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Deformations of corank 1 frontals

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We develop a Thom–Mather theory of frontals analogous to Ishikawa's theory of deformations of Legendrian singularities but at the frontal level, avoiding the use of the contact setting. In particular, we define concepts like frontal stability, versality of frontal unfoldings or frontal codimension. We prove several characterizations of stability, including a frontal Mather–Gaffney criterion, and of versality. We then define the method of reduction with which we show how to construct frontal versal unfoldings of plane curves and show how to construct stable unfoldings of corank 1 frontals with isolated instability which are not necessarily versal. We prove a frontal version of Mond's conjecture in dimension 1. Finally, we classify stable frontal multigerms and give a complete classification of corank 1 stable frontals from $\mathbb{C}^3$ to $\mathbb{C}^4$.

Keywords: classification; frontals; stability; versality

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1. Introduction

The study of frontal mappings has flourished rapidly in the last decade. Roughly speaking, a frontal is a mapping $f\colon N \to Z$ where N and Z are n and $(n+1)$-dimensional manifolds such that the image of N has a well-defined tangent hyperplane at each point. More precisely, f is a frontal if it admits a Legendrian lift $\tilde{f}\colon N \to PT^*Z$ such that $f = \pi \circ \tilde{f}$, where π is the canonical fibration. When the Legendrian lift is an immersion, we say that f is a wave front. The concept of frontals was first introduced by Fujimori *et al.* in [7] (see also [29]) and since then it has been of great interest to differential geometers, singularists and contact topologists. The fact of having a well-defined normal at each point allows one to study differential geometric properties and invariants in singular spaces [**4**, **16**, **21**, **24**, **25**], on the other hand, when studying contact and symplectic topology front singularities are unavoidable [**3**] and understanding the generic (or stable) situations is crucial.

In [**14**], Ishikawa developed the analogue of the Thom–Mather theory for corank 1 Legendrian singularities and he stated the main notions like infinitesimal deformations, stability, versality, etc. Our purpose in this paper is to construct a Thom–Mather theory of singularities of frontals, but downstairs, at the level of frontals, and thus, avoiding the use of the contact setting. In particular, we consider deformations that come from unfoldings F of the frontal f. We show that such unfoldings F come from a deformation of its Legendrian lift $\tilde{f}$ if and only if F is frontal as a mapping. Taking local charts of N and Z, we study map germs $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ under $\mathscr{A}$-equivalence, i.e. smooth changes of coordinates in source and target. Here, smooth means C^∞ when $\mathbb{K} = \mathbb{R}$ or holomorphic when $\mathbb{K} = \mathbb{C}$. The case of frontal surfaces ($n = 2$) was studied in a previous paper [**22**] where analytic/toplogical invariants were defined and characteristations of finite frontal codimension were given, amongst other interesting results on surfaces, using some of the definitions and results that will be given in this paper.

In § 3, we define the concept of frontal stability and versality. We define a frontal codimension and prove that a frontal is stable if and only if it has frontal codimension 0. We also give a characterization of versality analogous to Mather's versality theorem. Section 4 gives a geometric criterion for stability, a frontal Mather–Gaffney criterion which states that a frontal is stable if and only if it has isolated instability. Sections 5 and 6 are devoted to show how to construct stable frontals as frontal versal unfoldings of plane curves or as a well-defined sum of frontal unfoldings. We define the frontal reduction of an $\mathscr{A}_e$-versal unfolding of a plane curve and prove that it is, in fact, a versal frontal unfolding. As a by-product, we relate the frontal codimension of a plane curve with its $\mathscr{A}_e$-codimension and prove the frontal Mond conjecture (stated in [**22**]) in dimension 1, which says that the frontal codimension is less than or equal to the frontal Milnor number (the number of spheres in a stable deformation) with equality if the germ is quasi-homogeneous. We also give a method to construct stable unfoldings which are not necessarily versal. We then turn our attention to characterizing stability of frontal multigerms defining a frontal Kodaira–Spencer map which also yields a tangent space to the iso-singular locus (the manifold along which the frontal is trivial). Finally, we use our methods to obtain a complete list of stable 3-dimensional frontals in $\mathbb{C}^4$. Note that generic wave fronts were classified by Arnol'd in [**1**] and, on the other hand, Ishikawa classified stable Legendrian maps (which may have different projected frontals), but, until now, a complete classification of stable frontals was only known for $n = 1$ [**1**] and $n = 2$ [**23**].

For technical reasons in order to use Ishikawa's results, we restrict ourselves to the case of frontals whose Legendrian lift has corank 1.

2. Frontal map germs

Let W be a smooth manifold of dimension $2n+1$. A field of hyperplanes Δ over W is a contact structure for W if, for all $w \in W$, there exist an open neighbourhood $U \subseteq W$ of w and a $\sigma \in \Omega^1(U)$ such that

(1) $\operatorname{rk}\sigma_w = 1$;

(2) the fibre Δ_w of Δ at w is $\ker \sigma_w$;

(3) $(\sigma \wedge \mathrm{d}\sigma \wedge \overset{(n)}{\cdots} \wedge \mathrm{d}\sigma)_w \neq 0$.

We call σ the local contact form of W, and define a **contact manifold** as a pair (W, Δ), where Δ is a contact structure on W. Given a smooth manifold Z of dimension $n+1$, a locally trivial fibration $\pi\colon W \to Z$ is a **Legendrian fibration** for (W, Δ) if, for all $w \in W$,

$$(\mathrm{d}\pi_w)^{-1}(T_{\pi(w)}Z) \subseteq \ker \sigma_w.$$

Example 2.1. Let $W = PT^*Z$ be the projectivized cotangent bundle of a smooth manifold Z, and $(z, [\omega]) \in W$. We set for $i = 1, \dots, n+1$ the open subset $U_i = \{(z, [\omega] \in PT^*Z : \omega_i \neq 0)\}$, and define over U_i the differential 1-form

$$\alpha = \frac{\omega_1}{\omega_i}\,\mathrm{d}z^1 + \cdots + \mathrm{d}z^i + \cdots + \frac{\omega_{n+1}}{\omega_i}\,\mathrm{d}z^{n+1}.$$

The field of hyperplanes Δ given by $\Delta_w = \ker \alpha_w$ defines a contact structure over W, under which the canonical projection $W \to Z$ is a Legendrian fibration.

Definition 2.2. *Let $\pi\colon W \to Z$, $\pi'\colon W' \to Z'$ be Legendrian fibrations. A diffeomorphism $\Psi\colon W \to W'$ between contact manifolds is*

(1) *a **contactomorphism**, if $\Delta' = \mathrm{d}\Psi(\Delta)$;*

(2) *a **Legendrian diffeomorphism** if it is a contactomorphism and there exists a diffeomorphism $\psi\colon Z \to Z'$ such that $\psi \circ \pi = \pi' \circ \Psi$.*

We say W is contactomorphic to W' if there is a contactomorphism $\Psi\colon W \to W'$.

A well-known result by Darboux states that any two contact manifolds W, W' of the same dimension admit a local diffeomorphism $\Psi\colon W \to W'$ such that $\Delta' = \mathrm{d}\Psi(\Delta)$ (see e.g. [**27**], § 20.1). In particular, if $\dim W = 2n+1$, W is locally contactomorphic to the contact manifold described in example 2.1; therefore, we can restrict ourselves to the setting given in example 2.1.

Let $N \subseteq \mathbb{K}^{n+1}$ be an open subset. A mapping $F\colon N \to PT^*\mathbb{K}^{n+1}$ is **integral** if, for all $x \in N$, $\mathrm{d}F_x(T_xN) \subseteq \Delta_{F(x)}$.

Definition 2.3. *A smooth mapping $f\colon N^n \to Z^{n+1}$ is **frontal** if there exist an integral mapping $F\colon N \to PT^*Z$ and a Legendrian fibration $\pi\colon PT^*Z \to Z$ such that $f = \pi \circ F$. If F is an immersion, we say f is a **wave front**. Similarly, the image $X = f(N)$ of a frontal mapping $f\colon N \to Z$ (resp. a wave front) is also called a frontal (resp. wave front) in Z.*

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DEFINITION 2.4. *Let $S \subset N$ be a finite set. A smooth multigerm $f\colon (N, S) \to (Z, 0)$ is* ***frontal*** *(resp. wave front) if it has a frontal (resp. wave front) representative $f\colon N \to Z$. Similarly, a set germ $(X, 0)$ with $X \subset Z$ is* ***frontal*** *(resp. wave front) if it has a frontal (resp. wave front) representative.*

Let $F\colon N \to PT^*\mathbb{K}^{n+1}$ be an integral map and $f = \pi \circ F$: there exist $\nu_1, \ldots, \nu_{n+1} \in \mathscr{O}_n$ such that

$$0 = F^*\alpha = \sum_{i=1}^{n+1} \nu_1 \, \mathrm{d}(Z_i \circ f) = \sum_{i=1}^{n+1} \sum_{j=1}^{n} \nu_i \frac{\partial f_i}{\partial x_j} \, \mathrm{d}x^j, \tag{2.1}$$

where $Z_1, \ldots, Z_{n+1}$ are coordinates for $\mathbb{K}^{n+1}$. Setting $\nu = \nu_1 \, \mathrm{d}Z_1 + \cdots + \nu_{n+1} \, \mathrm{d}Z_{n+1}$, this is equivalent to $\nu(\mathrm{d}f \circ \xi) = 0$ for all $\xi \in \theta_n$. Since $PT^*\mathbb{K}^{n+1}$ is a locally trivial fibration, we can find for each pair $(z, [\omega]) \in PT^*\mathbb{K}^{n+1}$ an open neighbourhood $Z \subset \mathbb{K}^{n+1}$ of z and an open $U \subseteq \mathbb{K}P^{n+1}$ such that $\pi^{-1}(Z) \cong Z \times U$. Therefore, F is contact equivalent to the mapping $\tilde{f}(x) = (f(x), [\nu_x])$, known as the **Nash lift** of f.

If we assume that $\Sigma(f)$ is nowhere dense in N, the differential form ν is uniquely determined by f, giving us a one-to-one correspondence between f and $\tilde{f}$. Such a frontal map is known as a **proper frontal** map (according to Ishikawa [**15**]). We also define the **integral corank** of a proper frontal as the corank of its Nash lift.

For the rest of this article, we shall assume all frontal map germs are proper. Note that the notion of topological properness (i.e. the preimage of a compact subset is compact) is not used throughout this article.

EXAMPLE 2.5. Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ be the smooth map germ given by

$$f(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2, 2x_1^{p_1} + \cdots + 2x_n^{p_n}); \quad p_1, \ldots, p_n > 1$$

It is easy to see that f has corank n and the singular set $\Sigma(f)$ is nowhere dense in $\mathbb{K}^n$. Furthermore, the assumption that $p_1, \ldots, p_n > 1$ implies that the Jacobian ideal of f is generated by $x_1 x_2 \ldots x_n$, and thus it is a proper frontal map germ by proposition 2.6 below. In particular, the differential 1-form

$$\nu_{(x_1, \ldots, x_n)} = p_1 x_1^{p_1 - 2} \, \mathrm{d}X^1 + \cdots + p_n x_n^{p_n - 2} \, \mathrm{d}X^n - \mathrm{d}X^{n+1},$$

verifies that $\nu(\mathrm{d}f \circ \xi) = 0$ for all $\xi \in \theta_n$, and has corank equal to the number of p_i that are greater than 3. Therefore, the integral corank of f is also equal to the number of p_i greater than 3. In particular, f is a wave front when all p_i are equal to 3.

PROPOSITION 2.6 [**15**], lemma 2.3. *Let $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ be a map germ. If f is frontal, then the Jacobian ideal J_f of f is principal (i.e. it is generated by a single element). Conversely, if J_f is principal and $\Sigma(f)$ is nowhere dense in $(\mathbb{K}^n, S)$, then f is a proper frontal map germ.*

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If f has corank 1, we may choose local coordinates in the source and target such that

$$f(x,y)=(x,p(x,y),q(x,y));\quad x\in\mathbb{K}^{n-1},\ y\in\mathbb{K} \tag{2.2}$$

in which case J_f is the ideal generated by p_y and q_y, and we recover the following criterion by Nuño-Ballesteros [**23**]:

Corollary 2.7. *Let $f\colon(\mathbb{K}^n,S)\to(\mathbb{K}^{n+1},0)$ be a frontal map germ of corank 1, and choose coordinates in the source and target such that f is given as in equation (2.2). Then f is a frontal map germ if and only if either $p_y|q_y$ or $q_y|p_y$.*

We shall say that f is in **prenormal form** if it is given as in equation (2.2) with $q_y=\mu p_y$ for some $\mu\in\mathscr{O}_n$, in which case the Nash lift becomes

$$\tilde f=\left(f,\frac{\partial q}{\partial x_1}-\mu\frac{\partial p}{\partial x_1},\dots,\frac{\partial q}{\partial x_{n-1}}-\mu\frac{\partial p}{\partial x_{n-1}},\mu\right) \tag{2.3}$$

In particular, note that if $\operatorname{ord}_y(q)=\operatorname{ord}_y(p)+1$, then $\operatorname{ord}_y(\mu)=1$, and f is a wave front.

3. Lowering Legendrian equivalence

The first strides in the classification of frontal mappings were done by Arnol'd and his colleagues in a series of articles published in the 1970s and 1980s. In his work, he established a notion of equivalence native to Legendrian maps (known as *Legendrian equivalence*) and developed a classification of all simple, stable wave fronts (see [**27**], chapter 21).

Ishikawa extended Arnol'd's theory of Legendrian equivalence to the broader class of integral mappings in [**14**], defining a notion of infinitesimal stability and showing that an integral map of corank at most 1 is Legendrian stable if and only if it is infinitesimally stable. He also showed that all Legendrian stable integral mappings of corank at most 1 belong to a special family called open Whitney umbrellas, giving a characterization of stable umbrellas in terms of a certain $\mathbb{K}$-algebra Q.

The goal of this section is to formulate a notion of frontal stability and versality that does not require the use of contact geometry.

Remark 3.1. Let $f\colon N\subseteq\mathbb{K}^n\to\mathbb{K}^{n+1}$ be a proper frontal map with Nash lift $\tilde f=f\times[\nu]$, where $[\nu]\colon N\to P(\mathbb{K}^{n+1*})$ maps points in N to projective differential 1-forms $[\nu_x]$. There exists a $1\leqslant i\leqslant n+1$ such that ν_i is non-vanishing, so we can rewrite equation (2.1) as

$$d(Z_i\circ f)=-\frac{\nu_1}{\nu_i}\,d(Z_1\circ f)-\cdots-d\widehat{(Z_i\circ f)}-\cdots-\frac{\nu_{n+1}}{\nu_i}\,d(Z_{n+1}\circ f), \tag{3.1}$$

where the hat symbol denotes an ommited summand. We then define local coordinates X,Y,P on $PT^*\mathbb{K}^{n+1}$ such that $f_i=Y\circ f$ and

$$f_j=X_j\circ f,\quad P_j=\frac{\nu_j}{\nu_i}\quad(j=1,\dots,i-1);$$

$$f_{j+1}=X_j\circ f,\quad P_j=\frac{\nu_{j+1}}{\nu_i}\quad(j=i,\dots,n+1).$$

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These are known as the **Darboux coordinates** of $PT^*\mathbb{K}^{n+1}$. In particular, equation (3.1) implies that the mapping $X \circ f = (X_1 \circ f, \ldots, X_n \circ f)$ shares the same singular set with f. Therefore, $X \circ f\colon N \to \mathbb{K}^n$ is immersive outside of a nowhere dense subset K of U.

Definition 3.2. *Let $S, S' \subset \mathbb{K}^n$ be finite sets. Two integral map germs*

$$F\colon (\mathbb{K}^n, S) \to (PT^*\mathbb{K}^{n+1}, w), \quad F'\colon (\mathbb{K}^n, S') \to (PT^*\mathbb{K}^{n+1}, w')$$

*are **Legendre equivalent** if there exists a diffeomorphism $\phi\colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, S')$ and a Legendrian diffeomorphism $\Psi\colon (PT^*\mathbb{K}^{n+1}, w) \to (PT^*\mathbb{K}^{n+1}, w')$ such that $F' = \Psi \circ F \circ \phi^{-1}$.*

Arnol'd showed in [**27**], § 20.4 that a Legendrian diffeomorphism $\Psi\colon W \to W'$ is locally determined by a choice of Legendrian fibrations in the source and target, and a diffeomorphism ψ between the base spaces. Nonetheless, his proof was based on the fact that a Legendrian diffeomorphism preserves the fibres, and no explicit expression is given for Ψ.

Theorem 3.3. *Given a diffeomorphism $\psi\colon Z \to Z'$, the mapping*

$$\begin{aligned} T^*Z &\longrightarrow T^*Z' \\ (z, \omega) &\longmapsto (\psi(z), \omega \circ d\psi^{-1}_{\psi(z)}) \end{aligned}$$

induces a Legendrian diffeomorphism $\Psi\colon (PT^*Z, \Delta) \to (PT^*Z', \Delta')$.

Proof. Let $(z, \omega) \in T^*Z$: since ψ is a diffeomorphism, $\omega \circ d\psi^{-1}_{\psi(z)} \neq 0$ and Ψ is a well-defined diffeomorphism. Furthermore, it is clear that

$$\pi' \circ \Psi = \psi \circ \pi \tag{3.2}$$

by construction. Therefore, we only need to show that $\mathrm{d}\Psi_q(\Delta_q) = \Delta'_{\Psi(q)}$.

Let $q = (z, [\omega])$ and $v \in \Delta_q$. Since π is a submersion, $(\omega \circ \mathrm{d}\pi_q)(v) = 0$, and it follows from (3.2) that

$$(\omega \circ \mathrm{d}\psi^{-1}_{\psi(z)} \circ \mathrm{d}\pi'_{\Psi(q)})[\mathrm{d}\Psi_q(v)] = 0 \implies \mathrm{d}\Psi_q(v) \in \Delta'_{\Psi(q)}$$

Conversely, let $w \in \Delta'_{\Psi(q)}$. Since Ψ is a diffeomorphism, there exists a unique $v \in T_qPT^*Z$ such that $w = d\Psi_q(v)$. By definition of Δ', we have

$$(\omega \circ \mathrm{d}\psi^{-1}_{\psi(z)} \circ \mathrm{d}\pi'_{\Psi(q)})(w) = 0$$

By (3.2), this implies that $(\omega \circ \mathrm{d}\pi_q)(v) = 0$, from which follows that $w \in \mathrm{d}\Psi_q(\Delta_q)$. □

Remark 3.4. Let $\psi_t\colon (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{n+1}, 0)$ be a smooth 1-parameter family of diffeomorphisms. Given t in an open neighbourhood $U \subseteq \mathbb{K}$ of 0, we know by theorem 3.3 that we can lift ψ_t onto a Legendrian diffeomorphism $\Psi_t\colon (PT^*\mathbb{K}^{n+1}, w) \to (PT^*\mathbb{K}^{n+1}, 0)$. Since $\pi\colon PT^*\mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ is a fibre bundle and $\mathbb{K}^{n+1}$ is a paracompact Hausdorff space, π is a fibration (see [**26**], corollary 2.7.14), so it verifies

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the homotopy lifting property. Therefore, the 1-parameter family Ψ_t defined in this way is, indeed, a lift of the family ψ_t.

Corollary 3.5. *Let* $f, g\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$*:*

(1) *if* f *is* $\mathscr{A}$*-equivalent to* g *and* f *is frontal,* g *is frontal;*

(2) *if* f *and* g *are frontal,* $\tilde{f}$ *is Legendrian equivalent to* $\tilde{g}$ *if and only if* f *is* $\mathscr{A}$*-equivalent to* g*.*

Proof. Assume that f is frontal: there exists an integral map germ $F\colon (\mathbb{K}^n, S) \to PT^*\mathbb{K}^{n+1}$ such that $f = \pi \circ F$, where π is the canonical bundle projection. Now let $\phi\colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$, $\psi\colon (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{n+1}, 0)$ be diffeomorphisms such that $g = \psi \circ f \circ \phi^{-1}$: by theorem 3.3, we can lift ψ onto a Legendrian diffeomorphism $\Psi\colon PT^*\mathbb{K}^{n+1} \to PT^*\mathbb{K}^{n+1}$. Therefore, the map $G = \Psi \circ F \circ \phi^{-1}$ is an integral map such that $\pi \circ G = g$, and g is frontal. This proves the first item.

For the second item, the 'only if' is proved in a similar fashion. For the 'if', let $\phi\colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$ and $\Psi\colon (PT^*\mathbb{K}^{n+1}, w) \to (PT^*\mathbb{K}^{n+1}, w)$ be diffeomorphisms such that $\tilde{g} = \Psi \circ \tilde{f} \circ \phi^{-1}$, with Ψ Legendrian. By definition of Legendrian diffeomorphism, there exists a diffeomorphism $\psi\colon (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{n+1}, 0)$ such that $\pi \circ \Psi = \psi \circ \pi$, from which follows that

$$g = \pi \circ \tilde{g} = \pi \circ \Psi \circ \tilde{f} \circ \phi^{-1} = \psi \circ \pi \circ \tilde{f} \circ \phi^{-1} = \psi \circ f \circ \phi^{-1},$$

proving the second item. □

3.1. Unfolding frontal map germs

The theory of Legendrian equivalence describes homotopic deformations of a pair (π, F) via integral deformations, deformations (F_u) of F which are themselves integral for any fixed u. Nonetheless, frontal deformations often fail to preserve the frontal nature across the parameter space, as showcased in example 3.6 below.

Example 3.6. Let $\gamma\colon (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be the plane curve $t \mapsto (t^3, t^4)$. The 1-parameter deformation $\gamma_s(t) = (t^3 + st, t^4)$ verifies that γ_s is frontal for all $s \in \mathbb{K}$. If ω is a 1-form such that $\omega(\mathrm{d}\gamma_s \circ \partial t) = 0$ for all (t, s) in an open neighbourhood $U \subset \mathbb{K}^2$ of $(0, 0)$, a simple computation shows that ω must be given in the form

$$\omega_{(s,t)} = \alpha(t, s)(4t^3\,\mathrm{d}X - (3t^2 + s)\,\mathrm{d}Y)$$

for some $\alpha \in \mathscr{O}_2$. Therefore, $\tilde{\gamma}_s$ does not yield an integral deformation of $\tilde{\gamma}$ at $s = 0$.

Definition 3.7. *Let* $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ *be a frontal germ. An unfolding* $F\colon (\mathbb{K}^n \times \mathbb{K}^d, S \times \{0\}) \to (\mathbb{K}^{n+1} \times \mathbb{K}^d, 0)$ *of* f *is* ***frontal*** *if it is frontal as a map germ.*

Theorem 3.8. *Let* $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ *be a proper frontal map germ. A* d*-parameter unfolding* $F = (f_\lambda, \lambda)$ *of* f *is frontal if and only if* $\tilde{f}_\lambda$ *is an integral deformation of* $\tilde{f}$*.*

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Proof. Let F be a frontal d-parameter unfolding for f: there is a $\nu \in \Omega^1(F)$ such that $\nu(\mathrm{d}F \circ \eta) = 0$ for all $\eta \in \theta_{n+d}$. If we set $\nu_0 = \nu|_{\lambda=0}$, we can write

$$\nu_{(x,y,\lambda)} = (\nu_0)_{(x,y)} + \sum_{j=1}^{d} \lambda_j (\nu_j)_{(x,y,\lambda)}$$

for some $\nu_1, \ldots, \nu_j \in (\mathbb{K}^n, S) \to T^*\mathbb{K}^{n+1}$. Therefore, ν may be regarded as a d-parameter deformation of ν_0 and the Nash lift of f_λ,

$$(x, y, \lambda) \mapsto (f_\lambda(x, y), [\nu_{(x,y,\lambda)}]) \tag{3.3}$$

is an integral d-parameter deformation of $f \times [\nu_0]$. Since $f \times [\nu_0]$ is an integral map, $\nu_0(\mathrm{d}f \circ \xi) = 0$ for all $\xi \in \theta_n$. Properness of f then implies that $f \times [\nu_0] = \tilde{f}$, and thus the map germ (3.3) is an integral deformation of $\tilde{f}$.

Conversely, let $\tilde{f}_\lambda$ be an integral deformation of $\tilde{f}$. Taking coordinates (u, λ) in the source and Darboux coordinates in the target, the integrability condition becomes

$$\frac{\partial}{\partial u_j}(Y \circ f_\lambda) = (P_1 \circ \tilde{f}_\lambda)\frac{\partial}{\partial u_j}(X_1 \circ f_\lambda) + \cdots + (P_n \circ \tilde{f}_\lambda)\frac{\partial}{\partial u_j}(X_n \circ f_\lambda)$$

for $j = 1, \ldots, n$. Consider the differential form $\nu \in \Omega^1(F)$ given by

$$\sum_{j=1}^{n}(P_j \circ \tilde{f}_\lambda)\left(\mathrm{d}X^j - \sum_{k=1}^{d}\frac{\partial}{\partial \lambda_k}(X_j \circ f_\lambda)\,\mathrm{d}\lambda^k\right) - \mathrm{d}Y + \sum_{k=1}^{d}\frac{\partial}{\partial \lambda_k}(Y \circ f_\lambda)\,\mathrm{d}\lambda^k$$

Using the integrability condition above, we have

$$\begin{aligned}
\nu\left(\mathrm{d}F \circ \frac{\partial}{\partial u_i}\right) &= \sum_{j=1}^{n}(P_j \circ \tilde{f}_\lambda)\frac{\partial(X_j \circ f_\lambda)}{\partial u_i} - \frac{\partial(Y \circ f_\lambda)}{\partial u_i} = 0;\\
\nu\left(\mathrm{d}F \circ \frac{\partial}{\partial \lambda_i}\right) &= \sum_{j=1}^{n}(P_j \circ \tilde{f}_\lambda)\left(\frac{\partial(X_j \circ \tilde{f}_\lambda)}{\partial \lambda_i} - \frac{\partial(X_j \circ \tilde{f}_\lambda)}{\partial \lambda_i}\right)\\
&\qquad - \frac{\partial(Y \circ f_\lambda)}{\partial \lambda_i} + \frac{(Y \circ f_\lambda)}{\partial \lambda_i} = 0.
\end{aligned}$$

Therefore, $\nu(\mathrm{d}F \circ \xi) = 0$ for all $\xi \in \theta_{n+d}$ and F is frontal. □

REMARK 3.9. Properness of f is required for the 'if' direction, since $\widetilde{f_u}$ is not guaranteed to be a deformation of $\tilde{f}$, even if it is integral. Nonetheless, the 'only if' direction does not require properness.

The space of **infinitesimal integral deformations** of an integral $\tilde{f}$, defined by Ishikawa in [14], is given by

$$\theta_I(\tilde{f}) = \{v_0(\tilde{f}_t) : \tilde{f}_t \text{ integral }, \tilde{f}_0 = \tilde{f}\}; \quad v_0(\tilde{f}_t) = \left.\frac{\mathrm{d}\tilde{f}_t}{\mathrm{d}t}\right|_{t=0}.$$

This space is linear when $\tilde{f}$ has corank at most 1 [14], but it is known to have a conical structure in higher coranks. Counterexamples can be constructed using a

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similar procedure as in [**11**]. We also set $T\mathscr{L}_e\tilde{f}$ as the space of $\xi \in \theta_I(\tilde{f})$ given by $\xi = v_0(\Psi_t \circ \tilde{f} \circ \phi_t^{-1})$ for some 1-parameter families of diffeomorphisms $\phi\colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$ and $\Psi_t\colon (PT^*\mathbb{K}^{n+1}, w_0) \to (PT^*\mathbb{K}^{n+1}, w_0)$, Ψ_t Legendrian.

Definition 3.10. *Let $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ be a frontal map germ of integral corank at most* 1*. We define the space of* ***infinitesimal frontal deformations*** *of f as*

$$\mathscr{F}(f) = \{v_0(f_t) : (t, f_t) \text{ frontal}, f_0 = f\}.$$

As shown in theorem 3.12 below, $\mathscr{F}(f)$ is the linear projection of $\theta_I(\tilde{f})$. Therefore, if the integral corank of f is at most 1, $\mathscr{F}(f)$ is $\mathbb{K}$-linear; for this reason, any results involving $\mathscr{F}(f)$ will implicitly assume that f has integral corank at most 1. An alternative, direct proof is also given for corank 1 frontal map germs in remark 5.15 below.

Lemma 3.11. *Given a frontal map germ $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$, $T\mathscr{A}_e f \subseteq \mathscr{F}(f)$.*

Proof. Let $\phi_t\colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$, $\psi_t\colon (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{n+1}, 0)$ be two smooth 1-parameter families of diffeomorphisms and $f_t = \psi_t \circ f \circ \phi_t^{-1}$. It is clear by construction that the vector field germ given by f_t is in $T\mathscr{A}_e f$. By theorem 3.3, we can lift ψ_t onto a smooth 1-parameter family Ψ_t of Legendrian diffeomorphisms, in which case we can lift f_t onto an integral deformation $\widetilde{f_t} = \Psi_t \circ \tilde{f} \circ \phi_t^{-1}$. Using theorem 3.8, we then see that the unfolding $F = (f_t, t)$ is frontal. Therefore, the vector field germ given by f_t is in $\mathscr{F}(f)$, and thus $T\mathscr{A}_e f \subseteq \mathscr{F}(f)$. □

Theorem 3.12. *Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ be a proper frontal map germ and $\pi\colon PT^*\mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ be the canonical bundle projection. The mapping $t\pi\colon \theta_I(\tilde{f}) \to \mathscr{F}(f)$ given by $t\pi(\xi) = \mathrm{d}\pi \circ \xi$ is a $\mathbb{K}$-linear isomorphism and induces an isomorphism*

$$\Pi\colon \frac{\mathscr{F}(f)}{T\mathscr{A}_e f} \longrightarrow \frac{\theta_I(\tilde{f})}{T\mathscr{L}_e\tilde{f}}. \tag{3.4}$$

Proof. Let $\xi \in \theta_I(\tilde{f})$ and $\tilde{f}_t$ be an integral 1-parameter deformation of $\tilde{f}$ and $\xi = v_0(\tilde{f}_t)$: by theorem 3.8, $F(t,x) = (t, (\pi \circ \tilde{f}_t)(x))$ is a frontal 1-parameter unfolding of f. Furthermore, using the chain rule, we see that $v_0(\pi \circ \tilde{f}_t) = t\pi[v_0(\tilde{f}_t)]$, so $t\pi[\theta_I(\tilde{f})] \subseteq \mathscr{F}(f)$ and $t\pi\colon \theta_I(\tilde{f}) \to \mathscr{F}(f)$ is well defined. Conversely, let $\xi \in \mathscr{F}(f)$ and (t, f_t) be a frontal 1-parameter deformation of f with $\xi = v_0(f_t)$: by theorem 3.8, we can lift f_t onto an integral 1-parameter deformation $\tilde{f}_t$ of $\tilde{f}$. Using the chain rule, it then follows that $\xi \in t\pi[\theta_I(\tilde{f})]$, so $t\pi[\theta_I(\tilde{f})] = \mathscr{F}(f)$.

We move onto injectivity of $t\pi$. Let $\tilde{f}_t(x) = \tilde{f}(x) + t\tilde{h}(x,t)$ be an integral 1-parameter deformation of $\tilde{f}$ with $(\pi \circ \tilde{f}_t)(x) = f(x) + th(x,t)$. If we assume that $\xi = v_0(\tilde{f}_t) \in \ker t\pi$, then

$$0 = \left.\frac{\mathrm{d}f_t}{\mathrm{d}t}\right|_{t=0} = [h(x,t) + th_t(x,t)]_{t=0} = h(x,0) \implies h(x,t) = tg(x,t).$$

Our goal is to show that we can write $\tilde{h}(x,t) = t\tilde{g}(x,t)$ for some $\tilde{g}$, so that $v_0(\tilde{f}_t) = 0$ and thus $\ker t\pi = \{0\}$.

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Since $\tilde{f}_t$ is an integral deformation of $\tilde{f}$, it verifies the identity

$$\mathrm{d}(Y \circ f_t) = \sum_{j=1}^{n} (P_j \circ \tilde{f}_t)\, \mathrm{d}(X_j \circ f_t).$$

Taking the coefficient of $\mathrm{d}x^k$ on both sides of the equation and simplifying yields

$$t\frac{\partial(Y \circ g)}{\partial x_k} = \sum_{j=1}^{n} \left[(P_j \circ \tilde{h}) \frac{\partial(X_j \circ f_t)}{\partial x_k} + t(P_j \circ \tilde{f}) \frac{\partial(X_j \circ g)}{\partial x_k} \right].$$

Taking $t = 0$ gives us the homogeneous system of equations

$$0 = \sum_{j=1}^{n} \frac{\partial(X_i \circ f)}{\partial x_k}(x)(P_j \circ \tilde{h})(x, 0)$$

for $k = 1, \ldots, n$. Using the observation from remark 3.1 and the continuity of $P_1 \circ \tilde{h}, \ldots, P_n \circ \tilde{h}$, we conclude that $(P_1 \circ \tilde{h})(x, 0) = \cdots = (P_n \circ \tilde{h})(x, 0) = 0$ and thus $\tilde{h}(x, t) = t\tilde{g}(x, t)$.

It only remains to show that $t\pi(T\mathscr{L}_e\tilde{f}) = T\mathscr{A}_e f$. Let $\xi \in T\mathscr{L}_e\tilde{f}$: there exist 1-parameter families $\phi_t \colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$, $\Psi_t \colon (PT^*\mathbb{K}^{n+1}, w) \to (PT^*\mathbb{K}^{n+1}, w)$ of diffeomorphisms such that $\xi = v_0(\Psi_t \circ \tilde{f} \circ \phi_t^{-1})$, with Ψ_t Legendrian. Since Ψ_t is Legendrian for all t in a neighbourhood $U \subseteq \mathbb{K}$ of 0, there exists a 1-parameter family $\psi_t \colon (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{n+1}, 0)$ of diffeomorphisms such that $\pi \circ \Psi_t = \psi_t \circ \pi$ for all $t \in U$. We then have that $v_0(\psi_t \circ f \circ \phi_t^{-1}) = t\pi[v_0(\Psi_t \circ \tilde{f} \circ \phi_t^{-1})] = t\pi(\xi)$, hence $t\pi(\xi) \in T\mathscr{A}_e f$.

Conversely, if $\xi \in T\mathscr{A}_e f$, there exist 1-parameter families $\phi_t \colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$, $\psi_t \colon (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{n+1}, 0)$ of diffeomorphisms such that $\xi = v_0(\psi_t \circ f \circ \phi_t^{-1})$. Using theorem 3.3, there exists a 1-parameter family of Legendrian diffeomorphisms $\Psi_t \colon (PT^*\mathbb{K}^{n+1}, w) \to (PT^*\mathbb{K}^{n+1}, w)$ such that $\pi \circ \Psi_t = \psi_t \circ \pi$, and thus we can lift ξ onto $v_0(\Psi_t \circ \tilde{f} \circ \phi_t^{-1}) \in T\mathscr{L}_e\tilde{f}$, whose image via $t\pi$ is ξ. □

Remark 3.13. Let $f \colon (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ be a frontal map germ: theorem 3.12 states that $\mathscr{F}(f) = t\pi[\theta_I(\tilde{f})]$. Since $\tilde{f}$ has corank 1, a resut by Ishikawa [**14**] states that

$$\theta_I(\tilde{f}) = \{\xi \in \theta(\tilde{f}) : \xi^*\tilde{\alpha} = 0\},$$

wherein $\tilde{\alpha}$ denotes the natural lifting of the contact form in $PT^*\mathbb{K}^{n+1}$. Taking Darboux coordinates in $PT^*\mathbb{K}^{n+1}$,

$$\xi \in \mathscr{F}(f) \iff \mathrm{d}\xi_{n+1} - \sum_{i=1}^{n} (P_i \circ \tilde{f})\, \mathrm{d}\xi_i \in \mathscr{O}_n\, \mathrm{d}(f^*\mathscr{O}_n) \tag{3.5}$$

In particular, if f has corank 1 and it is given in prenormal form, equation (3.5) is equivalent to

$$\frac{\partial \xi_{n+1}}{\partial y} - \sum_{j=1}^{n-1} P_j \frac{\partial \xi_j}{\partial y} + \mu \frac{\partial \xi_n}{\partial y} \in \mathscr{O}_n\{p_y\},$$

where $P_1, \ldots, P_{n-1}$ are given as in equation (2.3).

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Definition 3.14. *The* ***frontal codimension*** *of f is defined as the dimension of $T^1_{\mathscr{F}_e} f = \mathscr{F}(f)/T\mathscr{A}_e f$. We say f is* ***$\mathscr{F}$-finite*** *or has* ***finite frontal codimension*** *if $\dim T^1_{\mathscr{F}_e} f < \infty$.*

3.2. Frontal versality and stability

In the previous subsection, we formulated the notions of integral deformation and Legendrian codimension purely in terms of frontal unfoldings. We now show that Ishikawa's results concerning the Legendrian stability and versality of pairs from [**14**] have a direct parallel in our theory of frontal deformations.

Definition 3.15. *A frontal map germ $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ is* ***stable as a frontal*** *or* ***$\mathscr{F}$-stable*** *if every frontal unfolding of f is $\mathscr{A}$-trivial.*

Corollary 3.16. *A frontal map germ $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ is stable as a frontal if and only if $\tilde{f}$ is Legendrian stable.*

Proof. Assume f is stable as a frontal and let $\tilde{f}_u$ be an integral deformation of $\tilde{f}$: by theorem 3.8, $\tilde{f}_u$ defines a frontal unfolding $F = (f_u, u)$ of f. Stability of f then implies that f_u is $\mathscr{A}$-equivalent to f. By corollary 3.5, this then implies that $\tilde{f}_u$ is Legendrian equivalent to $\tilde{f}$. Since the choice of $\tilde{f}_u$ was arbitrary, we conclude $\tilde{f}$ is Legendrian stable. The opposite direction is shown similarly. □

Corollary 3.17. *A frontal map germ $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ is $\mathscr{F}$-stable if and only if its $\mathscr{F}_e$-codimension is* 0*.*

Proof. Corollary 3.16 states that f is $\mathscr{F}$-stable if and only if its Nash lift $\tilde{f}$ is Legendrian stable. Since f has corank at most 1, so does $\tilde{f}$, and a result by Ishikawa [**14**] states that $\tilde{f}$ is Legendrian stable for the bundle projection π if and only if $\theta_I(\tilde{f}) = T\mathscr{L}_e\tilde{f}$. However, it follows from theorem 3.12 that this is equivalent to $\mathscr{F}(f) = T\mathscr{A}_e f$. □

Example 3.18. The following frontal hypersurfaces are stable as frontals:

(1) Cusp: $X^2 - Y^3 = 0$

(2) Folded Whitney umbrella: $Z^2 - X^2Y^3 = 0$, with $Y \geqslant 0$ in the real case.

Let $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ be a frontal map germ with d-parameter unfolding $F = (f_u, u)$, not necessarily frontal. Recall that the **pullback** of F by $h\colon (\mathbb{K}^l, 0) \to (\mathbb{K}^d, 0)$ is defined as the l-paramter unfolding

$$(h^*F)(x, v) = (f_{h(v)}(x), v).$$

Definition 3.19. *Let $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ be a frontal map germ. A frontal d-parameter unfolding F of f is* ***$\mathscr{F}$-versal*** *or* ***versal as a frontal*** *if, given any other frontal d-parameter unfolding G of f, there exists a diffeomorphism $h\colon (\mathbb{K}^d, 0) \to (\mathbb{K}^d, 0)$ such that G is equivalent to h^*F as unfoldings.*

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LEMMA 3.20. *Given a frontal map germ* $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$*, a frontal unfolding* $F = (f_u, u)$ *is* $\mathscr{F}$*-versal if and only if* $\tilde{f}_u$ *is a Legendre versal deformation of* $\tilde{f}$*.*

Proof. Assume F is a versal frontal unfolding of f and let $(\widetilde{g_u})$ be an s-parameter integral deformation of $\tilde{f}$. Theorem 3.8 implies that the s-parameter unfolding $G = (u, g_u)$ is frontal. By versality of F, there exist unfoldings $\mathcal{T}\colon (\mathbb{K}^{n+1} \times \mathbb{K}^d, 0) \to (\mathbb{K}^{n+1} \times \mathbb{K}^d, 0)$, $\mathcal{S}\colon (\mathbb{K}^n \times \mathbb{K}^d, S \times \{0\}) \to (\mathbb{K}^n \times \mathbb{K}^d, S \times \{0\})$ of the identity map germ and a smooth map germ $h\colon (\mathbb{K}^s, 0) \to (\mathbb{K}^d, 0)$ such that $G = \mathcal{T} \circ h^*F \circ \mathcal{S}^{-1}$.

Let $f\colon N \to Z$ be a representative of f which is a proper frontal map, and $F\colon \mathcal{N} \to \mathcal{Z}$ be a representative of F such that $\mathcal{N} \subseteq N \times \mathbb{K}^d$. A simple computation shows that $\Sigma(F) = \Sigma(f) \times \{0\}$; therefore, since $\Sigma(f)$ is nowhere dense in N, $\Sigma(F)$ is nowhere dense in $\mathcal{N}$ and F is a proper frontal map. Theorem 3.8 then states that f_u lifts into integral deformation of $\tilde{f}$. Now consider representatives $h^*F = (u, f_{h(u)})\colon \mathcal{N}_1 \to \mathcal{Z}_1$, $\mathcal{S} = (u, \sigma_u)\colon \mathcal{N}_1 \to \mathcal{N}_2$, $\mathcal{T} = (u, \tau_u)\colon \mathcal{Z}_1 \to \mathcal{Z}_2$ and $G\colon \mathcal{N}_2 \to \mathcal{Z}_2$ such that $G = \mathcal{T} \circ h^*F \circ \mathcal{S}^{-1}$ as mappings. Since (τ_u) is a smooth d-parameter family of diffeomorphisms, we can lift it onto a d-parameter family of smooth Legendrian diffeomorphisms $T_u\colon PT^*\mathcal{Z}_1 \to PT^*\mathcal{Z}_2$. Therefore,

$$\widetilde{g_u} = T_u \circ \widetilde{f_{h(u)}} \circ \sigma_u^{-1}$$

and $\widetilde{f_u}$ is a versal Legendrian deformation of $\tilde{f}$.

Conversely, let $\tilde{f}_u$ be a versal integral deformation of $\tilde{f}$ and $G = (g_u, u)$ be a frontal s-parameter unfolding of f. Theorem 3.8 implies that the s-parameter deformation $\widetilde{g_u}$ is integral. By versality of $\tilde{f}_u$, there exist smooth families of diffeomorphisms $T_u\colon (PT^*\mathbb{K}^{n+1}, w) \to (\mathbb{K}^{n+1}, w)$ and $\sigma_u\colon (\mathbb{K}^n, S) \to (\mathbb{K}^n, S)$ and a smooth map germ $h\colon (\mathbb{K}^s, 0) \to (\mathbb{K}^d, 0)$ verifying the following:

(1) T_u is a Legendrian diffeomorphism for all u;

(2) T_0 and σ_0 are the identity map germs;

(3) $\widetilde{g_u} = T_u \circ \tilde{f}_{h(u)} \circ \sigma_u$.

By item 1, we can find a smooth family of diffeomorphisms $\tau_u\colon (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{n+1}, 0)$ such that $\pi \circ T_u = \tau_u \circ \pi$ and τ_0 is the identity map germ. It follows that

$$\widetilde{g_u} = T_u \circ \tilde{f}_{h(u)} \circ \sigma_u \iff g_u = \tau_u \circ f_{h(u)} \circ \sigma_u.$$

If we now consider the unfoldings $\mathcal{T} = (\tau_u, u)$ and $\mathcal{S} = (\sigma_u, u)$, we have $G = \mathcal{T} \circ h^*F \circ \mathcal{S}$. We conclude that F is versal as a frontal. □

THEOREM 3.21 Frontal versality theorem. *Given a frontal map germ* $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$,

(1) *f admits a frontal versal unfolding if and only if it is* $\mathscr{F}$*-finite;*

(2) *a frontal unfolding* $F(u, x) = (u, f_u(x))$ *of f is versal as a frontal if and only if*

$$\mathscr{F}(f) = T\mathscr{A}_e f + \mathrm{Sp}_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_d\}, \quad \dot{F}_j = \left.\frac{\partial f_u}{\partial u_j}\right|_{u=0}.$$

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To show theorem 3.21, we shall make use of

THEOREM 3.22 Ishikawa's Legendre versality theorem [14]. *Given an integral* $\tilde{f}\colon (\mathbb{K}^n, S) \to (PT^*\mathbb{K}^{n+1}, w)$ *of corank at most* 1,

(1) $\tilde{f}$ *admits a versal Legendrian unfolding if and only if its Legendrian codimension is finite;*

(2) *a Legendrian unfolding* $\tilde{f}_u$ *of* $\tilde{f}$ *is versal if and only if*

$$\theta_I(\tilde{f}) = T\mathscr{L}_e\tilde{f} + \mathrm{Sp}_{\mathbb{K}}\left\{ \left.\frac{\partial \tilde{f}_u}{\partial u_1}\right|_{u=0}, \ldots, \left.\frac{\partial \tilde{f}_u}{\partial u_d}\right|_{u=0} \right\}. \tag{3.6}$$

Proof of theorem 3.21*.* By lemma 3.20, a frontal unfolding $F = (f_u, u)$ of f is versal as a frontal if and only if the smooth family $\widetilde{f_u}$ is a versal Legendre deformation of $\tilde{f}$. In particular, it follows from theorem 3.8 that $\tilde{f}$ admits a versal Legendrian deformation if and only if f admits a versal frontal unfolding. This fact shall be used to prove both items.

By theorem 3.22, f admits a $\mathscr{F}$-versal unfolding if and only if $\tilde{f}$ has finite Legendre codimension. However, it was proved in theorem 3.12 that this is equivalent to f being $\mathscr{F}$-finite. This shows the first item.

We move onto the second item. If F is $\mathscr{F}$-versal, $\tilde{f}_u$ is a Legendre versal unfolding of $\tilde{f}$ by lemma 3.20 and equation (3.6) holds. Computing the image via $t\pi$ on both sides of equation (3.6) and using theorem 3.12, we get

$$\begin{aligned}\mathscr{F}(f) &= T\mathscr{A}_e\tilde{f} + t\pi\left[\mathrm{Sp}_{\mathbb{K}}\left\{ \left.\frac{\partial \tilde{f}_u}{\partial u_1}\right|_{u=0}, \ldots, \left.\frac{\partial \tilde{f}_u}{\partial u_d}\right|_{u=0} \right\}\right] \\ &= T\mathscr{A}_e f + \mathrm{Sp}_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_d\}. \end{aligned} \tag{3.7}$$

Conversely, let us assume that (3.7) holds: using theorem 3.12, we see that (3.6) holds as well. Therefore, F is versal as a frontal. This shows the second item. □

4. A geometric criterion for $\mathscr{F}$-finiteness

The Mather–Gaffney criterion states that a smooth $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is $\mathscr{A}$-finite if and only if there is a finite representative $f\colon N \to Z$ with isolated instability. For example, the generic singularities for $n = 2$ are transversal double points, with Whitney umbrellas and triple points in the accumulation (see e.g. [20], § 4.7). This implies that generic frontal singularities such as the folded Whitney umbrella (see example 3.18) are not $\mathscr{A}$-finite, since it contains cuspidal edges near the origin. Nonetheless, cuspidal edges are generic within the subspace of frontal map germs $(\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$ [1], which suggests the existence of a Mather–Gaffney-type criterion for frontal hypersurfaces.

PROPOSITION 4.1. *A germ of analytic plane curve* $\gamma\colon (\mathbb{C}, S) \to (\mathbb{C}^2, 0)$ *is* $\mathscr{F}$*-finite (see definition* 3.14 *above) if and only if it is* $\mathscr{A}$*-finite.*

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Proof. If γ is $\mathscr{A}$-finite, it is clear that it is also $\mathscr{F}$-finite, since

$$\mathscr{F}(\gamma) \subseteq \theta(\gamma) \implies \dim \frac{\mathscr{F}(\gamma)}{T\mathscr{A}_e\gamma} \leqslant \dim \frac{\theta(\gamma)}{T\mathscr{A}_e\gamma} < \infty$$

Assume γ is $\mathscr{F}$-finite, and let $\gamma\colon N \to Z$ be a representative of γ. By the curve selection lemma [**2**], $\Sigma(\gamma)$ is an isolated subset in N, so we can assume (by shrinking N if necessary) that $\gamma(N\backslash S)$ is a smooth submanifold of Z and $\gamma^{-1}(\{0\}) = S$. By the Mather–Gaffney criterion, it then follows that γ is $\mathscr{A}$-finite, as stated. □

Given a frontal map $f\colon N \to Z$ and $z \in Z$, let $f_z\colon (N, f^{-1}(z)) \to (Z, z)$. We define $\mathscr{F}(f)$ as the sheaf of $\mathscr{O}_Z$-modules given by the stalk $\mathscr{F}(f)_z = \mathscr{F}(f_z)$. We also set θ_N (resp. θ_Z) as the sheaf of vector fields on N (resp. Z) and the quotient sheaves

$$\mathscr{T}^1_{\mathscr{R}_e} f = \frac{\mathscr{F}(f)}{tf(\theta_N)}; \quad \mathscr{T}^1_{\mathscr{F}_e} f = \frac{f_*\left(\mathscr{T}^1_{\mathscr{R}_e} f\right)}{\omega f(\theta_Z)};$$

Remark 4.2. If f is finite, we can take coordinates in N and W such that $\tilde{f}(x, y) = (x, f_n(x, y), \dots, f_{2n+1}(x, y))$. By [**13**], we have the identity

$$R_{\tilde{f}} := \left\{\lambda \in \mathscr{O}_N \colon \mathrm{d}\lambda \in \mathscr{O}_N \,\mathrm{d}\left(\tilde{f}^*\mathscr{O}_W\right)\right\} = \left(\frac{\partial}{\partial y}\right)^{-1} \mathscr{O}_N \left\{\frac{\partial \tilde{f}_n}{\partial y}, \dots, \frac{\partial \tilde{f}_{2n+1}}{\partial y}\right\}$$

which is a $\mathscr{O}_N$-finite algebra by [**12**]. Since f is finite, $R_{\tilde{f}}$ is $\mathscr{O}_Z$-finite.

Proposition 4.3 [**14**]. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a frontal map germ. If $\tilde{f}$ is $\mathscr{A}$-equivalent to an analytic $g\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{2n+1}, 0)$ (not necessarily integral) such that* $\operatorname{codim}_{\mathbb{C}} \Sigma(g) > 1$,

$$\frac{\theta_I(\tilde{f})}{T\mathscr{L}_e\tilde{f}} \cong_{\mathscr{O}_Z} \frac{R_{\tilde{f}}}{\mathscr{O}_Z\{1, \tilde{p}_1, \dots, \tilde{p}_n\}}$$

where $\tilde{p}_1, \dots, \tilde{p}_n$ are the coordinates of $\tilde{f}$ in the fibres of π.

Remark 4.4. Let f and $\tilde{f}$ be given as in the statement above. If we assume that f has corank 1 and is given as in equation (2.2), $\Sigma(\tilde{f}) = V(p_y, \mu_y)$.

Corollary 4.5. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a frontal map germ. If f is finite and* $\operatorname{codim} V(p_y, \lambda_y) > 1$, *there is a representative $f\colon N \to Z$ of f such that $\mathscr{T}^1_{\mathscr{F}_e} f$ is a coherent sheaf.*

Proof. Using proposition 4.3, we have

$$\frac{R_{\tilde{f}_w}}{\mathscr{O}_Z\{1, \tilde{p}_1, \dots, \tilde{p}_n\}} \cong_{\mathscr{O}_Z} \frac{\theta_I(\tilde{f}_w)}{T\mathscr{L}_e\tilde{f}_w} = (\mathscr{T}^1_{\mathscr{F}_e} f)_{\pi(w)}.$$

Since f is finite, $R_{\tilde{f}_w}$ is $\mathscr{O}_{Z,\pi(w)}$-finite, as shown in remark 4.2. Therefore, the stalk of $\mathscr{T}^1_{\mathscr{F}_e} f$ at $\pi(w)$ is finitely generated and $\mathscr{T}^1_{\mathscr{F}_e} f$ is of finite type.

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Let $V \subset Z$ be an open set and $\beta\colon \mathscr{O}^q_{Z\restriction V} \to (\mathscr{T}^1_{\mathscr{F}_e} f)_{\restriction V}$ an epimorphism of $\mathscr{O}_Z$-modules. Since $\mathscr{O}_Z$ is a Noetherian ring, every submodule of $\mathscr{O}^q_{Z\restriction V}$ is finitely generated. In particular, $\ker\beta$ is finitely generated. We then conclude that $\mathscr{T}^1_{\mathscr{F}_e} f$ is a coherent sheaf. □

THEOREM 4.6 Mather–Gaffney criterion for frontal maps. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a frontal map germ. If f is finite and* $\operatorname{codim}_{\mathbb{C}} \Sigma(\tilde{f}) > 1$, *$f$ is $\mathscr{F}$-finite if and only if there exists a representative $f\colon N' \to Z'$ of f such that the restriction $f\colon N'\backslash S \to Z'\backslash\{0\}$ is locally $\mathscr{F}$-stable.*

Proof. The case for $n = 1$ follows easily from the Mather–Gaffney criterion for $\mathscr{A}$-equivalence and proposition 4.1. Therefore, we assume $n > 1$.

Suppose first that f has finite $\mathscr{F}$-codimension: by corollary 4.5, $\mathscr{T}^1_{\mathscr{F}_e} f$ is a coherent sheaf. In addition,

$$\dim_{\mathbb{C}}(\mathscr{T}^1_{\mathscr{F}_e} f)_0 = \dim_{\mathbb{C}} T^1_{\mathscr{F}_e} f = \operatorname{codim}_{\mathscr{F}_e} f < \infty.$$

By Rückert's Nullstellensatz, there exists an open neighbourhood Z' of 0 in Z such that $\operatorname{supp} \mathscr{T}^1_{\mathscr{F}_e} f \cap Z \subseteq \{0\}$. Therefore, every other stalk of $\mathscr{T}^1_{\mathscr{F}_e} f$ is 0, and the restriction of f to $N'\backslash\{0\}$ is $\mathscr{F}$-stable, where $N' = f^{-1}(Z')$.

Conversely, suppose that there exists a representative $f\colon N' \to Z'$ such that the restriction $f\colon N'\backslash\{0\} \to Z'\backslash\{0\}$ is locally $\mathscr{F}$-stable. Given $z \in Z\backslash\{0\}$, $(\mathscr{T}^1_{\mathscr{F}_e} f)_z = 0$, so there exists an open neighbourhood U of 0 in Z such that $\operatorname{supp} \mathscr{T}^1_{\mathscr{F}_e} f \cap U \subseteq \{0\}$. By Rückert's Nullstellensatz, it follows that the dimension of the stalk of $\mathscr{T}^1_{\mathscr{F}_e} f$ at 0 is finite, but that dimension is equal to $\operatorname{codim}_{\mathscr{F}_e} f$. We conclude that the germ of f at 0 is $\mathscr{F}$-finite. □

5. Frontal reduction of a corank 1 map germ

In [**22**], we presented the notion of frontalization for a fold surface $f\colon (\mathbb{C}^2, S) \to (\mathbb{C}^3, 0)$, and proved that the frontalization process preserves some of the topological invariants of f. We also defined frontal versions of Mond's S_k, B_k, C_k and F_4 singularities (see [**18**]), observing that none of them are wave fronts. We now seek to describe a more general procedure to generate frontals using arbitrary corank 1 map germs.

EXAMPLE 5.1. Let $\gamma\colon (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be the parametrized curve $\gamma(t) = (t^3, t^4)$: the unfolding $\Gamma\colon (\mathbb{K}^3 \times \mathbb{K}, 0) \to (\mathbb{K}^3 \times \mathbb{K}^2, 0)$ given by

$$\Gamma(u, t) = (u, t^3 + u_1 t, t^4 + u_2 t + u_3 t^2) = (u, p(u, t), q(u, t))$$

is an $\mathscr{A}$-miniversal deformation for γ. By proposition 2.7 and since $\deg_t p_t < \deg_t q_t$, Γ is frontal if and only if $p_t | q_t$. If $\mu \in \mathscr{O}_1$ is such that $q_t = \mu p_t$, a simple computation

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then shows that the identity

$$4t^3 + u_2 + 2u_3t = (3t^2 + u_1)(\mu_1 t + \mu_0)$$

holds if and only if $u_2 = \mu_0 = 0$, $\mu_1 = 4/3$ and $2u_3 = 3u_1$. Setting $h(v) = (3v, 0, 2v)$, we obtain the unfolding

$$h^*\Gamma(t, v) = (v, t^3 + 3vt, t^4 + 2vt^2)$$

which is a swallowtail singularity.

In this section, we show that the frontal reduction of the versal unfolding of a plane curve is a $\mathscr{F}$-versal unfolding. The proof of this result gives a procedure to compute the frontal reduction of a given unfolding (versal or otherwise) via a system of polynomial equations, which may be solved using a computer algebra system such as Oscar or Singular.

Remark 5.2 Piuseux parametrization. Let $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be an analytic plane curve with isolated singularities. There exists a $f \in \mathbb{C}\{x, y\}$ such that $f \circ \gamma = 0$. By Piuseux's theorem (see e.g. [**28**], theorem 2.2.6, or [**5**], theorem 5.1.1), if $\alpha = \operatorname{ord} f$, $f(t^\alpha, t^{\alpha+1}h(t)) = 0$ for some $h \in \mathbb{C}\{t\}$. Therefore, γ is $\mathscr{A}$-equivalent to the plane curve

$$t \mapsto (t^\alpha, t^{\alpha+1}g(t)).$$

In particular, γ is $\mathscr{A}$-finite (and thus finitely determined) by the Mather–Gaffney criterion, so we can further assume that $g \in \mathbb{C}[t]$.

If $\mathbb{K} = \mathbb{R}$, it suffices to replace γ with its complexification $\gamma_{\mathbb{C}}$ in the argument above, as γ is analytic. Therefore, such a parametrization also exists in the real case.

Lemma 5.3. *Let $\gamma\colon (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be the plane curve from remark 5.2. There exists a smooth d-parameter deformation (g_w) of g such that*

$$\Gamma(u, v, w, t) = \left(u, v, w, t^\alpha + \sum_{j=1}^{\alpha-2} u_j t^j, \sum_{j=1}^{\alpha-1} v_j t^j + t^{\alpha+1} g_w(t)\right)$$

is a miniversal unfolding of γ.

Proof. Let $G = \{g_1, \ldots, g_d\} \subset \mathbb{K}[t]$ be a $\mathbb{K}$-basis for $T^1_{\mathscr{A}_e}\gamma$: by Martinet's theorem (see [**20**], theorem 7.2), a miniversal unfolding for γ is given by the expression

$$\Gamma(x, t) = (x, \gamma(t) + x_1 g_1(t) + \cdots + x_{n-1} g_{n-1}(t)). \tag{5.1}$$

A simple computation shows that

$$T\mathscr{A}_e\gamma \subseteq \mathscr{O}_1 \left\{ \begin{pmatrix} \alpha t^{\alpha-1} \\ (\alpha+1)t^\alpha q_0(t) + t^{\alpha+1} q_0'(t) \end{pmatrix} \right\} + \mathfrak{m}_1^\alpha \mathscr{O}_1^2. \tag{5.2}$$

Using equation (5.2), we may assume that $g_j(t) = (t^j, 0)$ and $g_{j+\alpha-2}(t) = (0, t^j)$ for $1 \leqslant j \leqslant \alpha - 2$. Setting $g_w(t) = g(t) + w_1 g_{2\alpha-1}(t) + \cdots + w_{d-2\alpha+1} g_d(t)$, equation

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(5.1) becomes

$$\Gamma(u,v,w,t)=\left(u,v,w,t^{\alpha}+\sum_{j=1}^{\alpha-2}u_jt^j,t^{\alpha+1}g_w(t)+\sum_{j=1}^{\alpha-1}v_jt^j\right),$$

as claimed. □

Remark 5.4. Let $h\colon(\mathbb{K}^r,0)\to(\mathbb{K}^d,0)$ be a smooth map germ and Γ be the unfolding from lemma 5.3. The pullback $h^*\Gamma$ is given by

$$(h^*\Gamma)(x,t)=\left(x,t^{\alpha}+\sum_{j=1}^{\alpha-2}u_j(x)t^j,\sum_{j=1}^{\alpha-1}v_j(x)t^j+t^{\alpha+1}g_{w(x)}(t)\right),$$

where $u_j(x)\equiv(u_j\circ h)(x)$, $v_j(x)\equiv(v_j\circ h)(x)$ and $w(x)\equiv(w\circ h)(x)$. As we saw in the proof of lemma 5.3,

$$g_w(t)=g(t)+w_1g_{2\alpha-1}(t)+\cdots+w_{d-2\alpha+1}g_d(t),$$

where g can be assumed to be a polynomial function (due to remark 5.2). Therefore, the component functions of $h^*\Gamma$ are elements of $\mathscr{O}_r[t]$, the algebra of polynomials on t with coefficients in $\mathscr{O}_r$.

Theorem 5.5. *If γ has a miniversal d-parameter unfolding Γ, there is a unique immersion $h\colon(\mathbb{K}^l,0)\to(\mathbb{K}^d,0)$ with the following properties:*

(1) *$h^*\Gamma$ is a frontal unfolding of γ;*

(2) *if $(h')^*\Gamma$ is frontal for any other $h'\colon(\mathbb{K}^{l'},0)\to(\mathbb{K}^d,0)$, $(h')^*\Gamma$ is equivalent as an unfolding to a pullback of $h^*\Gamma$.*

Therefore, $h^*\Gamma$ is a frontal miniversal unfolding.

We shall denote $h^*\Gamma$ as $\Gamma_{\mathscr{F}}$ and call it a *frontal reduction* of Γ.

Proof. Let Γ be the unfolding from lemma 5.3 and $d=\operatorname{codim}_{\mathscr{A}_e}\gamma$. We first want to show that there is an immersion $h\colon(\mathbb{K}^\ell,0)\to(\mathbb{K}^d,0)$ making $h^*\Gamma$ a frontal map germ; to do so, we shall derive a system of equations that determines whether a given pullback yields a frontal unfolding.

Let $(h^*\Gamma)(x,t)=(x,P(x,t),Q(x,t))$. By remark 5.4, $Q\in\mathscr{O}_r[t]$, so we can write $Q(x,t)=q_1(x)t+\cdots+q_\beta(x)t^\beta$. Since $h^*\Gamma$ is a corank 1 map germ, corollary 2.7 states that it is frontal if and only if either $P_t|Q_t$ or $Q_t|P_t$; in particular, we can assume that $\deg_t P_t\leqslant\deg_t Q_t$, allowing us to impose the condition $P_t|Q_t$ to $h^*\Gamma$.

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If $Q_t = \mu P_t$ for some $\mu \in \mathscr{O}_{r+1}$, there will exist $\mu_0, \dots, \mu_{\beta-\alpha}$ such that $\mu(x,t) = \mu_0(x) + \cdots + \mu_{\beta-\alpha}(x)t^{\beta-\alpha}$. Therefore, the identity $Q_t = \mu P_t$ is equivalent to

$$kq_k(x) = \sum_{i+j=k} iu_i(x)\mu_j(x) \tag{5.3}$$

for $k = 1, 2 \dots, \beta$. For $k \geqslant \alpha$, we may solve for $\mu_{k-\alpha}$ to get the expression

$$\mu_{k-\alpha}(x) = \frac{k}{\alpha}q_k(x) - \frac{1}{\alpha}\sum_{i+j=k} iu_i(x)\mu_j(x); \quad u_\alpha(x) \equiv 1.$$

The remaining terms define an immersion germ $h\colon (\mathbb{K}^{d-\alpha+1}, 0) \to (\mathbb{K}^d, 0)$ given by $h(u,w) = (u, v(u,w), w)$, which is the unique solution to equation (5.3) by construction. This proves item 1.

Let Λ be a frontal unfolding of γ: versality of Γ implies that Λ is equivalent to $(h')^*\Gamma$ for some $h'\colon (\mathbb{K}^r, 0) \to (\mathbb{K}^d, 0)$. Let $h\colon V \to U$ be a one-to-one representative of h, $\pi\colon U \to V$ be the projection

$$\pi(x_1, \dots, x_d) = (x_1, \dots, x_{\alpha-2}, x_{2\alpha-2}, \dots, x_d)$$

and $h'\colon V' \to U'$ be a representative of h'. Since $(h')^*\Gamma$ is frontal, h' verifies equation (5.3) and thus $h'(V') \subseteq h(V)$ by construction. Given $v' \in V'$, there exists a unique $v \in V$ such that

$$h'(v') = h(v) \implies (\pi \circ h')(v') = v \implies (h \circ \pi \circ h')(v') = h(v) = h'(v'),$$

and thus $(h')^*\Gamma = (h \circ \pi \circ h')^*\Gamma = (\pi \circ h')^*(h^*\Gamma)$. $\square$

EXAMPLE 5.6. Consider Arnol'd's E_8 singularity, $\gamma(t) = (t^3, t^5)$. A versal unfolding of this curve is given by

$$(u, v, w, t^3 + ut, t^5 + wt^4 + v_2t^2 + v_1t) = (u, v, w, p(u,t), q(v,w,t)).$$

The frontal reduction of this unfolding may now be computed using equation (5.3), which can be written in matrix form as

$$\begin{pmatrix} 5 \\ 4w \\ 0 \\ 2v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ u & 0 & 3 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix} \begin{pmatrix} \mu_2 \\ \mu_1 \\ \mu_0 \end{pmatrix} \implies \begin{pmatrix} \mu_2 \\ \mu_1 \\ \mu_0 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 15 \\ 12w \\ -5u \end{pmatrix}.$$

Since this system has five equations and only three unknowns, we can now solve for v, yielding $v_1 = -5/9u^2$ and $v_2 = 2/3w$.

REMARK 5.7. Let Γ be a miniversal unfolding of γ, and $\Gamma_{\mathscr{F}}$ be its frontal reduction. As shown in theorem 5.5, $\Gamma_{\mathscr{F}}$ is a miniversal frontal unfolding of γ. Setting $\Gamma_{\mathscr{F}}(x,u) = (\gamma_u(x), u)$, we can now consider the integral deformation $\tilde{\gamma}_u$ of $\tilde{\gamma}$. By theorem 3.20, miniversality of $\Gamma_{\mathscr{F}}$ implies that $\tilde{\gamma}_u$ is a Legendre miniversal deformation of $\tilde{\gamma}$. Therefore, the method of frontal reductions can also be used to generate miniversal Legendre deformations of corank 1 integral curves $f\colon (\mathbb{K}, 0) \to PT^*\mathbb{K}^2$.

Note that the proof of theorem 5.5 above employs a specific choice of coordinates for γ, as well as a specific miniversal unfolding. We now show that the method of frontal reductions does not depend on the choice of coordinates in the source and target for γ, or the choice of miniversal unfolding Γ.

COROLLARY 5.8. *Given two miniversal d-parameter unfoldings F, G of γ, $G_{\mathscr{F}}$ is $\mathscr{A}$-equivalent to $F_{\mathscr{F}}$.*

Proof. Since F is a miniversal unfolding, there exists a diffeomorphism $m\colon (\mathbb{K}^d, 0) \to (\mathbb{K}^d, 0)$ such that G is equivalent to m^*F as unfoldings of γ. If $h'\colon (\mathbb{K}^l, 0) \to (\mathbb{K}^d, 0)$ is the immersion such that $G_{\mathscr{F}} = (h')^*G$, then $G_{\mathscr{F}}$ is equivalent to $(h')^*m^*F$. In particular, $(h')^*m^*F$ is a frontal unfolding, so there exists a $p\colon (\mathbb{K}^l, 0) \to (\mathbb{K}^l, 0)$ such that $(h')^*m^*F$ is equivalent to $p^*F_{\mathscr{F}}$. Let $h\colon (\mathbb{K}^l, 0) \to (\mathbb{K}^d, 0)$ be the immersion such that $F_{\mathscr{F}} = h^*F$: if we now swap F and G in the argument above, we see that there exist a diffeomorphism $m'\colon (\mathbb{K}^d, 0) \to (\mathbb{K}^d, 0)$ and a smooth $p'\colon (\mathbb{K}^l, 0) \to (\mathbb{K}^l, 0)$ such that

$$F_{\mathscr{F}} \sim h^*(m')^*G \sim (p')^*G_{\mathscr{F}},$$

where $\sim$ denotes equivalence of unfoldings.

Since $G_{\mathscr{F}}$ is equivalent to $(h')^*m^*F$, we have the chain of equivalences

$$F_{\mathscr{F}} \sim (p')^*G_{\mathscr{F}} \sim (p')^*(h')^*m^*F \sim (p')^*p^*F_{\mathscr{F}} = (p \circ p')^*F_{\mathscr{F}}.$$

We wish to show that p' is a diffeomorphism, so that $G_{\mathscr{F}}$ is $\mathscr{A}$-equivalent to $(p')^*G_{\mathscr{F}}$ (hence to $F_{\mathscr{F}}$). Using the chain rule, we have

$$[(p \circ p')^* \dot{F_{\mathscr{F}}}]_j = \sum_{k=1}^{l} (\dot{F_{\mathscr{F}}})_k \frac{\partial (p \circ p')_k}{\partial u_j}(0).$$

By theorem 3.21, the classes of $\{(\dot{F_{\mathscr{F}}})_1, \ldots, (\dot{F_{\mathscr{F}}})_l\}$ and $\{[(p \circ p')^* \dot{F_{\mathscr{F}}}]_1, \ldots, [(p \circ p')^* \dot{F_{\mathscr{F}}}]_l\}$ form bases for the quotient vector space $\mathscr{F}(\gamma)/T\mathscr{A}_e\gamma$, hence the matrix

$$\left(\frac{\partial (p \circ p')_k}{\partial u_j}(0)\right) = \left(\frac{\partial p_k}{\partial u_j}(0)\right)\left(\frac{\partial p'_k}{\partial u_j}(0)\right)$$

is invertible, and so are its factors. It follows that p' is a diffeomorphism, as desired. □

COROLLARY 5.9. *Let $\gamma'\colon (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be a plane curve $\mathscr{A}$-equivalent to γ. If Γ and Γ' are the miniversal unfoldings of γ and γ', $\Gamma'_{\mathscr{F}}$ is $\mathscr{A}$-equivalent to $\Gamma_{\mathscr{F}}$.*

Proof. Let $\phi\colon (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ and $\psi\colon (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ be diffeomorphisms such that $\gamma' = \psi \circ \gamma \circ \phi^{-1}$. We consider the unfolding $\overline{\Gamma}$ of γ' given by

$$(\psi \times \mathrm{id}) \circ \Gamma \circ (\phi^{-1} \times \mathrm{id}).$$

If $h\colon (\mathbb{K}^l, 0) \to (\mathbb{K}^d, 0)$ is the immersion such that $\Gamma_{\mathscr{F}} = h^*\Gamma$, then $h^*\overline{\Gamma}$ is $\mathscr{A}$-equivalent to $\Gamma_{\mathscr{F}}$. In particular, $h^*\overline{\Gamma}$ is a frontal unfolding of γ', so $h^*\overline{\Gamma}$ is $\mathscr{A}$-equivalent to $\Gamma'_{\mathscr{F}}$ since $\Gamma'_{\mathscr{F}}$ is a stable unfolding of γ'. □

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Remark 5.10. While the method of frontal reductions successfully turns $\mathscr{A}$-versal unfoldings into $\mathscr{F}$-versal unfoldings, the same does not hold for stable unfoldings. For example, given the plane curve $\gamma(t) = (t^2, t^{2k+1})$, $k > 1$, a stable unfolding of γ is given by $f(u, t) = (u, t^2, t^{2k+1} + ut)$. However, the only pullback that can turn f into a frontal map germ is $u(s) = 0$, giving us γ, which is not stable by hypothesis.

A more general method to compute stable unfoldings will be given in § 6.

Corollary 5.11. *Given* $\gamma\colon (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$,

$$\operatorname{codim}_{\mathscr{F}_e} \gamma = \operatorname{codim}_{\mathscr{A}_e} \gamma - \operatorname{mult}(\gamma) + 1$$

Consequently, if $\gamma(\mathbb{K}, 0)$ *is the zero locus of some analytic* $g \in \mathscr{O}_2$,

$$\operatorname{codim}_{\mathscr{F}_e} \gamma = \tau(g) - \operatorname{ord}(g) - \frac{1}{2}\mu(g) + 1.$$

Proof. In the proof of theorem 5.5, we see that $l = d - \alpha + 1$, where $d = \operatorname{codim}_{\mathscr{A}_e} \gamma$ and $\alpha = \operatorname{mult}(\gamma)$. Since $h^*\Gamma$ is a miniversal l-parameter unfolding, $\operatorname{codim}_{\mathscr{F}_e} \gamma = l$, giving the first identity.

Now assume $\mathbb{K} = \mathbb{C}$: Milnor's formula [**17**] states that the delta invariant $\delta(g)$ and the Milnor number $\mu(g)$ of g are related via the identity $2\delta(g) = \mu(g)$, since γ is a mono-germ. On the other hand, a result in [**8**] states that $\operatorname{codim}_{\mathscr{A}_e} \gamma = \tau(g) - \delta(g) = \tau(g) - 1/2\mu(g)$, τ being the Tjurina number, hence yielding the expression

$$\operatorname{codim}_{\mathscr{F}_e} \gamma = \tau(g) - \frac{1}{2}\mu(g) - \operatorname{mult}(\gamma) + 1.$$

In particular, the order of g is equal to $\operatorname{mult}(\gamma)$ (see [**5**], corollary 5.1.6). For $\mathbb{K} = \mathbb{R}$, simply note that $\mu(g) = \mu(g_{\mathbb{C}})$, $\operatorname{ord}(g) = \operatorname{ord}(g_{\mathbb{C}})$ and $\tau(g) = \tau(g_{\mathbb{C}})$, where $g_{\mathbb{C}}$ is the complexification of g. □

Example 5.12. Let $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be the A_{2k} singularity, with normalization $\gamma(t) = (t^2, t^{2k+1})$. Direct computations show that

$$\frac{\theta(\gamma)}{T\mathscr{A}_e\gamma} \cong \operatorname{Sp}\{(0, t^{2\ell+1}) : 0 \leqslant \ell < k\}; \quad \frac{\mathscr{F}(\gamma)}{T\mathscr{A}_e\gamma} \cong \operatorname{Sp}\{(0, t^{2\ell+1}) : 1 \leqslant \ell < k\},$$

from which follows that its $\mathscr{A}_e$-codimension is k and its $\mathscr{F}_e$-codimension is $k - 1$. Therefore, we have $\operatorname{codim}_{\mathscr{F}_e} \gamma = k - 1 = k - 2 + 1 = \operatorname{codim}_{\mathscr{A}_e} \gamma - \operatorname{mult}(\gamma) + 1$, as expected.

The image of γ is given as the zero locus of the function $g(x, y) = y^2 - x^{2k+1}$. Using the second expression for the frontal codimension, we have

$$\tau(g) - \frac{1}{2}\mu(g) = \operatorname{codim}_{\mathscr{F}_e} \gamma + \operatorname{ord}(g) - 1 = k - 1 + 2 - 1 = k$$

as expected, since both the Tjurina and Milnor numbers of g are $2k$.

In [**22**], § 5, we introduced the notion of frontal Milnor number $\mu_{\mathscr{F}}$ for a frontal multi-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. This analytic invariant was defined in a similar fashion to Mond's image Milnor number [**19**], only changing smooth stabilizations

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for frontal ones. We then conjectured that $\mu_{\mathscr{F}}$ verified an adapted version of Mond's conjecture, which we called *Mond's frontal conjecture*.

Applying [**22**], proposition 5.10 to corollary 5.11, we can now prove Mond's frontal conjecture in dimension 1.

COROLLARY 5.13. *Given a plane curve* $\gamma\colon (\mathbb{C}, S) \to (\mathbb{C}^2, 0)$, $\mu_{\mathscr{F}}(\gamma) \geqslant \operatorname{codim}_{\mathscr{F}}(\gamma)$, *with equality if* γ *is quasi-homogeneous.*

Proof. Let γ be a non-constant analytic plane curve. By the curve selection lemma [**2**], γ has an isolated singularity at the origin, so it is $\mathscr{A}$-finite and

$$\mu_I(\gamma) \geqslant \operatorname{codim}_{\mathscr{A}_e}(\gamma),$$

with equality if γ is quasi-homogeneous (see [**19**]). By corollary 5.11, γ is $\mathscr{F}$-finite and $\operatorname{codim}_{\mathscr{A}_e}(\gamma) = \operatorname{codim}_{\mathscr{F}_e}(\gamma) + \operatorname{mult}(\gamma) - 1$. Using [**22**], proposition 5.10 and conservation of multiplicity (see e.g. [**20**], corollary E.4), $\mu_{\mathscr{F}}(\gamma) = \mu_I(\gamma) - \operatorname{mult}(\gamma) + 1$, as stated above. Therefore,

$$\mu_{\mathscr{F}}(\gamma) + \operatorname{mult}(\gamma) - 1 = \mu_I(\gamma) \geqslant \operatorname{codim}_{\mathscr{A}}(\gamma) = \operatorname{codim}_{\mathscr{F}}(\gamma) + \operatorname{mult}(\gamma) - 1,$$

with equality if γ is quasi-homogeneous. □

Now let $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ be a corank 1 frontal map germ with isolated frontal instability. We can choose coordinates in the source and target such that

$$f(x, y) = (x, p(x, y), q(x, y)); \quad q_y = \mu p_y; \quad (x, y) \in \mathbb{K}^{n-1} \times \mathbb{K},$$

for some $p, q, \mu \in \mathscr{O}_n$. We then set S' as the projection on the y coordinate of S and consider the *generic slice* $\gamma\colon (\mathbb{K}, S') \to (\mathbb{K}^2, 0)$ of f, given by $\gamma(t) = (p(0, t), q(0, t))$. Since f has isolated frontal instabilities, γ is $\mathscr{A}$-finite (see proposition 4.1 above) and we may consider a versal unfolding Γ of γ with frontal reduction

$$\Gamma_{\mathscr{F}}\colon (\mathbb{K}^d \times \mathbb{K}, S' \times \{0\}) \to (\mathbb{K}^d \times \mathbb{K}^2, 0).$$

It is not true in general that the sum of two frontal mappings is frontal (e.g. $(x, y) \mapsto (x, y^3, y^4)$ and $(x, y) \mapsto (x, xy, 0)$), but we can still construct a *frontal sum* operator that yields a frontal mapping given two frontal mappings with corank at most 1. Let $p', q', \mu' \in \mathscr{O}_{d+1}$ such that

$$\Gamma_{\mathscr{F}}(u, y) = (u, p'(u, y), q'(u, y)); \quad q'_y = \mu p'_y:$$

we define the *frontal sum* $F\colon (\mathbb{K}^d \times \mathbb{K}^n, \{0\} \times S) \to (\mathbb{K}^d \times \mathbb{K}^{n+1}, 0)$ of f and $\Gamma_{\mathscr{F}}$ as $F(u, x, y) = (u, x, P(u, x, y), Q(u, x, y))$, where

$$\begin{aligned} P(u, x, y) &= p(x, y) + p'(u, y) - p(0, y); \\ Q(u, x, y) &= \int_0^y (\mu(x, s) + \mu'(u, s) - \mu(0, s)) P_s(u, x, s) \, \mathrm{d}s. \end{aligned} \tag{5.4}$$

This map germ constitutes an unfolding of both f and $\Gamma_{\mathscr{F}}$ by construction. Stability of $\Gamma_{\mathscr{F}}$ then implies that F is stable. Therefore, frontal sums allow us to construct stable frontal unfoldings that are not necessarily versal.

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EXAMPLE 5.14 Frontalized fold surfaces. Let $f\colon (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ be a frontal fold surface given in the form

$$f(x, y) = (x, y^2, a_1(x)y^3 + a_2(x)y^5 + \cdots + a_n(x)y^{2n+1} + y^{2n+3});$$

wherein we assume $a_0, \ldots, a_n \in \mathbb{K}[x]$. The function $t \mapsto f(0, t)$ has order $2n + 3$, so f can be seen as a smooth 1-parameter unfolding of the curve

$$\gamma(t) = (t^2, t^{2n+3} + a_n(0)t^{2n+1} + \cdots + a_1(0)t^3).$$

A frontal miniversal unfolding for γ is given by

$$\Gamma(u, t) = (u, t^2, t^{2n+3} + u_n t^{2n+1} + \cdots + u_1 t^3),$$

and we can recover f by setting $u_j(x) = a_j(x)$. Taking $(u, x) \mapsto (0, u_1 + a_1(x), \ldots, u_n + a_n(x))$ gives the stable unfolding

$$F(u, x, t) = (u, t^2, t^{2n+3} + [u_n + a_n(x)]t^{2n+1} + \cdots + [u_1 + a_1(x)]t^3).$$

REMARK 5.15. The *frontal sum* defined in (5.4) can be used to show that $\mathscr{F}(f)$ is linear when f has corank at most 1: first, since f is a corank 1 frontal, we take coordinates in the source and target such that

$$f(x, y) = (x, p(x, y), q(x, y)); \quad q_y = \mu p_y,$$

and consider the generic slice $\gamma(t) = (p(0, t), q(0, t))$.

Let $\xi, \eta \in \mathscr{F}(f)$ with respective integral $\mathscr{F}$-curves $F = (f_u, u)$, $G = (g_u, u)$. Since F and G are unfoldings of f, they may also be regarded as unfoldings of γ. We then consider the frontal sum $H = (u, v, h_{(u,v)})$ of F and G, and set $\hat{H} = (w, \hat{h}_w) = (w, h_{(w,w)})$. Note that the image of $\hat{H}$ is simply the intersection of the image of H with the hypersurface of equation $u = v$, so $\hat{H}$ is frontal. Using the chain rule and Leibniz's integral rule, we see that

$$\left.\begin{matrix} P_w = P_u + P_v \\ Q_w = Q_u + Q_v \end{matrix}\right\} \implies \left.\frac{\partial \hat{h}_w}{\partial w}\right|_{w=0} = \xi + \eta$$

and thus $\xi + \eta \in \mathscr{F}(f)$.

6. Stability of frontal map germs

In § 5, we described a method to generate $\mathscr{F}$-versal unfoldings of analytic plane curves using pullbacks. Nonetheless, as pointed out in remark 5.10, the pullback of a stable unfolding is generally not stable as a frontal.

In this section, we describe a technique to generate stable frontal unfoldings, not too dissimilar to the method Mather used to generate all stable map germs. We also give a classification of all $\mathscr{F}$-stable proper frontal map germs $(\mathbb{C}^3, S) \to (\mathbb{C}^4, 0)$ of corank 1 in § 6.2, aided by Hefez and Hernandes' normal form theorem for plane curves [**9**, **10**].

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Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a frontal map germ and $\xi \in \mathscr{F}(f)$. By definition of $\mathscr{F}(f)$, ξ is given by a frontal 1-parameter unfolding $F = (f_t, t)$ of f; this is, F verifies that

$$\mathrm{d}(Y \circ f_t) = \sum_{i=1}^{n} p_i \,\mathrm{d}(X_i \circ f_t) + p_0 \,\mathrm{d}t$$

for some $p_0, \ldots, p_n \in \mathscr{O}_{n+1}$. If we now consider the vector field germ $\lambda\xi$ with $\lambda \in \mathscr{O}_n$, $\lambda\xi$ is given by the 1-parameter unfolding $(\lambda f_t, t)$. This unfolding is frontal if and only if

$$\mathrm{d}(Y \circ \lambda f_t) = \sum_{i=1}^{n} q_i \,\mathrm{d}(X_i \circ \lambda f_t) + q_0 \,\mathrm{d}t \tag{6.1}$$

for some $q_0, \ldots, q_n \in \mathscr{O}_{n+1}$. Expanding on both sides of the equality and rearranging, we see that equation (6.1) is equivalent to

$$\lambda \sum_{i=1}^{n} (q_i - p_i) \,\mathrm{d}(X_i \circ f_t) + (q_0 - \lambda p_0) \,\mathrm{d}t = \Big[(Y \circ f_t) - \sum_{i=1}^{n} q_i (X_i \circ f_t)\Big] \,\mathrm{d}\lambda.$$

Therefore, the ring $R_f = \{\lambda \in \mathscr{O}_n : \mathrm{d}\lambda \in \mathscr{O}_n \,\mathrm{d}(f^*\mathscr{O}_{n+1})\}$ acts on $\mathscr{F}(f)$ via the usual action. In particular, $f^*\mathscr{O}_{n+1} \subseteq R_f$, so $\mathscr{F}(f)$ is an $\mathscr{O}_{n+1}$-module via the action $h\xi = (h \circ f)\xi$.

If we assume that f has integral corank 1 (so that $\mathscr{F}(f)$ is a $\mathbb{K}$-vector space), we can define the $\mathbb{K}$-vector spaces

$$T\mathscr{K}_{\mathscr{F}e} f = tf(\theta_n) + \mathfrak{m}_{n+1}\mathscr{F}(f); \quad T^1_{\mathscr{K}_{\mathscr{F}e}} f = \frac{\mathscr{F}(f)}{T\mathscr{K}_{\mathscr{F}e} f}.$$

We also define the **frontal $\mathscr{K}_e$-codimension** $\operatorname{codim}_{\mathscr{K}_{\mathscr{F}e}} f$ of f as the dimension of $T^1_{\mathscr{K}_{\mathscr{F}e}} f$ in $\mathbb{K}$, and will say that f is **$\mathscr{K}_{\mathscr{F}e}$-finite** if $\operatorname{codim}_{\mathscr{K}_{\mathscr{F}e}} f < \infty$.

Remark 6.1. The space $\mathscr{F}(f)$ is not generally a $\mathscr{O}_n$-module via the usual action: consider the plane curve $\gamma\colon (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ given by $\gamma(t) = (t^2, t^3)$. Using remark 3.13, we see that $(0, 1) \in \mathscr{F}(\gamma)$, but $(0, t) = t(0, 1) \notin \mathscr{F}(\gamma)$.

Recall that the Kodaira–Spencer map is defined as the mapping $\overline{\omega} f\colon T_0\mathbb{K}^{n+1} \to T^1_{\mathscr{K}_e} f$ sending $v \in T_0\mathbb{K}^{n+1}$ onto $\omega f(\eta)$, where $\eta \in \theta_{n+1}$ is such that $\eta_0 = v$. Since f is frontal, the image of ωf is contained within $\mathscr{F}(f)$, and the target space becomes $T^1_{\mathscr{K}_{\mathscr{F}e}} f$. Similarly, the kernel of this $\overline{\omega} f$ becomes

$$\tau(f) := (\overline{\omega} f)^{-1}[T\mathscr{K}_{\mathscr{F}e} f]|_0,$$

since no element in $T\mathscr{K}_e f \backslash \mathscr{F}(f)$ has a preimage.

Lemma 6.2. *The map germ f is $\mathscr{F}$-stable if and only if $\overline{\omega} f$ is surjective.*

Proof. Assume f is $\mathscr{F}$-stable and let $\zeta \in \mathscr{F}(f)$: there exist $\xi \in \theta_n$ and $\eta \in \theta_{n+1}$ such that $\zeta = tf(\xi) + \omega f(\eta)$. Setting $v = \eta_0$, it follows that $\overline{\omega} f(v) \equiv \zeta \mod T\mathscr{K}_{\mathscr{F}e} f$, and surjectivity of $\overline{\omega} f$ follows.

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Conversely, assume $\overline{\omega}f$ is surjective: we have the identity

$$T\mathscr{A}_e f + \mathfrak{m}_{n+1}\mathscr{F}(f) = \mathscr{F}(f). \tag{6.2}$$

Set $V' = \mathscr{F}(f)/tf(\theta_{n,S})$ and denote by $p\colon \mathscr{F}(f) \to V'$ the quotient projection. We may then write equation (6.2) as

$$(\pi \circ \omega f)(\theta_{n+1}) + \mathfrak{m}_{n+1}V' = V' \implies \frac{V'}{\mathfrak{m}_{n+1}V'} \lesssim (\pi \circ \omega f)(\theta_{n+1}).$$

Since $(p \circ \omega f)(\theta_{n+1})$ is finitely generated over $\mathscr{O}_{n+1}$, so is $V'/\mathfrak{m}_{n+1}V'$. This implies that $V'/\mathfrak{m}_{n+1}V'$ is finitely generated over $\mathbb{K}$, so V' is finitely generated over $\mathscr{O}_{n+1}$ by Weierstrass' preparation theorem. Since $\mathscr{O}_{n+1}$ is a local ring, Nakayama's lemma implies that $V' = (\pi \circ \omega f)(\theta_{n+1})$, which is equivalent to $\mathscr{F}(f) = T\mathscr{A}_e f$, and frontal stability follows. □

THEOREM 6.3. *A frontal* $f\colon (\mathbb{K}^n, S) \to (\mathbb{K}^{n+1}, 0)$ *with branches* $f_1, \ldots, f_r$ *is* $\mathscr{F}$*-stable if and only if* $f_1, \ldots, f_r$ *are* $\mathscr{F}$*-stable and the vector subspaces* $\tau(f_1), \ldots, \tau(f_r) \subseteq T_0\mathbb{K}^{n+1}$ *meet in general position.*

Proof. Let g be either f or one of its branches. By lemma 6.2, g is $\mathscr{F}$-stable if and only if $\overline{\omega}g$ is surjective; this is,

$$\frac{\mathscr{F}(g)}{T\mathscr{K}_{\mathscr{F}_e}g} \cong \frac{T_0\mathbb{K}^{n+1}}{\ker \overline{\omega}g} = \frac{T_0\mathbb{K}^{n+1}}{\tau(g)} \tag{6.3}$$

Let $S = \{s_1, \ldots, s_r\}$, the ring isomorphism $\mathscr{O}_{n,S} \to \mathscr{O}_{n,s_1} \oplus \cdots \oplus \mathscr{O}_{n,s_r}$ induces a module isomorphism $\mathscr{F}(f) \to \mathscr{F}(f_1) \oplus \ldots \mathscr{F}(f_r)$, which in turn induces an isomorphism

$$\frac{\mathscr{F}(f)}{T\mathscr{K}_{\mathscr{F}_e}f} \longrightarrow \frac{\mathscr{F}(f_1)}{T\mathscr{K}_{\mathscr{F}_e}f_1} \oplus \cdots \oplus \frac{\mathscr{F}(f_r)}{T\mathscr{K}_{\mathscr{F}_e}f_r} \tag{6.4}$$

On the other hand, the spaces $\tau(f_i)$ meet in general position if and only if the canonical map

$$T_0\mathbb{K}^{n+1} \longrightarrow \frac{T_0\mathbb{K}^{n+1}}{\tau(f_1)} \oplus \cdots \oplus \frac{T_0\mathbb{K}^{n+1}}{\tau(f_r)} \tag{6.5}$$

is surjective. The statement then follows from (6.3)–(6.5). □

We now use Ephraim's theorem to give a geometric interpretation to $\tau(f_i)$, $i = 1, \ldots, r$. Recall that the isosingular locus $\operatorname{Iso}(D, x_0)$ of a complex space $D \subseteq W$ at x_0 is defined as the germ at x_0 of the set of points $x \in D$ such that (D, x) is diffeomorphic to (D, x_0). Ephraim [**6**] showed that $\operatorname{Iso}(D, x_0)$ is a germ of smooth submanifold of (W, x_0) and its tangent space at x_0 is given by the evaluation at x_0 of the elements in the space

$$\operatorname{Der}(-\log(D, x_0)) = \{\xi \in \theta_W : \xi(I) \subseteq I\}$$

where $I \subset \mathscr{O}_W$ is the ideal of map germs vanishing on (D, x_0). We shall now use this result to give a geometric interpretation to the space $\tau(f)$.

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Proposition 6.4. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a finite, frontal map germ with integral corank* 1*. If f is $\mathscr{F}$-stable and* $\operatorname{codim}\Sigma(\tilde{f}) > 1$*, $\tau(f)$ is the tangent space at* 0 *of* $\operatorname{Iso}(f(\mathbb{C}^n, S))$*.*

To prove this result, we shall make use of the following

Lemma 6.5 cf. [20]. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a finite, frontal map germ with integral corank* 1 *and $\xi \in \theta_{n+1}$. If f is $\mathscr{F}$-finite and* $\operatorname{codim} V(p_y, \mu_y) > 1$*,*

$$\operatorname{Der}(-\log f) = \operatorname{Lift}(f) := \{\eta \in \theta_{n+1} : \omega f(\eta) = tf(\xi) \textit{ for some } \xi \in \theta_n\}.$$

Proof of proposition 6.4. By Ephraim's theorem [6], the tangent space to $\operatorname{Iso}(f(\mathbb{C}^n, S))$ at 0 is given by the evaluation at 0 of the elements in $\operatorname{Der}(-\log f)$. Using lemma 6.5, $\operatorname{Der}(-\log f)$ is the space of elements in θ_{n+1} that are liftable via f. Therefore, we only need to show that the evaluation of 0 of this space coincides with $\tau(f)$.

Let $\eta \in \operatorname{Lift}(f)$: there exists a $\xi \in \theta_n$ such that $\omega f(\eta) = tf(\xi) \in T\mathscr{K}_{\mathscr{F}_e} f$, so $\eta|_0 \in \tau(f)$. Conversely, if $\eta \in \theta_{n+1}$ verifies that $\eta|_0 \in \tau(f)$, there exist $\xi \in \theta_n$, $\zeta \in \mathscr{F}(f)$ such that

$$\omega f(\eta) = tf(\xi) + (f^*\beta)\zeta$$

for some $\beta \in \mathfrak{m}_{n+1}$. Since f is $\mathscr{F}$-stable, $\mathscr{F}(f) = T\mathscr{A}_e f$, which implies that

$$(f^*\mathfrak{m}_{n+1})\mathscr{F}(f) = (f^*\mathfrak{m}_{n+1})[tf(\theta_n) + \omega f(\theta_{n+1})] \subseteq tf(\mathfrak{m}_n\theta_n) + \omega f(\mathfrak{m}_{n+1}\theta_{n+1})$$

Therefore, there exist $\xi' \in \mathfrak{m}_n\theta_n$ and $\eta' \in \mathfrak{m}_{n+1}\theta_{n+1}$ such that

$$(f^*\beta)\zeta = tf(\xi') + \omega f(\eta') \implies \omega f(\eta - \eta') = tf(\xi + \xi')$$

and $\eta - \eta' \in \operatorname{Lift}(f)$. In particular, if $s \in S$, $(\eta - \eta')|_0 = \omega f(\eta - \eta')|_s = v - 0 = v$, thus finishing the proof. $\square$

6.1. Generating stable frontal unfoldings

The generation of stable unfoldings in Thom–Mather's theory of smooth deformations is done by computing the $\mathscr{K}_e$-tangent space of a smooth map germ $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ of rank 0. If $\mathfrak{m}_n\theta(f)/T\mathscr{K}_e f$ is generated over $\mathbb{K}$ by the classes of $g_1, \dots, g_s \in \mathscr{O}_n$, Martinet's theorem ([20], theorem 7.2) states that the map germ

$$F(u, x) = (u, f(x) + u_1 g_1(x) + \cdots + u_s g_s(x))$$

is a stable unfolding of f. While such a result fails to yield frontal unfoldings of frontal map germs, if f has corank 1, we can still make use of the frontal sum operation defined in § 5 to formulate a frontal version of Martinet's theorem.

Lemma 6.6. *Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ be a frontal map germ of integral corank* 1 *with frontal unfolding $F = (u, f_u)$, and (u, y) be local coordinates on $(\mathbb{K}^d \times \mathbb{K}^{n+1}, 0)$.*

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There is an $\mathscr{O}_{n+d+1}$-linear isomorphism

$$\beta\colon \frac{\mathscr{F}(F)}{T\mathscr{K}_{\mathscr{F}e}F} \longrightarrow \frac{\mathscr{F}(f)}{T\mathscr{K}_{\mathscr{F}e}f}$$

induced by the $\mathscr{O}_{n+d}$-linear epimorphism $\beta_0\colon \theta(F) \to \theta(f)$ sending ∂y_i onto ∂y_i for $i = 1, \ldots, n+1$ and ∂u_j onto $-\dot{F}_j$ for $j = 1, \ldots, d$.

Proof. In [20], lemma 5.5, it is shown that β_0 induces a $\mathscr{O}_{n+d}$-linear isomorphism $\beta_1\colon T^1_{\mathscr{K}_e}F \to T^1_{\mathscr{K}_e}f$. In particular, we can consider β_0 as a $\mathscr{O}_{n+d+1}$-epimorphism via F^*. Note that $T\mathscr{K}_{\mathscr{F}e}g = T\mathscr{K}_e g \cap \mathscr{F}(g)$ for any frontal map germ g with integral corank 1, so it suffices to show that β_0 sends $\mathscr{F}(F)$ onto $\mathscr{F}(f)$.

Let $\xi \in \theta(F)$ with integral $\mathscr{F}$-curve F_t: the integral $\mathscr{F}$-curve for $\beta_0(\xi)$ is given by

$$f_t = i^*(\pi \circ F_t); \quad \pi(t, u, y) = (t, y); \quad i(x) = (0, x).$$

In particular, if (t, F_t) is a frontal, (t, f_t) is also frontal, since the image of (t, f_t) is embedded within the image of (t, F_t). Conversely, given a frontal unfolding (t, f_t) of f, the map (t, u, f_t) is a frontal unfolding of F with $f_t = i^*(\pi \circ F_t)$, hence $\beta_0(\mathscr{F}(F)) = \mathscr{F}(f)$. □

As a consequence of lemma 6.6, if $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ is a stable frontal map germ, it is either the versal unfolding of some frontal map germ of rank 0 or a prism (i.e. a trivial unfolding) thereof.

THEOREM 6.7. *Let $\gamma\colon (\mathbb{K}, 0) \to (\mathbb{K}^2, 0)$ be the plane curve from remark 5.2, and*

$$T_j(t) = (t^j, B_j(t)), \quad B_j(t) = j \int_0^t s^{j-1}\mu(s)\, \mathrm{d}s.$$

If $\mathscr{F}_0(\gamma) = \mathscr{F}(\gamma) \cap \mathfrak{m}_1\theta(\gamma)$, then

$$\mathrm{Sp}_{\mathbb{K}}\{T_1, \ldots, T_{\alpha-2}\} \hookrightarrow \frac{\mathscr{F}_0(\gamma)}{T\mathscr{K}_{\mathscr{F}e}\gamma} \hookrightarrow \mathrm{Sp}_{\mathbb{K}}\{T_1, \ldots, T_{\alpha-2}, (0, t^\alpha), \ldots, (0, t^{2\alpha-1})\}.$$

Proof. Let $\xi = (a, b) \in \theta(\gamma)$: by remark 3.13, $\xi \in \mathscr{F}(\gamma)$ if and only if $b' - \mu a' \in \mathfrak{m}_1^{\alpha-1}$, which in turn is equivalent to assuming that $b' - \mu a' \equiv \lambda_1 T_1' + \cdots + \lambda_{\alpha-2}T_{\alpha-2}' \mod \mathfrak{m}_1^{\alpha-1}$ for some $\lambda_1, \ldots, \lambda_{\alpha-2} \in \mathbb{K}$. Therefore,

$$\mathscr{F}(\gamma) = \mathbb{K} \oplus \mathrm{Sp}_{\mathbb{K}}\{T_1, \ldots, T_{\alpha-1}\} \oplus \mathfrak{m}_1^\alpha\theta(\gamma). \tag{6.6}$$

A simple computation shows that $T\mathscr{K}_{\mathscr{F}e}\gamma \subseteq \mathfrak{m}_1^{\alpha-1}\theta(\gamma)$, hence $T_j \notin T\mathscr{K}_{\mathscr{F}e}\gamma$ for $j < \alpha - 1$. However, $T_{\alpha-1} \in t\gamma(\theta_1)$, giving the first monomorphism. For the second monomorphism, first note that γ is finitely determined, so there exists a $k > 0$ such that $\mathfrak{m}_1^{k+1}\theta(\gamma) \subseteq T\mathscr{A}_e\gamma \subseteq T\mathscr{K}_{\mathscr{F}e}\gamma$. If $j = \alpha, \ldots, k$, there exist $l > 0$ and $0 \leqslant \beta < \alpha$ such that $j = l\alpha + \beta$. Using equation (6.6), we see that

$$(t^j, 0) = (t^\alpha)^l(t^\beta, 0) = (t^\alpha)^l T_\beta(t) + (t^\alpha)^l(0, B_\beta(t)) \in \mathfrak{m}_2\mathscr{F}(\gamma) \subseteq T\mathscr{K}_{\mathscr{F}e}\gamma.$$

Similarly, $(0, t^j) \in T\mathscr{K}_{\mathscr{F}e}\gamma$ for all $j \geqslant 2\alpha$. □

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If we now consider the 1-parameter unfolding $\Gamma_j(u,t) = (u, \gamma(t) + uT_j(t))$,

$$\frac{\partial}{\partial t}(t^{\alpha+1}h(t) + uB_j(t)) = \mu(t)\frac{\partial}{\partial t}(t^\alpha + jut^j)$$

and Γ_j is frontal due to corollary 2.7. Similarly, if we set $\Gamma_k(u,t) = (u, \gamma(t) + ut^\alpha k(t))$ with $k \in \mathscr{O}_1^2$,

$$Q_t(u,t) = \frac{\partial}{\partial t}(t^{\alpha+1}h(t) + ut^\alpha k_2(t)) = t^{\alpha-1}(\alpha\mu(t) + uk_2(t) + utk_2'(t));$$
$$P_t(u,t) = \frac{\partial}{\partial t}(t^\alpha + ut^\alpha k_1(t)) = t^{\alpha-1}(\alpha + \alpha uk_1(t) + tk_1'(t)).$$

Since $\alpha + \alpha uk_1(t) + tk_1'(t)$ is a unit, $P_t \,|\, Q_t$ and Γ_k is also frontal.

If $\mathscr{F}_0(\gamma) = T\mathscr{K}_{\mathscr{F}e}\gamma + \mathrm{Sp}_{\mathbb{K}}\{T_{j_1}, \dots, T_{j_d}, k_1, \dots, k_b\}$ for some $k_1, \dots, k_b \in \mathfrak{m}_1^\alpha \mathscr{O}_1^2$, we consider the $(d+b)$-parameter frontal unfolding

$$F(u,t) = \Gamma_{j_1}(u_1,t)\# \dots \#\Gamma_{j_d}(u_d,t)\#\Gamma_{k_1}(u_{d+1},t)\# \dots \#\Gamma_{k_b}(u_{d+b},t), \tag{6.7}$$

where $\#$ denotes the frontal sum operation defined in equation (5.4).

EXAMPLE 6.8. Let $f\colon (\mathbb{K},0) \to (\mathbb{K}^2,0)$ be the plane curve $f(t) = (t^3, t^5)$, which verifies that $\mathscr{F}_0(f) = T\mathscr{K}_{\mathscr{F}e}f \oplus \mathrm{Sp}_{\mathbb{K}}\{(9t, 5t^3), (0, t^4)\}$. We then consider the 1-parameter unfoldings

$$F_1(t,v) = (v, t^3, t^5 + vt^4); \quad F_2(t,u) = (u, t^3 + 9ut, t^5 + 5ut^3),$$

whose frontal sum is

$$F(t,u,v) = \left(u, v, t^3 + 9ut, t^5 + 5ut^3 + \frac{1}{3}vt^4 + 6uvt^2\right).$$

This unfolding is $\mathscr{A}$-equivalent to the $A_{3,1}$ singularity from [**14**], example 4.2.

THEOREM 6.9. *The map germ $F\colon (\mathbb{K}^d \times \mathbb{K}^b \times \mathbb{K}, 0) \to (\mathbb{K}^d \times \mathbb{K}^b \times \mathbb{K}^2, 0)$ defined in equation (6.7) is stable as a frontal. Moreover, if the $\mathbb{K}$-codimension of $T\mathscr{K}_{\mathscr{F}_e}f$ over $\mathscr{F}_0(f)$ is $d+b$, every other stable frontal unfolding of f must have at least $d+b$ parameters.*

Proof. It is clear by definition of $T\mathscr{K}_{\mathscr{F}e}F$ that

$$\mathscr{F}_0(F) \supseteq T\mathscr{K}_{\mathscr{F}e}F \supseteq T\mathscr{A}_eF \cap \mathfrak{m}_{d+b+1}\theta(F),$$

so F is $\mathscr{F}$-stable if and only if $\mathscr{F}_0(F) = T\mathscr{K}_{\mathscr{F}e}F$. By lemma 6.6, this is equivalent to

$$\mathscr{F}_0(f) = T\mathscr{K}_{\mathscr{F}e}f + \mathrm{Sp}_{\mathbb{K}}\left\{-\dot{F}_1, \dots, -\dot{F}_{d+b}\right\}.$$

It follows from the definition of frontal sum that

$$\dot{F}_i(t) = (P_{u_i}(0,t), Q_{u_i}(0,t)) = \begin{cases} \dot{\Gamma}_{j_i}(t) & \text{if } i \leqslant d; \\ \dot{\Gamma}_{k_i}(t) & \text{if } i > d, \end{cases}$$

and thus F is stable. $\square$

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6.2. Corank 1 stable frontal map germs in dimension 3

By theorem 6.3, a frontal multigerm $f\colon(\mathbb{K}^3,S)\to(\mathbb{K}^4,0)$ is $\mathscr{F}$-stable if and only if its branches $f_1,\dots,f_r$ are $\mathscr{F}$-stable and $\tau(f_1),\dots,\tau(f_r)$ meet in general position. Therefore, we only need to classify the stable monogerms.

By lemma 6.6, every $\mathscr{F}$-stable monogerm with corank 1 is a versal unfolding of an irreducible analytic plane curve γ with $\mathscr{F}_e$-codimension at most 2. In particular, if $\gamma(\mathbb{C},0)$ is the zero locus of some analytic $g\in\mathscr{O}_2$, $\tau(g)-\delta(g)\leqslant\operatorname{ord}(g)+1$ due to corollary 5.11. A consequence of theorem 6.7 is that $\operatorname{codim}_{\mathscr{K}_{\mathscr{F}e}}\gamma\geqslant\operatorname{ord}(g)$, meaning that $\operatorname{ord}(g)$ must be at most 4.

If $\operatorname{ord}(g)=2$, it follows from a result by Zariski [**30**] that $g(x,y)=x^2-y^{2n+1}$. For $n=0,1$, this yields an $\mathscr{F}$-stable plane curve; for $n>1$, we can unfold $\gamma(t)$ into

$$\Gamma_n(u,t)=(u,t^2,t^{2n+1}+ut^3),$$

which is stable.

The cases $\operatorname{ord}(g)=3$ and $\operatorname{ord}(g)=4$ will be examined using Hefez and Hernandes' classification of analytic plane curves from [**10**]. Every analytic plane curve has an associated invariant $\Sigma=\langle v_0,\dots,v_g\rangle$, known as the **semigroup of values**. If the curve is irreducible, its delta invariant δ is equal to

$$\frac{1}{2}\left[1-v_0-\sum_{i=1}^{g}v_i\left(1-\frac{\operatorname{GCD}(v_0,\dots,v_{i-1})}{\operatorname{GCD}(v_0,\dots,v_i)}\right)\right],$$

regardless of its analytic family. Therefore, the expression $\tau-\delta$ only depends on τ.

For $\operatorname{ord}(g)=3$, Σ is given by $\langle 3,v_1\rangle$ with $v_1>3$, so $\delta=v_1-1$. If $\tau=2(v_1-1)$, $\tau-\delta=v_1-1<4$, so $g(x,y)$ is either x^3-y^4 or x^3-y^5. The case $\tau=2v_1-j-1$ with $j\geqslant 2$ implies that $\tau<\delta$, which is impossible.

For $\operatorname{ord}(g)=4$, Σ can be either $\langle 4,v_1\rangle$ or $\langle 4,v_1,v_2\rangle$. If $\Sigma=\langle 4,v_1\rangle$, v_1 is coprime with 4, so $\delta=3/2(v_1-1)$ and we have two possible values for τ:

(1) if $\tau=3(v_1-1)$, $\tau-\delta=3/2(v_1-1)\leqslant 5$, which implies that $\tau<\delta$;

(2) if $\tau=3v_1-j-2$ with $j>1$,

$$\tau-\delta=\frac{1}{2}(3v_1-2j-1)\leqslant 5\implies j\geqslant\frac{1}{2}(3v_1-11).$$

Since $j\leqslant v_1/2$, it follows that $v_1\geqslant 3v_1-11$, giving us $\gamma(t)=(t^4,t^5+t^7)$.

If $\Sigma=\langle 4,v_1,v_2\rangle$, $\operatorname{GCD}(4,v_1)=2$ and $\operatorname{GCD}(4,v_1,v_2)=1$, which implies that $v_1\geqslant 6$ and $v_2\geqslant 2v_1$. Using

$$\delta=\frac{1}{2}(v_2+v_1-3);\quad \tau=v_2+\frac{1}{2}v_1-2,$$

it follows that $\tau-\delta=(v_2-1)/2>5$. Since we are only interested in the case $\tau-\delta\leqslant 5$, we can ignore this case.

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Table 1. *Stable proper frontal map germs* $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$. *The notation* $A_{i,j}$ *is due to Ishikawa [14].*

Plane curve		Versal frontal unfolding
(t^2, t^3)	$A_{2,0}$	(u, v, t^2, t^3)
(t^2, t^5)	$A_{2,1}$	$(u, v, t^2, t^5 + ut^3)$
(t^3, t^4)	$A_{3,0}$	$(u, v, t^3 + 3ut, 3t^4 + 2ut^2)$
(t^3, t^5)	$A_{3,1}$	$(u, v, t^3 + tu, t^5 + vt^4 + 2uvt^2 - 5u^2t)$
$(t^4, t^5 + t^7)$	$A_{4,0}$	$(u, v, t^4 + 8tu, t^7 + t^5 + t^3v(5 - 14v) + t^2u(5 - 42v) - 28tu^2)$

For the remaining cases, the possible values for $\tau - \delta$ fall into one of the following categories:

$$\frac{3(v_1 - 1)}{2} + k - \left[\frac{v_1}{4}\right]; \quad \frac{3(v_1 - 1)}{2} - 2j + 1; \quad \frac{3(v_1 - 1)}{2} - 2j + 2,$$

for $2 \leqslant j \leqslant [v_1/4]$ and $1 \leqslant k \leqslant [v_1/4] - j$. If $\tau - \delta \leqslant 5$, then $v_1 \geqslant 7$, which is not possible.

THEOREM 6.10. Table 1 *shows all stable proper frontal map germs* $(\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ *of corank* 1 *together with the plane curves of which they are versal unfoldings. All stable frontal multigerms are obtained by transverse self-intersections of these mono-germs, as shown in theorem* 6.3.

Proof. The discussion conducted throughout this subsection shows that the only plane curves of frontal codimension less than or equal to 2 are (t^2, t^3), (t^2, t^5), (t^2, t^7), (t^3, t^4), (t^3, t^5) and $(t^4, t^5 + t^7)$. The curve (t^2, t^3) is easily checked to be stable as a frontal. The family of curves (t^2, t^{2k+1}) for $k > 1$ unfolds into $(s, t) \mapsto (s, t^2, t^{2k+1} + st^3)$, which is $\mathscr{A}$-equivalent to the folded Whitney umbrella $(s, t) \mapsto (s, t^2, st^3)$, which is stable as a frontal [**22**, **23**].

The curves (t^3, t^4) and $(t^4, t^5 + t^7)$ unfold into the swallowtail and butterfly singularities ($A_{3,0}$ and $A_{4,0}$ in Table 1), both of which are stable wave fronts ([**27**]). The E_8 singularity unfolds into Ishikawa's $A_{3,1}$ singularity [**14**]. □

CONJECTURE 6.11. *Any stable proper frontal map germ* $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ *of corank* 1 *corresponds to one of Ishikawa's* $A_{i,j}$ *singularities, where*

$$i = \dim \frac{\tilde{f}^* \mathscr{O}_{2n+1}}{f^* \mathfrak{m}_{n+1}} \in \{2, \ldots, n\}; \quad j + 1 = \dim \frac{\mathscr{O}_n}{\tilde{f}^* \mathfrak{m}_{2n+1}} \in \left\{1, \ldots, \left[\frac{n}{2}\right] + 1\right\}$$

and square brackets denote the floor function. All stable frontal multigerms are obtained by transverse self-intersections of these mono-germs, as shown in theorem 6.3.

The algebra $\tilde{f}^* \mathscr{O}_{2n+1} / f^* \mathfrak{m}_{n+1}$ was introduced by Ishikawa in [**14**] in order to give a characterization of Legendrian stability.

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