σ_i has a χ-Whittaker functional, there exists a vector $v \in V_{\sigma_i}$ such that $\langle \Phi(v), \widetilde{\lambda} \rangle = \langle v, \widetilde{\lambda} \rangle$ is non-zero. Consequently

$$\begin{aligned} \langle c_{\sigma_i}\Phi(v), \widetilde{\lambda} \rangle &= \langle v, \widetilde{\lambda} \rangle \\ &= \langle \Phi(v), \widetilde{\lambda} \rangle \end{aligned}$$

implies $c_{\sigma_i} = 1$. Now it follows from the definition of ω_w that

$$\begin{aligned} \omega'_w(\widetilde{a}_\sigma) &= \widetilde{\sigma}(\widetilde{a}_\sigma^{-1})\Phi'^{-1} w\widetilde{\sigma}(\widetilde{a}_\sigma)\Phi'|V_{\sigma_i} \\ &= c^{-1}_{\widetilde{a}_\sigma \cdot \sigma_i} \cdot \omega_w(\widetilde{a}_\sigma). \end{aligned}$$

We now fix Φ' so that for each $\sigma \subset \tau$

$$c_{\widetilde{a}_\sigma \cdot \sigma_i} = \omega_w(\widetilde{a}_\sigma),$$

and consequently $\omega'_w(\widetilde{a}_\sigma) = 1$. Set $\Phi_0 = \Phi'$. The uniqueness of Φ_0 is now clear as well, proving the corollary.

Again let $\sigma_i \subset \tau$ be the unique subrepresentation of τ which has a χ-Whittaker functional. Then every other $\sigma \subset \tau$ has a χ_{a_σ}-Whittaker functional ($\sigma = \widetilde{a}_\sigma \cdot \sigma_i$) and again by Corollary 2.7 of **[11]** σ is the unique element in the L-packet which has such a functional. Fix a χ-Whittaker functional $\widetilde{\lambda}$. Using dual representations, let

$$\widetilde{\lambda}_\sigma = \widetilde{\sigma}'(\widetilde{a}_\sigma)\widetilde{\lambda} \quad \text{and} \quad \widetilde{\lambda}_{w\sigma} = (w\widetilde{\sigma})'(\widetilde{a}_\sigma)\widetilde{\lambda}$$

be the corresponding χ_{a_σ}-Whittaker functionals for $\widetilde{\sigma}$ and $w\widetilde{\sigma}$, respectively. We now prove the following characterization of Φ_0.

LEMMA 3.8. *The operator Φ_0 is such that for every $\sigma \subset \tau$*

$$\langle \Phi_0(v), \widetilde{\lambda}_{w\sigma} \rangle = \langle v, \widetilde{\lambda}_\sigma \rangle \qquad (v \in V_\sigma).$$

Proof. Let ϕ be any isomorphism $\phi:\tau \cong w\tau$, and let b_ϕ be a nonzero complex number such that

$$b_\phi\langle \phi(v), \widetilde{\lambda} \rangle = \langle v, \widetilde{\lambda} \rangle,$$

where v is in the space of σ_i. We then have

$$\begin{aligned} \langle w\widetilde{\sigma}(\widetilde{a}_\sigma^{-1})\phi(v), \widetilde{\lambda} \rangle &= \omega_{w,\sigma}(\widetilde{a}_\sigma^{-1})\langle \phi(\widetilde{\sigma}(\widetilde{a}_\sigma^{-1})v), \widetilde{\lambda} \rangle \\ &= b_\phi^{-1}\omega_{w,\widetilde{a}_\sigma \cdot \sigma_i}(\widetilde{a}_\sigma^{-1})\langle \widetilde{\sigma}(\widetilde{a}_\sigma^{-1})v, \widetilde{\lambda} \rangle \\ &= b_\phi^{-1}\omega^{-1}_{w,\sigma_i}(\widetilde{a}_\sigma)\langle \widetilde{\sigma}(\widetilde{a}_\sigma^{-1})v, \widetilde{\lambda} \rangle. \end{aligned}$$

To complete the lemma one has now only to check that

$$\Phi_0|V_\sigma = b_\phi \omega_{w,\sigma_i}(\tilde{a}_\sigma)\phi|V_\sigma.$$

But this is just the definition of Φ_0.

References

1. J. Arthur, *On a family of distributions obtained from Eisenstein series I: Applications of the Paley-Wiener theorem*, Amer. J. Math. *104* (1982), 1243-1288.

2. ——— *On a family of distributions obtained from Eisenstein series II: Explicit formulas*, Amer. J. Math. *104* (1982), 1289-1336.

3. ——— *The trace formula for reductive groups*, Lectures for Journées Automorphes, Dijon University.

4. W. Casselman and J. A. Shalika, *The unramified principal series of p-adic groups II*, Comp. Math. *41* (1980), 207-231.

5. S. S. Gelbart and A. W. Knapp, *Irreducible constituents of principal series of $SL_n(k)$*, Duke Math. J. *48* (1981), 313-326.

6. ——— *L-indistinguishability and R-groups for the special linear group*, Advances in Math. *43* (1982), 101-121.

7. H. Jacquet, *From GL_2 to GL_n*, 1975 U. S. - Japan Seminar on Number Theory, Ann Arbor.

8. ——— *Generic representations* (Lecture notes in Math. 587, Springer-Verlag, 1977), 91-101.

9. H. Jacquet, I. I. Piatetski-Shapiro and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. *105* (1983), 367-464.

10. A. W. Knapp and G. Zuckerman, *Normalizing factors, tempered representations and L-groups*, "Automorphic Forms, Representations, and *L*-functions", Proc. Symposia Pure Math. *33*, Part 1, 93-105 (American Mathematical Society, Providence, R. I., 1979).

11. J. P. Labesse and R. P. Langlands, *L-indistinguishability for $SL(2)$*, Can. J. Math. *31* (1979), 726-785.

12. R. P. Langlands, *On the functional equations satisfied by Eisenstein series* (Lecture notes in Math. 544, Springer-Verlag, 1976).

13. ——— *Les débuts d'une formule des traces stables*, printed notes. L'École Normale Supérieure de Jeunes Filles, Paris (1980).

14. F. Shahidi, *On certain L-functions*, Amer. J. Math. *103* (1981), 297-355.

15. ——— *Local coefficients and normalization of intertwining operators for $GL(n)$*, Comp. Math. *48* (1983), 271-295.

16. ——— *Fourier transforms of intertwining operators and Plancherel measures for $GL(n)$*, Amer. J. Math., to appear.

17. D. Shelstad, *Orbital integrals and a family of groups attached to a real reductive group*, Ann. Scient. Ec. Norm. Sup. *12* (1979), 1-31.

18. ——— *L-indistinguishability for real groups*, Math. Ann. *259* (1982), 385-430.

19. A. J. Silberger, *Introduction to harmonic analysis on reductive p-adic groups*, Math. Notes of Princeton University Press *23* (Princeton, N. J., 1979).

Purdue University,
West Lafayette, Indiana