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On boolean near-rings

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It is well-known that a boolean ring is commutative. In this note we show that a distributively generated boolean near-ring is multiplicatively commutative, and therefore a ring. This is accomplished by using subdirect sum representations of near-rings.

1. Introduction

It is well-known that a boolean ring is isomorphic to a subdirect sum of fields $I/(2)$. The purpose of this note is to extend the above result to distributively generated near-rings. Also, examples will be given to show that the result does not hold for arbitrary near-rings.

2. Definitions and basic information

A (left) *near-ring* R is a system with two binary operations, addition and multiplication, such that

(i) the elements of R form a group $(R, +)$ under addition,

(ii) the elements of R form a multiplicative semigroup,

(iii) $x(y+z) = xy + xz$, for all $x, y, z \in R$.

In particular, if R contains a multiplicative semigroup S whose elements generate $(R, +)$ and satisfy

(iv) $(x+y)s = xs + ys$, for all $x, y \in R$ and $s \in S$,

we say that R is a *distributively generated* (d.g.) near-ring.

The most natural example of a near-ring is given by the set R of all mappings of an additive group (not necessarily abelian) into itself. If the mappings are added by adding images and multiplication is iteration,

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then the system $(R, +, \cdot)$ is a near-ring. If S is a multiplicative semigroup of endomorphisms of R and R' is the sub-near-ring generated by S, then R' is a d.g. near-ring. Other examples of d.g. near-rings may be found in [6].

An element a of R is *right (anti-right) distributive* if $(b+c)a = ba + ca$ $\big((b+c)a = ca + ba\big)$ for all $b, c \in R$. It follows at once that an element a is right distributive if and only if $(-a)$ is anti-right distributive. In particular, any element of a d.g. near-ring is a finite sum of right and anti-right distributive elements.

The kernels of near-ring homomorphisms are called *ideals*. Blackett [3] showed that K is an ideal of a near-ring N if and only if K is a normal subgroup of $(N, +)$ that satisfies

(i) $NK \subseteq K$ and

(ii) $(m+k)n - mn \in K$, for all $m, n \in N$ and $k \in K$.

Distributively generated near-rings allow a much stronger structure theory and representation theory than near-rings in general. Many of the fundamental theorems on rings can be generalized to d.g. near-rings. See, for example, [1], [7], [8] and [9].

3. Subdirect sums of near-rings

The theory of subdirect sum representation of rings carries over almost word for word to near-rings [5]. A nonzero near-ring R is *subdirectly irreducible* if and only if the intersection of all the nonzero ideals of R is nonzero. The near-ring analogue of Birkhoff's [2] fundamental result for rings can be stated as follows.

THEOREM 1. [5] *Each near-ring R is isomorphic to a subdirect sum of subdirectly irreducible near-rings.*

4. The main result.

In this section we shall prove that every d.g. boolean near-ring is a ring. To facilitate the discussion we first prove two lemmas.

LEMMA 1. *Let R be a d.g. boolean near-ring and let x, y, z, w be elements in R such that x and y are right distributive, z is anti-right distributive and w is any element in R. Then the following*

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statements hold.

(i) $x + x = 0$,

(ii) $xy = yx$,

(iii) $xz = zx$,

(iv) $xw = wx$,

(v) $A_x = \{r \in R : xr = 0\}$ *is an ideal of* R ,

(vi) *If* $A_x = 0$, *then* x *is an identity and* $(R, +)$ *is abelian.*

Proof. *(i)* follows from the expansion of $(x + x)^2$. Since xy is right distributive, *(ii)* is the consequence of *(i)* and the expansion of $(x + y)^2$. For *(iii)*, observe that $(-z)$ is right distributive and hence $x(-z) = (-z)x$. Since x is right distributive $(-z)x = -(zx)$. It follows that $xz = zx$ because $x(-z) = -(xz)$ is always valid. Since every element of a d.g. near-ring is a finite sum of right and anti-right distributive elements, we have, by using *(ii)* and *(iii)*, that

$$xw = x(w_1+w_2+\ldots+w_n) = xw_1 + xw_2 + \ldots + xw_n$$
$$= w_1x + w_2x + \ldots + w_nx$$
$$= wx .$$

The proof of *(v)* follows from *(iv)* and the definition of an ideal. If $A_x = 0$, then x is a left identity. For if not, there exists $y \in R$ such that $y \neq 0$ and $xy \neq y$. Thus $x(xy-y) = 0$ and $A_x \neq 0$, which is a contradiction. By *(iv)* and *(i)*, x is a two-sided identity and $x + x = 0$. If r is any element of R then $r + r = r(x+x) = 0$. Thus each element of $(R, +)$ is of order two and $(R, +)$ is abelian.

LEMMA 2. *If* R *is a subdirectly irreducible d.g. boolean near-ring then* R *is a boolean ring with an identity.*

Proof. Suppose for each right distributive element x in R , $A_x \neq 0$. Since R is subdirectly irreducible and each A_x is an ideal of R , we have that $\bigcap A_x = A \neq 0$. Let $w \neq 0$ be an element in A . Thus $xw = 0$ for each right distributive element x in R . Since $xw = wx = 0$, it follows that $wz = 0$ if z is anti-right distributive. Furthermore, if

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y is any element in R, then $wy = w(y_1+y_2+\ldots+y_n) = wy_1+wy_2+\ldots+wy_n = 0$. This implies that $A_w = R$. But then $w = ww = 0$, contradicting the fact that $w \neq 0$. Thus there exists a right distributive element x in R such that $A_x = 0$. By *(vi)* of Lemma 1, $(R, +)$ is abelian and x is an identity. Now that R is a ring follows from [*6*, p. 93].

We are now ready to prove the main result of this note.

THEOREM 2. *Every d.g. boolean near-ring R is a boolean ring.*

Proof. By Theorem 1, R is isomorphic to a subdirect sum of subdirectly irreducible near-rings R_i. Now each R_i is a homomorphic image of R and therefore a d.g. boolean near-ring [*6*]. By Lemma 2, each R_i is a boolean ring and hence $(R_i, +)$ is abelian. It follows that $(R, +)$ is abelian and hence [*6*, p. 93] R is a ring.

5. General boolean near-rings

Let G be an additive group (not necessarily abelian). Define for each $x \in G$, $xy = y$ for each $y \in G$. Then $(G, +, \cdot)$ is a boolean near-ring. Other interesting examples of boolean near-rings which are not rings can be found in [*4*] and [*11*]. Clearly there exist boolean near-rings which are not boolean rings. Thus we conclude that Theorem 2 cannot be extended to arbitrary near-rings.

6. Remark

A ring R is said to be a *p-ring* if p is a fixed prime and $x^p = x$, $px = 0$ for each x in R. Thus a boolean ring is a 2-ring. McCoy and Montgomery [*10*] showed that a p-ring R is isomorphic to a subdirect sum of fields $I/(p)$. In the light of the result of this paper one naturally asks that whether the result of McCoy and Montgomery can be extended to distributively generated near-rings. This question is still open.

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