SOME PROPERTIES OF COMPOSITIONS AND THEIR APPLICATION TO THE BALLOT PROBLEM

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1. Introduction and summary. This paper is a continuation of two papers [4], [5] and brings out the solution of the ballot problem in its general form.

In [5], Narayana has considered a generalised occupancy problem which can be viewed as a problem in compositions of integers. In what follows, we use the definitions of [6]. Furthermore, we say that an r-composition $(t_1(m), \ldots, t_r(m))$ of m dominates an r-composition $(t_1(n), \ldots, t_r(n))$ of n $(m \geq n)$ if and only if

$$(1) \qquad \sum_{\alpha=1}^{i} t_\alpha(m) \geq \sum_{\alpha=1}^{i} t_\alpha(n), \qquad \text{for } i = 1, \ldots, r.$$

Evidently $\sum_{\alpha=1}^{r} t_\alpha(m) = m$ and $\sum_{\alpha=1}^{r} t_\alpha(n) = n$. For integers $n_1, \ldots, n_k$ such that $n_1 \geq \ldots \geq n_k$, we are required in [5] to determine the number of r-compositions of n_1 that dominate r-compositions of n_2, that in turn dominate r-compositions of n_3, and so on. In other words, we are looking for the number of elements in the set $C = C(n_1, \ldots, n_k; r)$ $= \{(t_1(n_1), \ldots, t_r(n_1)), \ldots, (t_1(n_k), \ldots, t_r(n_k)) :$

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(2) $$\sum_{\alpha=1}^{i} t_\alpha(n_j) \geq \sum_{\alpha=1}^{i} t_\alpha(n_{j+1})$$

for $i = 1, \ldots, r$ and $j = 1, \ldots, k-1\}$.

Now, corresponding to C, consider the set of lattice paths in a k-dimensional Euclidean space with axes X_ℓ's such that the $(k(i-1)+\ell)$th segment $(i = 1, \ldots, r$ and $\ell = 1, \ldots, k)$ of any path is the distance $t_i(n_\ell)$ on X_ℓ. Suppose a step in a path represents k consecutive segments beginning from the segment on X_1. Then the set consists of paths from the origin to $(n_1, \ldots, n_k)$ not crossing the region bounded by $X_k = 0$, $X_j = X_{j+1}$, $j = 1, \ldots, k-1$ and having exactly r components. Denote this set by $L^*(n_1, \ldots, n_k; r)$ or briefly L^*. Thus the above construction has established a 1:1 correspondence between C and L^*. Letting $N\{.\}$ represent the number of elements in the set $\{.\}$, it is shown in [5] that,

(3) $$N\{C\} = N\{L^*\} = (n_1, \ldots, n_k)_r$$

where

(4) $$(n_1, \ldots, n_k)_r = \begin{vmatrix} \binom{n_1-1}{r-1} & \binom{n_2-1}{r} & \cdots & \binom{n_k-1}{r+k-2} \\ \binom{n_1-1}{r-2} & \binom{n_2-1}{r-1} & \cdots & \binom{n_k-1}{r+k-3} \\ \vdots & \vdots & \vdots & \vdots \\ \binom{n_1-1}{r-k} & \binom{n_2-1}{r-k+1} & \cdots & \binom{n_k-1}{r-1} \end{vmatrix} .$$

The determinant (4) plays an important role in this paper. It is also proved that $(n_1, \ldots, n_k)_r$ satisfies the following:

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$$(5) \qquad \sum_{x_k \geq r}^{n_k - 1} \; \sum_{x_{k-1} \geq x_k}^{n_{k-1} - 1} \cdots \sum_{x_1 \geq x_2}^{n_1 - 1} (x_1, \ldots, x_k)_r = (n_1, \ldots, n_k)_{r+1} .$$

In section 2, we define a partial order on C and establish an isomorphism between C and a set of compositions of $M > n_1 - r + k$, which is specified later. We also show that for $r = 2$, it leads to an interesting correspondence among two different sets of lattice paths and a set of lattice points. As a special case, the number of r-compositions of n that are [p]-dominated by a given r-composition of m has been evaluated (for definition see [4] section 4). Section 3 deals with the application of results in section 2, in order to provide a solution to a generalised class of the ballot problem [2, p. 66]. Finally, some identities which arise as a natural consequence of the above are included in the last section.

2. Isomorphism between two sets of compositions. Recalling the definition of a composition vector [6] (that is, defining

$$T_i^*(n_j) = \sum_{\alpha=1}^{i} t_\alpha(n_j)),$$

it is remarked that the set C is trivially in 1:1 correspondence with the set $T^* = T^*(n_1, \ldots, n_k; r) = \{ (T_1^*(n_1), \ldots, T_r^*(n_1)), \ldots, (T_1^*(n_k), \ldots, T_r^*(n_k)):$

$$(6) \qquad T_i^*(n_j) \geq T_i^*(n_{j+1})$$

for $i = 1, \ldots, r$ and $j = 1, \ldots, k-1\}$. Because of this correspondence and somewhat relative advantage of T^* over C, we refer frequently to T^* instead of C.

Definition: Given two elements $\tau_1^* = \{ (T_{11}^*(n_1), \ldots, T_{r1}^*(n_1)), \ldots (T_{11}^*(n_k), \ldots, T_{r1}^*(n_k)): T_{i1}^*(n_j) \geq T_{i1}^*(n_{j+1})$ for all i and j$\}$ and $\tau_2^* = \{ (T_{12}^*(n_1), \ldots, T_{r2}^*(n_1)), \ldots, (T_{12}^*(n_k), \ldots, T_{r2}^*(n_k)): T_{i2}^*(n_j) \geq T_{i2}^*(n_{j+1})$ for all i and j$\}$

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of T^*, we say that τ_1^* dominates τ_2^* if and only if

$$T_{i1}^*(n_j) \geq T_{i2}^*(n_j) \text{ for } i = 1, \ldots, r \text{ and } j = 1, \ldots, k .$$

It can be shown that this relation is a partial order defined on elements of T^*. Proceeding in a manner as in [6], we can also prove the following lemma.

LEMMA. The elements in T^* form a distributive lattice.

Next, it is easy to verify that the inequalities (6) for all i and j are satisfied if and only if

$$(7) \qquad T_{i+1\,j} \geq T_{ij} \text{ for } i = 1, \ldots, r-1 \text{ and } j = 1, \ldots, k ,$$

where $T_{ij} = T_i^*(n_j) - i - j + k + 1$ and $T_{rj} = n_j - r - j + k + 1$. Since $T_{ik} < \ldots < T_{i1}$ follows from (7), we now consider $k+1$ composition vectors $(T_{ik}, \ldots, T_{i1}, M)$ for $i = 1, \ldots, r$ where $M > n_1 - r + k$ is a constant. Because of the inequalities in (7), we notice that $M > T_{i1}$ for $i = 1, \ldots, r$. Let $T = T(n_1, \ldots, n_k; r)$ be the set

$$\{(T_{rk}, \ldots, T_{r1}, M), \ldots, (T_{1k}, \ldots, T_{11}, M):$$

$$T_{i+1\,j} \geq T_{ij} \text{ for } i = 1, \ldots, r-1 \text{ and } j = 1, \ldots, k\} .$$

In terms of compositions, it may be seen that T is the set such that $(k+1)$-composition $(n_k - r+1, n_{k-1} - n_k + 1, \ldots, n_1 - n_2 + 1, M - n_1 + r - k)$ of M dominates $(k+1)$-compositions of M, each of which again dominates $(k+1)$-compositions of M and so on. Using the simple transform $a_{ij} = T_i^*(n_j) - i$, define the set $S = S(n_1, \ldots, n_k; r)$ to be $\{(a_{rk} = n_k - r, \ldots, a_{r1} = n_1 - r), \ldots, (a_{1k}, \ldots, a_{11}):$

$$(8) \qquad a_{i+1\,j} \geq a_{ij} \quad \text{for } i = 1, \ldots, r-1 \text{ and } j = 1, \ldots, k\}$$

where $(a_{ik}, \ldots, a_{i1})$ is a vector of non-negative, nondecreasing

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integers. The relation of domination on T^* through 1:1 transformations used above is extended to T and S and therefore we have the theorem.

THEOREM 1. Sets T^*, T and S are isomorphic distributive lattices.

We have shown in section 1 that $T^*(n_1, \ldots, n_k; 2)$ through $C(n_1, \ldots, n_k; 2)$ is 1:1 to $L^*(n_1, \ldots, n_k; 2)$. Also $T^*(n_1, \ldots, n_k; 2)$ is 1:1 to $S(n_1, \ldots, n_k; 2)$ by the theorem. But $S(n_1, \ldots, n_k; 2)$ is the set $\{(a_k, \ldots, a_1)\}$ of all vectors of non-negative and nondecreasing integers such that

$$(9) \qquad 0 \leq a_j \leq n_j - 2 \qquad \text{for } j = 1, \ldots, k.$$

Now, using the construction of lattice paths from non-negative nondecreasing vectors [4, p. 253], we notice that $S(n_1, \ldots, n_k; 2)$ is 1:1 to the set $L(n_1, \ldots, n_k)$ of lattice paths from $(0, 0)$ to $(n_1 - 1, k)$ not crossing the boundary given by the points $(0, 0,)$, $(1, 1)$, $(n_1 - n_2 + 1, 2)$, $(n_1 - n_3 + 1, 3), \ldots, (n_1 - n_k + 1, k)$, Here we have two remarks to offer:

(a) The above lattice paths are equivalent to paths from $(0, 0)$ to (n_i, k) not touching the same boundary;

(b) The set of paths are, in general, also equivalent to paths from $(0, 0)$ to $(n_1 + e - 2, k)$ [or $(n_1 + e - 1, k)$] not crossing [or not touching] the boundary $(0, 0)$, $(e, 1)$, $(n_1 - n_2 + e, 2)$, $(n_1 - n_3 + e; 3), \ldots, (n_1 - n_k + e, k)$, where e is a positive integer. From section 1,

$$(10) \qquad N\{L^*(n_1, \ldots, n_k; 2)\} = (n_1, \ldots, n_k)_2 .$$

We observe from (4) and (5) that

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$$(11)\qquad \sum_{x_k=1}^{n_k-1}\ \sum_{x_{k-1}\geq x_k}^{n_{k-1}-1} \cdots \sum_{x_1\geq x_2}^{n_1-1} 1 = (n_1,\ldots,n_k)_2 .$$

The expression on the left hand side of (11) represents the number of lattice points in the region $R(n_1,\ldots,n_k)$ of k-dimensional Euclidean space bounded by hyperplanes

$$X_k = \frac{1}{2},\quad X_k = n_k - \frac{1}{2},\quad X_{k-1} = X_k - \frac{1}{2},\quad X_{k-1} = n_{k-1} - \frac{1}{2},\ \ldots$$

$$X_1 = X_2 - \frac{1}{2},\quad X_1 = n_1 - \frac{1}{2}.$$

Thus, as a corollary of Theorem 1, we see that:

COROLLARY 1. $N\{L^*(n_1,\ldots,n_k;2)\} = N\{L(n_1,\ldots,n_k)\} = N\{R(n_1,\ldots,n_k)\} = (n_1,\ldots,n_k)_2$.

It is not difficult to observe that $T(n_1,\ldots,n_k;2)$ represent the set of $(k+1)$-composition vectors $(T_1,\ldots,T_k,M)$ which are dominated by the $(k+1)$-composition vector

$$(n_k-1, n_{k-1},\ldots,n_1+k-2, M),\quad M > n_1 + k - 2 .$$

Thus according to [6], the number $N\{T(n_1,\ldots,n_k;2)\}$ is give by D_k in the recursive formula

$$(12)\qquad \begin{cases} D_0 = 1 \\ D_u = \displaystyle\sum_{\alpha=1}^{u} (-1)^{\alpha+1} \binom{n_{k-u+\alpha}+u-2}{\alpha} D_{u-\alpha} . \end{cases}$$

We know from Theorem 1 that

$$N\{T(n_1,\ldots,n_k;2)\} = (n_1,\ldots,n_k)_2 .$$

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Therefore, we have

COROLLARY 2. A solution of D_u in (12) is $(n_{k-u+1}, n_{k-u+2}, \ldots, n_k)_2$.

A direct proof is also possible. We indicate it here. Using induction, D_u can be written as the determinant

$$(13)\qquad \begin{vmatrix} \binom{n_{k-u+1}+u-2}{1} & \binom{n_{k-u+2}+u-2}{2} & \cdots & \binom{n_k+u-2}{u} \\ \binom{n_{k-u+1}+u-3}{0} & \binom{n_{k-u+2}+u-3}{1} & \cdots & \binom{n_k+u-3}{u-1} \\ 0 & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & \binom{n_k-1}{1} \end{vmatrix} .$$

Subtraction of the i^{th} row from the (i-1)st row $(i = 2, \ldots, u)$, and repetition of this process reduces (13) to $(n_{k-u+1}, n_{k-u+2}, \ldots, n_k)_2$.

We now consider a problem, the solution of which is obtained with the help of Corollary 1. It is required to determine the number of r-compositions of n that are [s]-dominated by the r-composition $(t_1(m), \ldots, t_r(m))$ of m $(m \geq sn)$ [4, page 254]. The r-composition $(t_1(m), \ldots, t_r(m))$ of m [s]-dominates an r-composition $(t_1(n), \ldots, t_r(n))$ of n if and only if

$$(14)\qquad T_i(m) \geq sT_i(n)$$

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for $i = 1, \ldots, r$. Inequalities (14) are equivalent to

$$\left[\frac{T_i(m)}{s}\right] \geq T_i(n)$$

for $i = 1, \ldots, r$, where $[z]$ is the greatest integer less than or equal to z. Thus we are interested in the set of r-compositions of n that are dominated by the r-composition

$$\left(\left[\frac{T_1(m)}{s}\right], \left[\frac{T_2(m)}{s}\right] - \left[\frac{T_1(m)}{s}\right], \ldots, \left[\frac{m}{s}\right] - \left[\frac{T_{r-1}(m)}{s}\right]\right)$$

of $[\frac{m}{s}]$. Transforming the set to the set of non-negative and non-decreasing vectors, as done earlier, we observe that the above set is 1:1 with the set of vectors $(a_1, \ldots, a_{r-1})$ such that

(i) a_i's are non-negative integers,

(ii) $a_1 \leq \ldots \leq a_{r-1}$,

(iii) $0 \leq a_i \leq \min\left(\left[\frac{T_i(m)}{s}\right] - i, n-r\right)$ for $i = 1, \ldots, r-1$.

From the discussion following Theorem 1, we can get the number of such vectors, which is stated as a theorem.

THEOREM 2. The number of r-compositions of n that are [s]-dominated by the r-composition $(t_1(m), \ldots, t_r(m))$ of m $(m \geq sn)$ is

$$\min\left(\left[\frac{T_{r-1}(m)}{s}\right] - r+1, n-r\right) + 2,$$

$$\min\left(\left[\frac{T_{r-2}(m)}{s}\right] - r+2, n-r\right) + 2, \ldots,$$

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$$\min\left(\left[\frac{T_1(m)}{s}\right] - 1,\ n-r \quad + 2\right)_2 .$$

3. Generalised ballot problems. The ballot problem [2, p. 66] and its extension have been discussed by several authors [1], [4], [7]. We state it as follows:

If in a ballot, Candidate A scores a votes and Candidate B scores b votes, where $a > b\mu$, μ being a positive integer, what is the probability that at each instant A's vote exceeds μ times B's vote?

Representing each vote for A by a unit horizontal step and each vote for B by a unit vertical step, one of the solutions suggested in [4] uses the correspondence between lattice paths and non-negative non-decreasing vectors. In fact, the ballot problem with two candidates, in a generalised form, involves counting of lattice paths not touching a certain boundary which lies to the left of the paths. Recalling remarks (a) and (b), we have obtained the solution to such a problem in Section 2. In this context, we present below two theorems, the proof of which obviously follows from the preceeding results.

THEOREM 3. Let x and y respectively represent votes for A and B at a particular instant. Suppose that A scores a votes and B scores b votes such that $a > b\mu + \nu$, μ and ν being non-negative numbers. The number of ways in which $x > y\mu + \nu$ happens is given by

$$(a-[\mu+\nu]+1,\ a-[2\mu+\nu]+1, \ldots, a-[b\mu+\nu]+1)_2 .$$

At this point we note that Takács [7] gives a solution for general μ and $\nu = 0$. When μ is a positive integer and $\nu = 0$, the ballot problem reduces to the case stated at the beginning of this section. Therefore, the required number is $(a-\mu+1, \ldots, a-b\mu+1)_2 = (a, b, \mu)$ say. We have to show that

$$(a, b, \mu) = \frac{a-b\mu}{a+b}\binom{a+b}{b} .$$

For $b = 1$, the result is true for all a and μ. Adding

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each row to the previous row in the determinant (a, b, μ), we obtain

$$(a, b, \mu) = (a+1, b, \mu) - (a+1, b-1, \mu) .$$

Hence

$$(12) \quad (a+1, b, \mu) = (a+1, b-1, \mu) + (a, b, \mu)$$

$$= (a+1, b-1, \mu) + (a, b-1, \mu) + (a-1, b, \mu)$$

$$= (a+1, b-1, \mu) + (a, b-1, \mu) + \ldots + (b\mu+1, b-1, \mu) ,$$

because $(b\mu, b, \mu) = 0$. Applying induction, we get from (12) that

$$(a+1, b, \mu) = \sum_{\alpha=b\mu+1}^{a+1} (\alpha, b-1, \mu) = \sum_{\alpha=b\mu+1}^{a+1} \frac{\alpha-(b-1)\mu}{\alpha+b-1} \binom{\alpha+b-1}{b-1}$$

$$= \sum_{\alpha=b\mu+1}^{a+1} \binom{\alpha+b-2}{b-1} - \mu \sum_{\alpha=b\mu+1}^{a+1} \binom{\alpha+b-2}{b-2}$$

$$= \binom{a+b}{b} - \mu\binom{a+b}{b-1} = \frac{a-b\mu+1}{a+b+1} \binom{a+b+1}{b} ,$$

and the result follows.

Another variation of the ballot problem is given below, and the result will be used in the next section.

THEOREM 4. For A and B having a and b votes respectively, where $a > b_1\mu_1 + \nu_1 + (b-b_1)\mu_2 + \nu_2$, μ_1, ν_1, μ_2 and ν_2 being non-negative numbers and $b_1 \leq b$ a non-negative integer, the number of ways in which $x > y\mu_1 + \nu_1$ when $0 \leq y \leq b_1$, and $x > b_1\mu_1 + \nu_1 + (y-b_1)\mu_2 + \nu_2$ when $b_1 \leq y \leq b$ can happen is

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$$(a - [\mu_1 + \nu_1] + 1, \ldots, a - [b_1\mu_1 + \nu_1] + 1,$$

$$a - [b_1\mu_1 + \nu_1 + \mu_2 + \nu_2] + 1, \ldots,$$

$$a - [b_1\mu_1 + \nu_1 + (b-b_1)\mu_2 + \nu_2] + 1)_2 .$$

The above theorems illustrate the use of the results developed in Section 2, in some simple boundary cases.

4. Some combinatorial identities. The two A. P. case of [4, p. 256-258] is a special case of Theorem 4, with μ_1, ν_1, μ_2 and ν_2 as non-negative integers. Using the same notation as in [4], we therefore get

$$(13)\quad N\{A_{p,q}(a+1,b+1;\, c+1,d+1)\} = N_{p,q}(a+1,b+1;\, c+1,d+1)$$

$$= (a+(p-1)b+c+(q-1)d+2,\ a+(p-1)b+c+(q-2)d+2;\ \ldots,$$

$$a+(p-1)b+c+2,\ a+(p-1)b+2,\ a+(p-2)b+2, \ldots, a+2)_2$$

$$= \sum_{k=0}^{q} (-1)^k \frac{a+1}{a+1+(p+q-k)(b+1)} \binom{a+1+(p+q-k)(b+1)}{p+q-k} .$$

$$\frac{(q-k+1)b-c-(q-k)d}{(q-k+1)b-c-qd} \binom{(q-k+1)b-c-qd}{k}$$

by Theorem 4 and Theorem 3 of [4]. Put $b = 1$, $c = 1$, $d = 0$. Then $A_{p,q}(a+1, 2, 2, 1)$ is 1:1 with the set of paths from (0, 0) to $(p+a, p+q)$ not touching the line $x + q + 1 = y$, and the number of such paths is equal to

$$\binom{2p+q+a}{p+q} - \binom{2p+q+a}{p-1}$$

by [3]. Therefore, we have an identity

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(14) $(\underbrace{a+p+2,\ldots,a+p+2}_{q},a+p+1,a+p,\ldots,a+2)_2$

$$= \sum_{k=0}^{q} (-1)^k \frac{a+1}{a+1+2(p+q-k)} \binom{a+1+2(p+q-k)}{p+q-k} \binom{q-k}{k}$$

$$= \binom{2p+q+a}{p+q} - \binom{2p+q+a}{p-1} .$$

Consider $(p+a, p+q)$ as the origin, $x = p+a$, $y = p+q$ as x-axis and y-axis respectively, such that the old origin becomes $(p+q, p+a)$. Thus the previous set of paths is the same as the set of paths from $(0, 0)$ to $(p+q, p+a)$ not touching $x + a + 1 = y$. The number in the latter set gives rise to the identity

(15) $(\underbrace{p+q+2,\ldots,p+q+2}_{a},\ p+q+1,\ldots,q+2)_2$

$$= \sum_{k=0}^{a} (-1)^k \frac{q+1}{q+1+2(p+a-k)} \binom{q+1+2(p+a-k)}{p+a-k} \binom{a-k}{k}$$

$$= \binom{2p+q+a}{p+a} - \binom{2p+q+a}{p-1} .$$

Either from the remark preceding (15) or from the obvious identity $\binom{2p+q+a}{p+q} = \binom{2p+q+a}{p+a}$, we see that (14) equals (15). We can show that

(16) $$\sum_{k=0}^{p+q} (-1)^k \frac{a+1}{a+1+2(p+q-k)} \binom{a+1+2(p+q-k)}{p+q-k} \binom{q-k}{k}$$

$$= \sum_{k=0}^{p+a} (-1)^k \frac{q+1}{q+1+2(p+a-k)} \binom{q+1+2(p+a-k)}{p+a-k} \binom{a-k}{k}$$

$$= \binom{2p+q+a}{p+q} = \binom{2p+q+a}{p+a}$$

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by formula (17) in [9]. Therefore

$$(17) \quad \sum_{k=q+1}^{p+q} (-1)^k \frac{a+1}{a+1+2(p+q-k)} \binom{a+1+2(p+q-k)}{p+q-k} \binom{q-k}{k}$$

$$= \sum_{k=a+1}^{p+a} (-1)^k \frac{q+1}{q+1+2(p+a-k)} \binom{q+1+2(p+a-k)}{p+a-k} \binom{a-k}{k}$$

$$= \binom{2p+q+a}{p-1} .$$

Perhaps some of the identities might have been proved or can be proved directly. A less obvious identity arises as follows. In the ballot problem stated in Theorem 3, set $\nu = 0$ and $\mu = \frac{a}{b+1}$ where a and b+1 are relatively prime numbers. Then an application of the result of Theorem 2 of [8] yields

$$\left(a - \left[\frac{a}{b+1}\right] + 1,\ a - \left[\frac{2a}{b+1}\right] + 1,\ \ldots,\ a - \left[\frac{ba}{b+1}\right] + 1\right)_2$$

$$= \frac{1}{a+b+1} \binom{a+b+1}{a} .$$

In conclusion, we remark that the solution in the form of a determinant might not reduce to a simpler expression, except in some special cases.

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