PERSPECTIVE SIMPLEXES[†]

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(Received 1 July 1960)

Introduction

The main purpose of this paper is to prove the proposition: "*A* set of r mutually perspective (m.p.) $(s-1)$-simplexes have the same $[s-2]$ (say x) of perspectivity, if and only if their $\binom{r}{2}$ centres of perspectivity (c.p.) lie in an $[r-2]$ (say y); there then arises another such set of s m.p. $(r-1)$-simplexes, having the same rs vertices, which have y as their common $[r-2]$ of perspectivity such that their $\binom{s}{2}$ c.p. lie in x." The proposition is true in any $[k]$ for $k = s-1, s, \cdots, r+s-2$ $(r \leqq s)$. The configuration of the proposition in $[r+s-2]$ arises from the incidences of any $r+s$ arbitrary primes therein and is therefore invariant under the symmetric group of permutations of $r+s$ objects, and that in $[r+s-3]$ is self-dual and therefore self-polar for a quadric therein. Some special cases of some interest for $r = s$ are deduced. The treatment is an illustration of the elegance of the Möbius Barycentric Calculus ([15], pp. 136—143; [1], p. 71).

1. Proof of the proposition

(a) Let P_{iu} be the rs vertices of the r m.p. $(s-1)$-simplexes (P_i), x their common $[s-2]$ of perspectivity, P_{uv} the trace in x of an edge $P_{iu}P_{iv}$ of one (P_i) of them, and P_{ij} the centre of perspectivity of a pair (P_i), (P_j) of them $(i, j = 1, \cdots, r;\ u, v = r+1, \cdots, r+s)$. Their r correspondig edges $P_{iu}P_{iv}$ obviously concur at P_{uv}.

By using the same letters for the symbols of points ([4], p. 115; [7]—[13]), we may then take

$$(1) \qquad P_{uv} = P_{iu} - P_{iv} = P_{ju} - P_{jv} = P_{ku} - P_{kv} = \cdots$$

and therefore

$$(2) \qquad P_{ij} = P_{iu} - P_{ju} = P_{iv} - P_{jv} = P_{iw} - P_{jw} \cdots.$$

[†] The former editor wishes to apologise for the delay in publication of this paper.

[*] This paper was originally written at the Indian Institute of Technology, Kharagpur, but revised at the University of Sydney under the direction of Professor T. G. Room.

Every 3 points P_{ij}, P_{jk}, P_{ik} are evidently collinear in a line L_{ijk} (say), and therefore every 4 such lines L_{ijk}, L_{jkl}, L_{kli}, L_{lij} or 6 points P_{ij}, P_{jk}, P_{ki}, P_{li}, P_{lj}, P_{lk} are coplanar, and so on. Thus the $\binom{r}{2}$ points P_{ij} lie by $\binom{3}{2}$s or by threes in $\binom{r}{3}$ $[3-2]$s or lines, by $\binom{4}{2}$s or sixes in $\binom{r}{4}$ $[4-2]$s or planes, $\cdots$, and by $\binom{r}{2}$s or all in $\binom{r}{r}$ or one $[r-2]$ (say y).

Conversely, the relations (2) imply (1), too, and hence follows the first part of the proposition, viz.

A set of r *m.p.* $(s-1)$*-simplexes have an* $[s-2]$, x *of perspectivity common, if and only if their* $\binom{r}{2}$ *c.p. all lie in an* $[r-2]$, y.

(b) Again we may look at the picture in a different way by constructing s $(r-1)$-simplexes (P_u) formed of the same rs vertices, and notice that every pair (P_u), (P_v) of them are in perspective with centre of perspectivity at P_{uv} such that P_{ij} is the common trace of their s corresponding edges in y. That proves the second part of the proposition, viz.

There arises another set of s *m.p.* $(r-1)$*-simplexes, having the same* $r\,s$ *vertices which have* y *as their common* $[r-2]$ *of perspectivity such that their* $\binom{s}{2}$ *c.p. lie in* x.

(c) Further we observe that the r $(s-1)$-simplexes (P_i) or s $(r-1)$-simplexes (P_u) may lie in any $[k]$ for $k = s-1, s, \cdots, r+s-2$ $(r \leqq s)$ and the proof of the proposition holds good in all these r spaces. Hence:

The proposition is true in all the r *spaces* $[k]$.

2. Configuration

(a) The rs points P_{iu}, $\binom{r}{2}$ P_{ij} and $\binom{s}{2}$ P_{uv} may be observed to form a figure of $rs+\binom{r}{2}+\binom{s}{2} = \binom{r+s}{2}$ points P_{ht} $(h, t = 1, \cdots, r+s)$ lying by threes on $\binom{r+s}{3}$ lines, $r+s-2$ through each point, as if it arises in $[r+s-2]$ from a prime section p [14] of a simplex (X) in $[r+s-1]$, and therefore forms a picture of incidences of $r+s$ $[r+s-3]$ sections of the $r+s$ prime faces of (X) by p. Hence: *The configuration of the proposition in* $[r+s-2]$ *forms a picture of incidences of* $r+s$ *arbitrary primes therein.*

(b) We may now revise (as suggested by Prof. Room) the proof of the proposition by taking the $\binom{r+s}{2}$ points of the configuration on the edges of the simplex $(X) = X_1 \cdots X_{r+s}$ as follows:

$$\text{If } P_{iu} = X_i - X_u, \tag{3}$$

then

$$P_{uv} = P_{iu} - P_{iv} = X_v - X_u, \tag{4}$$

$$P_{ij} = P_{iu} - P_{ju} = X_i - X_j. \tag{5}$$

All the points $P_{ht} = X_h - X_t$ of the figure obviously lie in the prime p whose equation, referred to (X), is

$$(6) \qquad \sum x_h = 0.$$

The $\binom{s}{2}$ points P_{uv} lie in the $[s-2]$, x, given by the $r+1$ equations

$$(7) \qquad \sum x_u = 0 = x_i.$$

The $\binom{r}{2}$ points P_{ij} lie in the $[r-2]$, y, given by the $s+1$ equations

$$(8) \qquad \sum x_i = 0 = x_u.$$

(c) We may thus split the vertices of the simplex (X) into any two sets. Hence:

The configuration of the proposition is equivalent to that of $r-p$ *m.p.* $(s+p-1)$*-simplexes having a common* $[s+p-2]$, x', *of perspectivity such that their* $\binom{r-p}{2}$ *c.p. lie in an* $[r-p-2]$, y', *or to that of* $s+p$ *m.p.* $(r-p-1)$*-simplexes having* y' *as their common* $[r-p-2]$ *of perspectivity such that their* $\binom{s+p}{2}$ *c.p. lie in* x'. *The proposition is now true in any* $[k']$ *for* $k' = s+p-1$, $s+p, \cdots, r-s-2$.

d) In particular, *the configuration is equivalent to that of a pair of perspective* $(r+s-3)$*-simplexes which form a self-dual figure in* $[r+s-3]$ ([2], *pp.* 128, 251). Hence: *The figure arising from a pair of perspective* $(r+s-3)$*-simplexes always splits into that of* r *m.p.* $(s-1)$*-simplexes having the same* $[s-2]$, x, *of perspectivity or* s *m.p.* $(r-1)$*-simplexes whose* $\binom{s}{2}$ *c.p. lie in* x.

3. Group

From the preceding section now follows that: *The configuration of the proposition is invariant under the symmetric group of permutations of* $r+s$ *objects.* For the order of the $r+s$ vertices of the simplex (X) does not affect the number of its edges and therefore that of their intersections P_{ht} with the prime p.

4. Quadric

The self-dual character of the configuration (§ 2d) in $[r+s-3]$ suggests that it is self-polar for a quadric Q therein, as pointed out by Prof. Room.

We may take a quadric Q' in $[r+s-1]$ for which the simplex (X) is self-polar and the prime p (§ 2b) is tangent to it at a point $P(p_1, \cdots, p_{r+s})$. The equation of Q', referred to (X), is then found to be (cf. [14])

$$(9) \qquad \sum x_h^2/p_h = 0, \quad \sum p_h = 0.$$

The section of Q' by p is an $(r+s-3)$-cone C $(r+s>4)$ with vertex at P such that a point P_{ht} in p on an edge X_hX_t of (X) is conjugate for C to the $[r+s-4]$ section p_{ht} of its opposite $[r+s-3]$ by p. That is, the joins of P to P_{ht} and p_{ht} are polar of each other w.r.t. C.

Thus the figure, obtained as a section of (X) by p, projects from P on to a $[r+s-3]$, q, into one self-polar for the quadric section Q of C by q. This figure is the same as the configuration of the proposition such that the pair of perspective simplexes, equivalent to it (§ 2d), are polar reciprocal of each other for Q.

In other words, if the coordinate-system (cf. [14]) in q depending on $r+s$ parameters x_h be such that

a) $(x_1, \cdots, x_{r+s})$ are coordinates of a point only if $\sum x_h = 0$,

b) $(x_1, \cdots, x_{r+s})$ and $(x_1+kp_1, \cdots, x_{r+s}+kp_{r+s})$ represent the same point for all finite values of k and $\sum p_h = 0$, then the $\binom{r+s}{2}$ points P_{ht}, each having 2 coordinates 1, -1 and the rest all zeros, form the figure, under consideration, selfpolar for the quadric Q given by the same equation as (9).

5. Special cases for $r = s$

(a) We may now state the proposition as follows:

A set of r m.p. $(r-1)$-simplexes have the same $[r-2]$, x, of perspectivity, if and only if their $\binom{r}{2}$ c.p. lie in an $[r-2]$, y; then there arises another such set of r m.p. $(r-1)$-simplexes, having the same r^2 vertices, which have y as their common $[r-2]$ of perspectivity such that their $\binom{r}{2}$ c.p. lie in x. The proposition is true in any $[k]$ for $k = r-1, r, \cdots, 2r-2$.

In particular, $r = 3$ give us 2 such triads of m.p. triangles. Figure 1 illustrates $(P) = P_1P_2P_3$ $(P = A, B, C)$ and $(k) = A_kB_kC_k$ $(k = 1, 2, 3)$ as the said triads of triangles (cf. [3], p. 36), $x = M_{12}M_{23}M_{31}$, $y = XYZ$ being their respective axes of perspectivity such that X, Y, Z are the c.p. of the first triad and M_{12}, M_{23}, M_{31} of the second. This holds in [4], solid and plane.

(b) A further specialized case arises when the third triangle of a triad of m.p. triangles, having the same axis of perspectivity, is derived from the other two. For example, if $A_1A_2A_3$, $B_1B_2B_3$ be a pair of perspective triangles and the third triangle is formed of the 3 points of intersection $C_i = A_jB_k \cdot A_kB_j$ $(i, j, k = 1, 2, 3)$, the 3 triangles (P) form one triad satisfying the required conditions and the second triad (k) follow ([3], p. 45; [6]) as illustrated below in Figure 2.

This specialized proposition is true in solid and plane only.

(c) For the dual configuration, general as well as special, in a plane, reference may be made to Baker ([5], pp. 350—351), and that in $[s-1]$ may be stated as follows:

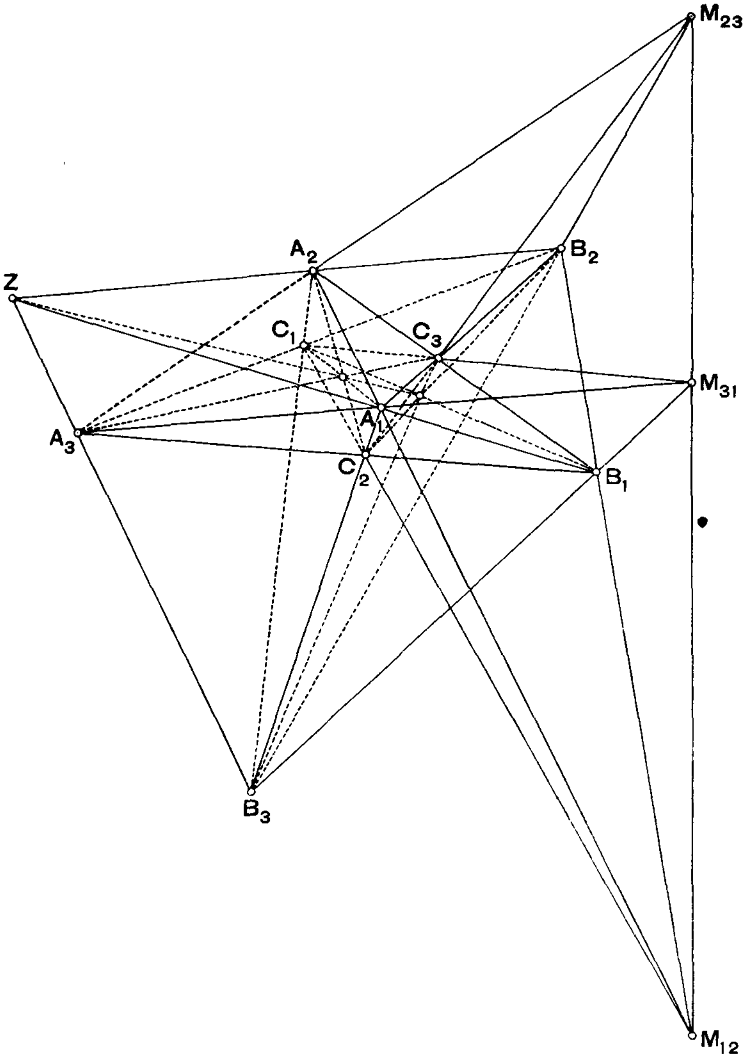

Figure 1

A set of r m.p. simplexes in $[s-1]$ *have the same centre X of perspectivity if and only if their* $\binom{r}{2}$ *primes of perspectivity have an* $[s-r]$ *common or* concur *when* $r = s$ *at a point Y, and there then arises another such set of r m.p. simplexes, having the same* r^2 *prime faces, which have Y as their common centre of perspectivity such that their* $\binom{r}{2}$ *primes of perspectivity* concur *at X.*

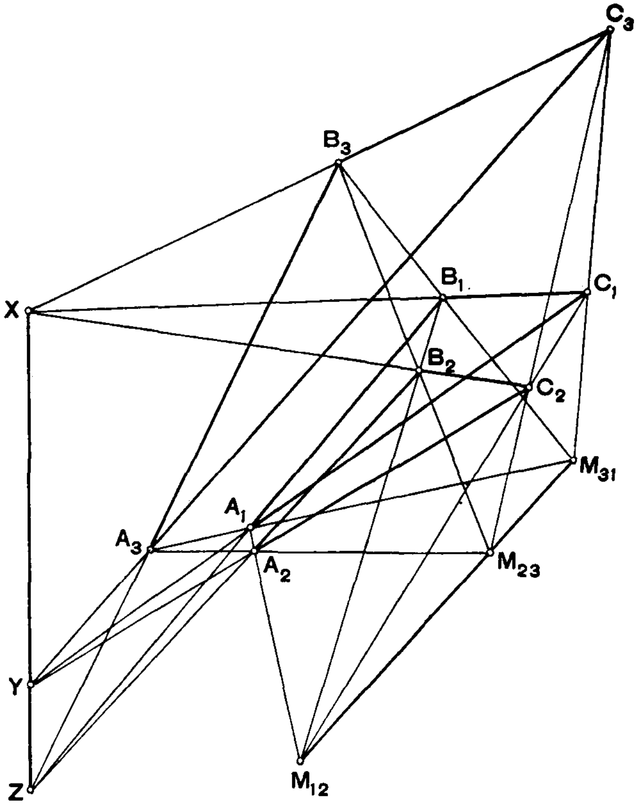

Figure 2

My thanks are due to Professor T. G. Room for his kind suggestions, and to Mr. R. K. Maithel (student at the Indian Institute of Technology, Kharagpur) for tracing the figures.

References

[1], [2], [3], [4] H. F. Baker, Principles of Geometry, vols. 1, 2, 3, 4 (Cambridge, 1929, 1930, 1934, 1940).

[5] H. F. Baker, Introduction to plane geometry (Cambridge, 1943).

[6] N. A. Court, Desargues and his strange theorem, *Scripta Math.* 20 (1954), 1—20.

[7] H. S. M. Coxeter, Twelve points in *PG* (5,3) with 95040 selftransformations, *Proc. Roy. Soc.* A 247 (1958), 279—293.

[8] S. R. Mandan, Commutative law in four dimensional space S_4. *East Panjab. Uni. Res. Bull.* 14 (1951), 31—32.

[9] S. R. Mandan, Projective tetrahedra in a 4-space, *J. Sci. & Engg. Res.* 3 (1959), 169—174.

[10] S. R. Mandan, Desargues' theorem in n-space, *J. Australian Math. Soc.* 1 (1960), 311—318.

[11] T. G. Room, Some configurations based on five general planes in space of ten dimensions (and the double-ten of planes and lines in space of four dimensions), *Proc. Lond. Math. Soc.* (2) 28 (1927), 312—346.

[12] T. G. Room, An extension of the theorem of the fifth associated line, *ibid*, 31 (1929), 455—486.
[13] T. G. Room, Cards and Cubes (A study in incidence geometry), *Sydney Uni. Sci. Jour.* 17 (1938) 41—46.
[14] T. G. Room, The orthocentre, perspective triangles, and the double-six, *Australian Math. Teacher* 3 (1947), 42—47.
[15] A. N. Whitehead, A treatise on universal algebra 1 (Cambridge, 1898).

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