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SHARP INEQUALITIES FOR THE VARIATION OF THE DISCRETE MAXIMAL FUNCTION

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Abstract

In this paper we establish new optimal bounds for the derivative of some discrete maximal functions, in both the centred and uncentred versions. In particular, we solve a question originally posed by Bober *et al.* ['On a discrete version of Tanaka's theorem for maximal functions', *Proc. Amer. Math. Soc.* **140** (2012), 1669–1680].

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1. Introduction

1.1. Background. Let M denote the centred Hardy–Littlewood maximal operator on $\mathbb{R}^d$, that is, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$,

$$Mf(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)|\, dy, \tag{1.1}$$

where $B_r(x)$ is the ball centred at x with radius r and $m(B_r(x))$ is its d-dimensional Lebesgue measure. From classical results in harmonic analysis, $M : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a bounded operator for $1 < p \le \infty$ and, for $p = 1$, $M : L^1(\mathbb{R}^d) \to L^{1,\infty}(\mathbb{R}^d)$ is also bounded. In 1997, Kinnunen [11] showed that $M : W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$ is bounded for $1 < p \le \infty$ and that was the starting point for the study of the regularity of maximal operators acting on Sobolev functions. This result was later extended to multilinear, local and fractional contexts in [7, 12, 13]. Due to the lack of reflexivity of L^1, results for $p = 1$ are subtler. For example, in [10, Question 1], Hajłasz and Onninen asked whether the operator $f \mapsto |\nabla Mf|$ is bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$. Progress on this question (and its variant for functions of bounded variation) has been restricted to dimension $d = 1$.

Let $\widetilde{M}$ denote the uncentred maximal operator (defined similarly to (1.1), with the supremum taken over all balls containing the point x in its closure). Refining the work

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of Tanaka [18], Aldaz and Pérez Lázaro [2] showed that if f is of bounded variation then $\widetilde{M}f$ is absolutely continuous and

$$\operatorname{Var} \widetilde{M}f \leq \operatorname{Var} f, \tag{1.2}$$

where $\operatorname{Var} f$ denotes the total variation of f. The inequality (1.2) is sharp. Kurka [14] considered the centred maximal operator in dimension $d = 1$ and proved that

$$\operatorname{Var} Mf \leq C \operatorname{Var} f \quad \text{with } C = 240\,004. \tag{1.3}$$

It is currently unknown if one can reduce the constant to $C = 1$ in the centred case. Other interesting works related to this theory are [1, 4, 8, 9, 16, 17].

1.2. Discrete setting. In this paper we consider similar questions in the discrete setting. Let us start with some definitions.

We denote a vector $\vec{n} \in \mathbb{Z}^d$ by $\vec{n} = (n_1, n_2, \dots, n_d)$. For a function $f : \mathbb{Z}^d \to \mathbb{R}$, we define its ℓ^p-norm as usual:

$$\|f\|_{\ell^p(\mathbb{Z}^d)} = \Big(\sum_{\vec{n}\in\mathbb{Z}^d} |f(\vec{n})|^p\Big)^{1/p} \quad \text{if } 1 \leq p < \infty; \quad \|f\|_{\ell^\infty(\mathbb{Z}^d)} = \sup_{\vec{n}\in\mathbb{Z}^d} |f(\vec{n})|.$$

We define its total variation $\operatorname{Var} f$ by

$$\operatorname{Var} f = \sum_{i=1}^{d} \sum_{\vec{n}\in\mathbb{Z}^d} |f(\vec{n} + \vec{e}_i) - f(\vec{n})|,$$

where $\vec{e}_i = (0, 0, \dots, 1, \dots, 0)$ is the canonical ith basis vector. Also, we say that a function $f : \mathbb{Z}^d \to \mathbb{R}$ is a *delta function* if there exist $\vec{p} \in \mathbb{Z}^d$ and $k \in \mathbb{R}$ such that

$$f(\vec{p}) = k \quad \text{and} \quad f(\vec{n}) = 0 \quad \text{for all } \vec{n} \in \mathbb{Z}^d \setminus \{\vec{p}\}.$$

1.2.1. A sharp inequality in dimension one. For $f : \mathbb{Z} \to \mathbb{R}$, we define its centred Hardy–Littlewood maximal function $Mf : \mathbb{Z} \to \mathbb{R}^+$ as

$$Mf(n) = \sup_{r\in\mathbb{Z}^+} \frac{1}{(2r+1)} \sum_{k=-r}^{r} |f(n+k)|,$$

while the uncentred maximal function $\widetilde{M}f : \mathbb{Z} \to \mathbb{R}^+$ is given by

$$\widetilde{M}f(n) = \sup_{r,s\in\mathbb{Z}^+} \frac{1}{(r+s+1)} \sum_{k=-r}^{s} |f(n+k)|.$$

In [3], Bober *et al.* proved the inequalities

$$\operatorname{Var} \widetilde{M}f \leq \operatorname{Var} f \leq 2\|f\|_{\ell^1(\mathbb{Z})} \tag{1.4}$$

and

$$\operatorname{Var} Mf \leq \Big(2 + \frac{146}{315}\Big)\|f\|_{\ell^1(\mathbb{Z})}. \tag{1.5}$$

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The leftmost inequality in (1.4) is the discrete analogue of (1.2). The rightmost inequality in (1.4) is simply the triangle inequality. Both inequalities in (1.4) are sharp (for example, equality is attained if f is a delta function). On the other hand, (1.5) is not optimal, and it was asked in [3] whether the sharp constant for (1.5) is in fact $C = 2$. Our first result answers this question affirmatively and characterises the extremal functions.

THEOREM 1.1. *Let $f : \mathbb{Z} \to \mathbb{R}$ be a function in $\ell^1(\mathbb{Z})$. Then*

$$\operatorname{Var} Mf \le 2\|f\|_{\ell^1(\mathbb{Z})}, \tag{1.6}$$

and the constant $C = 2$ is the best possible. Moreover, the equality is attained if and only if f is a delta function.

REMARK 1.2. In [19], Temur proved the analogue of (1.3) in the discrete setting:

$$\operatorname{Var} Mf \le C \operatorname{Var} f \tag{1.7}$$

with constant $C = (72\,000)2^{12} + 4$. This inequality is qualitatively stronger than (1.6) (in fact, $\operatorname{Var} f$ should be seen as the natural object to be on the right-hand side), but it does not imply (1.6). By the triangle inequality, (1.6) suggests that it may be possible to prove (1.7) with constant $C = 1$, but this is currently an open problem.

1.2.2. Sharp inequalities in higher dimensions. We now aim to extend Theorem 1.1 to higher dimensions. In order to do so, we first recall the notion of maximal operators associated to regular convex sets as considered in [5].

Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set with Lipschitz boundary, and such that $\vec{0} \in \operatorname{int}(\Omega)$ and $\pm\vec{e}_i \in \overline{\Omega}$ for $1 \le i \le d$. For $r > 0$, write

$$\overline{\Omega}_r(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^d : r^{-1}(\vec{x} - \vec{x}_0) \in \overline{\Omega}\} \quad \text{and set } \overline{\Omega}_0(\vec{x}_0) = \{\vec{x}_0\}.$$

Whenever $\vec{x}_0 = \vec{0}$, we shall write $\overline{\Omega}_r = \overline{\Omega}_r(\vec{0})$ for simplicity. This object plays the role of the 'ball of centre $\vec{x}_0$ and radius r' in our maximal operators below. For instance, to work with regular ℓ^p-balls, consider $\Omega = \Omega_{\ell^p} = \{\vec{x} \in \mathbb{R}^d : \|\vec{x}\|_p < 1\}$, where $\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_d|^p)^{1/p}$ for $\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ and $1 \le p < \infty$, and $\|\vec{x}\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_d|\}$.

Given $f : \mathbb{Z}^d \to \mathbb{R}$, we denote by $A_r f(\vec{n})$ the average of $|f|$ over the Ω-ball of centre $\vec{n}$ and radius r, that is,

$$A_r f(\vec{n}) = \frac{1}{N(r)} \sum_{\vec{m} \in \overline{\Omega}_r} |f(\vec{n} + \vec{m})|,$$

where $N(\vec{x}, r)$ is the number of lattice points in the set $\overline{\Omega}_r(\vec{x})$ (and $N(r) := N(\vec{0}, r)$). We denote by M_Ω the discrete centred maximal operator associated to Ω,

$$M_\Omega f(\vec{n}) = \sup_{r \ge 0} A_r f(\vec{n}) = \sup_{r \ge 0} \frac{1}{N(r)} \sum_{\vec{m} \in \overline{\Omega}_r} |f(\vec{n} + \vec{m})|,$$

and we denote by $\widetilde{M}_\Omega$ its uncentred version

$$\widetilde{M}_\Omega f(\vec{n}) = \sup_{\overline{\Omega}_r(\vec{x}_0)\ni\vec{n}} A_r f(\vec{x}_0) = \sup_{\overline{\Omega}_r(\vec{x}_0)\ni\vec{n}} \frac{1}{N(\vec{x}_0, r)} \sum_{\vec{m}\in\overline{\Omega}_r(\vec{x}_0)} |f(\vec{m})|.$$

It should be understood throughout the rest of the paper that we always consider Ω-balls with at least one lattice point. These convex Ω-balls have roughly the same behaviour as the regular Euclidean balls from the geometric and arithmetic points of view, in the sense that for large radii, the number of lattice points inside the Ω-ball is roughly equal to the volume of the Ω-ball (see [15, Ch. VI, Section 2, Theorem 2]).

Given $f \in \ell^1_{\mathrm{loc}}(\mathbb{Z}^d)$, the discrete centred maximal operator associated to Ω_{ℓ^p} is

$$M_p f(\vec{n}) = M_{\Omega_{\ell^p}} f(\vec{n}) \quad \text{for } 1 \le p < \infty \quad \text{and} \quad Mf(\vec{n}) = M_{\Omega_{\ell^\infty}} f(\vec{n}) \quad \text{for } p = \infty.$$

Analogously, we denote by $\widetilde{M}_p f$ and $\widetilde{M} f$ the uncentred versions of the discrete maximal operators associated to Ω_{ℓ^p} for $1 \le p \le \infty$. Note that in dimension $d = 1$ we have $M_p = M$ and $\widetilde{M}_p = \widetilde{M}$ for all $1 \le p \le \infty$.

In [5], Carneiro and Hughes showed that, for any regular set Ω as above and $f : \mathbb{Z}^d \to \mathbb{R}$, there exist constants $C(\Omega, d)$ and $\widetilde{C}(\Omega, d)$ such that

$$\operatorname{Var} M_\Omega f \le C(\Omega, d) \|f\|_{\ell^1(\mathbb{Z}^d)} \tag{1.8}$$

and

$$\operatorname{Var} \widetilde{M}_\Omega f \le \widetilde{C}(\Omega, d) \|f\|_{\ell^1(\mathbb{Z}^d)}. \tag{1.9}$$

Inequalities (1.8) and (1.9) were extended to a fractional setting in [6, Theorem 3]. Here we extend Theorem 1.1 to higher dimensions in two distinct ways. We find the sharp form of (1.8) when $d \ge 1$ and $\Omega = \Omega_{\ell^1}$ (a rhombus), and the sharp form of (1.9) when $d \ge 1$ and $\Omega = \Omega_{\ell^\infty}$ (a regular cube). As we shall see below, we use different techniques in the proofs of these two extensions, taking into consideration the geometry of the chosen sets Ω.

For $d \ge 1$ and $k \ge 0$, we write $N_{1,d}(k) = |\overline{(\Omega_{\ell^1})_k}| = |\{\vec{x} \in \mathbb{Z}^d : \|\vec{x}\|_1 \le k\}|$.

Theorem 1.3. *Let $d \ge 2$ and $f : \mathbb{Z}^d \to \mathbb{R}$ be a function in $\ell^1(\mathbb{Z}^d)$. Then*

$$\operatorname{Var} M_1 f \le 2d\Big(1 + \sum_{k\ge1} \frac{N_{1,d-1}(k) - N_{1,d-1}(k-1)}{N_{1,d}(k)}\Big)\|f\|_{\ell^1(\mathbb{Z}^d)} =: C(d)\|f\|_{\ell^1(\mathbb{Z}^d)}, \tag{1.10}$$

and the constant $C(d)$ is the best possible. Moreover, the equality is attained if and only if f is a delta function.

Remark 1.4. Note that $C(d) < \infty$, because there exists a constant C such that

$$N_{1,d}(k) = Ck^d + O(k^{d-1}),$$

where $C = m(\Omega_{\ell^1})$ (see [15, Ch. VI. Section 2, Theorem 2]). For sufficiently large k,

$$\frac{N_{1,d-1}(k) - N_{1,d-1}(k-1)}{N_{1,d}(k)} \sim \frac{1}{k^2}.$$

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In particular, for $d = 2$,

$$C(2) = 4 + 8 \sum_{k \geq 1} \frac{1}{k^2 + (k+1)^2}.$$

Our proof of Theorem 1.3 is the natural extension of the proof of Theorem 1.1 but we decided to present Theorem 1.1 separately since it contains the essential idea with less technical details. The next result is the sharp version of (1.9) for the discrete uncentred maximal operator with respect to cubes (that is, ℓ^∞-balls). This proof follows a different strategy from Theorems 1.1 and 1.3.

Theorem 1.5. *Let $d \geq 1$ and $f : \mathbb{Z}^d \to \mathbb{R}$ be a function in $\ell^1(\mathbb{Z}^d)$. Then*

$$\begin{aligned} \operatorname{Var} \widetilde{M} f &\leq 2d\bigg(1 + \sum_{k \geq 1} \frac{1}{k}\bigg(\Big(\frac{2}{k+1} + \frac{2k-1}{k}\Big)^{d-1} - \Big(\frac{2k-1}{k}\Big)^{d-1}\bigg)\bigg)\|f\|_{\ell^1(\mathbb{Z}^d)} \\ &=: \widetilde{C}(d)\|f\|_{\ell^1(\mathbb{Z}^d)}, \end{aligned} \tag{1.11}$$

and the constant $\widetilde{C}(d)$ is the best possible. Moreover, the equality is attained if and only if f is a delta function.

Remark 1.6. In particular, $\widetilde{C}(1) = 2$ (and we recover (1.4)) and $\widetilde{C}(2) = 12$.

For the proofs of these three theorems we may assume throughout the rest of the paper, without loss of generality, that $f \geq 0$.

2. Proof of Theorem 1.1

Since $f \in \ell^1(\mathbb{Z})$, for all $n \in \mathbb{Z}$ there exists $r_n \in \mathbb{Z}$ such that $Mf(n) = A_{r_n} f(n)$. Define

$$X^- = \{n \in \mathbb{Z} : Mf(n) \geq Mf(n+1)\} \quad \text{and} \quad X^+ = \{n \in \mathbb{Z} : Mf(n+1) > Mf(n)\}.$$

Then

$$\begin{aligned} \operatorname{Var} Mf &= \sum_{n \in \mathbb{Z}} |Mf(n) - Mf(n+1)| \\ &= \sum_{n \in X^-} (Mf(n) - Mf(n+1)) + \sum_{n \in X^+} (Mf(n+1) - Mf(n)) \\ &\leq \sum_{n \in X^-} (A_{r_n} f(n) - A_{r_n+1} f(n+1)) + \sum_{n \in X^+} (A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n)). \end{aligned} \tag{2.1}$$

Given $p \in \mathbb{Z}$, we find the maximal contribution of $f(p)$ to the right-hand side of (2.1).

Case 1: $n \in X^-$ and $n \geq p$. If $p < n - r_n$, the contribution of $f(p)$ to the term $A_{r_n} f(n) - A_{r_n+1} f(n+1)$ is 0; if $n - r_n \leq p$, the contribution is

$$\begin{aligned} \frac{1}{2r_n + 1} - \frac{1}{2r_n + 3} &= \frac{2}{(2r_n + 1)(2r_n + 3)} \\ &\leq \frac{2}{(2(n-p) + 1)(2(n-p) + 3)} = \frac{1}{2(n-p) + 1} - \frac{1}{2(n-p) + 3}, \end{aligned}$$

and equality is attained if and only if $r_n = n - p$.

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Case 2: $n \in X^+$ and $n \geq p$. If $p < n + 1 - r_{n+1}$, the contribution of $f(p)$ to $A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n)$ is nonpositive; if $n + 1 - r_{n+1} \leq p$, the contribution is

$$\begin{aligned}\frac{1}{2r_{n+1}+1} - \frac{1}{2r_{n+1}+3} &= \frac{2}{(2r_{n+1}+1)(2r_{n+1}+3)} \\ &\leq \frac{2}{(2(n+1-p)+1)(2(n+1-p)+3)} \\ &= \frac{1}{2(n+1-p)+1} - \frac{1}{2(n+1-p)+3} \\ &< \frac{1}{2(n-p)+1} - \frac{1}{2(n-p)+3}.\end{aligned}$$

Case 3: $n \in X^-$ and $n < p$. If $p > n + r_n$, the contribution of $f(p)$ to the term $A_{r_n} f(n) - A_{r_n+1} f(n+1)$ is nonpositive; if $n + r_n \geq p$, the contribution is

$$\begin{aligned}\frac{1}{2r_n+1} - \frac{1}{2r_n+3} &= \frac{2}{(2r_n+1)(2r_n+3)} \\ &\leq \frac{2}{(2(p-n)+1)(2(p-n)+3)} \\ &= \frac{1}{2(p-n)+1} - \frac{1}{2(p-n)+3} \\ &< \frac{1}{2(p-n-1)+1} - \frac{1}{2(p-n-1)+3}.\end{aligned}$$

Case 4: $n \in X^+$ and $n < p$. If $p > n + 1 + r_{n+1}$, the contribution of $f(p)$ to $A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n)$ is 0; if $n + 1 + r_{n+1} \geq p$, the contribution is

$$\begin{aligned}\frac{1}{2r_{n+1}+1} - \frac{1}{2r_{n+1}+3} &= \frac{2}{(2r_{n+1}+1)(2r_{n+1}+3)} \\ &\leq \frac{2}{(2(p-n-1)+1)(2(p-n-1)+3)} \\ &= \frac{1}{2(p-n-1)+1} - \frac{1}{2(p-n-1)+3},\end{aligned}$$

and equality is attained if and only if $r_{n+1} = p - n - 1$.

Conclusion. The contribution of $f(p)$ to the right-hand side of (2.1) is bounded by

$$\sum_{n \geq p} \Big(\frac{1}{2(n-p)+1} - \frac{1}{2(n-p)+3} \Big) + \sum_{n<p} \Big(\frac{1}{2(p-n-1)+1} - \frac{1}{2(p-n-1)+3} \Big) = 2.$$

Since p is arbitrary, this establishes (1.6). If f is a delta function, we can easily see that

$$\operatorname{Var} Mf = 2\|f\|_{\ell^1(\mathbb{Z})}.$$

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On the other hand, let $f : \mathbb{Z} \to \mathbb{R}$ be a function such that $\operatorname{Var} Mf = 2\|f\|_{\ell^1(\mathbb{Z})}$ and $f \geq 0$. If $P = \{t \in \mathbb{Z} : f(t) \neq 0\}$, then

$$\operatorname{Var} Mf = 2 \sum_{t \in P} f(t)$$

and, given $t_1 \in P$, the contribution of $f(t_1)$ to (2.1) is 2. Therefore, by the previous analysis, for all $n \geq t_1$ we must have $n \in X^-$ and $r_n = n - t_1$. If we take $t_2 \in P$, the same conclusion holds, so $t_1 = t_2$ and therefore $P = \{t_1\}$. This proves that f is a delta function and the proof is concluded.

3. Proof of Theorem 1.3

3.1. Preliminaries. Since $f \in \ell^1(\mathbb{Z}^d)$, there exists $r_{\vec{n}} \in \mathbb{Z}$ with $M_1 f(\vec{n}) = A_{r_{\vec{n}}} f(\vec{n})$. For all $\vec{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$, define

$$|\vec{m}|_1 = \sum_{i=1}^{d} |m_i|$$

and, for $1 \leq j \leq d$, set $I_j = \{l \subset \mathbb{Z}^d : l \text{ is a line parallel to the vector } \vec{e}_j\}$,

$$X_j^- = \{\vec{n} \in \mathbb{Z}^d : M_1 f(\vec{n}) \geq M_1 f(\vec{n} + \vec{e}_j)\} \quad \text{and} \quad X_j^+ = \{\vec{n} \in \mathbb{Z}^d : M_1 f(\vec{n} + \vec{e}_j) > M_1 f(\vec{n})\}.$$

We then have

$$\begin{aligned}
\operatorname{Var} M_1 f &= \sum_{\vec{n} \in \mathbb{Z}^d} \sum_{j=1}^{d} |M_1 f(\vec{n}) - M_1 f(\vec{n} + \vec{e}_j)| \\
&\leq \sum_{j=1}^{d} \sum_{l \in I_j} \sum_{\vec{n} \in l \cap X_j^-} (A_{r_{\vec{n}}} f(\vec{n}) - A_{r_{\vec{n}}+1} f(\vec{n} + \vec{e}_j)) \\
&\quad + \sum_{j=1}^{d} \sum_{l \in I_j} \sum_{\vec{n} \in l \cap X_j^+} (A_{r_{\vec{n}+\vec{e}_j}} f(\vec{n} + \vec{e}_j) - A_{r_{\vec{n}+\vec{e}_j}+1} f(\vec{n})). \qquad (3.1)
\end{aligned}$$

For a fixed point $\vec{p} = (p_1, p_2, \ldots, p_d) \in \mathbb{Z}^d$, we want to evaluate the maximal contribution of $f(\vec{p})$ to the right-hand side of (3.1).

3.2. Auxiliary results. We now prove the following lemma of arithmetical character, which will be particularly useful in the rest of the proof.

Lemma 3.1. *If $d \geq 1$, then*

$$N_{1,d}(k)^2 > N_{1,d}(k+1) N_{1,d}(k-1) \quad \textit{for all } k \geq 1.$$

Proof. We prove this via induction. For $d = 1$, we have $N_{1,1}(k) = 2k + 1$ and therefore

$$N_{1,1}(k)^2 = 4k^2 + 4k + 1 > (2k+3)(2k-1) = N_{1,1}(k+1) N_{1,1}(k-1).$$

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Since $N_{1,d}(k) = |\{(x_1, \dots, x_d) \in \mathbb{Z}^d : |x_1| + \cdots + |x_d| \le k\}|$, fixing the value of the last variable, we can verify that

$$N_{1,d}(k) = N_{1,d-1}(k) + 2\sum_{j=0}^{k-1} N_{1,d-1}(j). \tag{3.2}$$

Now, let us assume that the result is true for d, that is,

$$N_{1,d}(k)^2 > N_{1,d}(k+1)N_{1,d}(k-1) \quad \text{for all } k \ge 1. \tag{3.3}$$

For simplicity, write $g(k) := N_{1,d}(k)$ and $f(k) := N_{1,d+1}(k)$ for all $k \ge 0$. Thus, by (3.3),

$$\frac{g(1)}{g(0)} > \frac{g(2)}{g(1)} > \cdots > \frac{g(k)}{g(k-1)} > \frac{g(k+1)}{g(k)} > \cdots \tag{3.4}$$

and, by (3.2),

$$f(k) = g(k) + 2\sum_{j=0}^{k-1} g(j) \quad \text{for all } k \ge 0.$$

The latter implies that

$$f(k+1) - f(k) = g(k+1) + g(k) \quad \text{for all } k \ge 0.$$

Therefore, by (3.4),

$$\frac{g(k+1)}{g(k)} > \frac{g(k+2)+g(k+1)}{g(k+1)+g(k)} \quad \text{and} \quad \frac{g(k+1) + 2\sum_{j=1}^{k} g(j)}{g(k) + 2\sum_{j=1}^{k} g(j-1)} > \frac{g(k+1)}{g(k)}.$$

Combining these inequalities, we arrive at

$$\begin{aligned}\frac{f(k+1)}{f(k)} &\ge \frac{g(k+1) + 2\sum_{j=1}^{k} g(j)}{g(k) + 2\sum_{j=1}^{k} g(j-1)} > \frac{g(k+1)}{g(k)} > \frac{g(k+2)+g(k+1)}{g(k+1)+g(k)} \\ &= \frac{f(k+2)-f(k+1)}{f(k+1)-f(k)}\end{aligned}$$

and hence

$$\frac{f(k+1)-f(k)}{f(k)} > \frac{f(k+2)-f(k+1)}{f(k+1)}.$$

This implies that

$$\frac{f(k+1)}{f(k)} > \frac{f(k+2)}{f(k+1)} \quad \text{for all } k \ge 0,$$

which gives the result for $d+1$ and establishes the lemma by induction. □

COROLLARY 3.2. *If $d \ge 1$,*

$$\frac{1}{N_{1,d}(k)} - \frac{1}{N_{1,d}(k+1)} > \frac{1}{N_{1,d}(k+1)} - \frac{1}{N_{1,d}(k+2)} \quad \textit{for all } k \ge 0. \tag{3.5}$$

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Proof. We notice that (3.5) is equivalent to

$$\frac{N_{1,d}(k+1)}{N_{1,d}(k)} + \frac{N_{1,d}(k+1)}{N_{1,d}(k+2)} > 2.$$

This follows from Lemma 3.1 and the arithmetic–geometric mean inequality because

$$\frac{N_{1,d}(k+1)}{N_{1,d}(k)} + \frac{N_{1,d}(k+1)}{N_{1,d}(k+2)} > \frac{N_{1,d}(k+2)}{N_{1,d}(k+1)} + \frac{N_{1,d}(k+1)}{N_{1,d}(k+2)} \geq 2. \qquad \square$$

3.3. Proof of Theorem 1.3. Let us simplify notation by writing $N_1(k) := N_{1,d}(k)$. Given $1 \leq j \leq d$, using Corollary 3.2, we make the following observations.

Case 1: $\vec{n} \in X_j^-$ and $n_j \geq p_j$. If $|\vec{n} - \vec{p}|_1 > r_{\vec{n}}$, the contribution of $f(\vec{p})$ to $A_{r_{\vec{n}}} f(\vec{n}) - A_{r_{\vec{n}}+1} f(\vec{n} + \vec{e}_j)$ is nonpositive; if $|\vec{n} - \vec{p}|_1 \leq r_{\vec{n}}$, the contribution is

$$\begin{aligned}
\frac{1}{N_1(r_{\vec{n}})} - \frac{1}{N_1(r_{\vec{n}}+1)} &\leq \frac{1}{N_1(|\vec{n}-\vec{p}|_1)} - \frac{1}{N_1(|\vec{n}-\vec{p}|_1+1)} \\
&= \frac{1}{N_1(|\vec{n}-\vec{p}|_1)} - \frac{1}{N(|\vec{n}+\vec{e}_j-\vec{p}|_1)}.
\end{aligned}$$

The equality is attained if and only if $r_{\vec{n}} = |\vec{n} - \vec{p}|_1$.

Case 2: $\vec{n} \in X_j^+$ and $n_j \geq p_j$. If $|\vec{n} + \vec{e}_j - \vec{p}|_1 > r_{\vec{n}+\vec{e}_j}$, the contribution of $f(\vec{p})$ to $A_{r_{\vec{n}+\vec{e}_j}} f(\vec{n} + \vec{e}_j) - A_{r_{\vec{n}+\vec{e}_j}+1} f(\vec{n})$ is nonpositive; if $|\vec{n} + \vec{e}_j - \vec{p}|_1 \leq r_{\vec{n}+\vec{e}_j}$, the contribution is

$$\begin{aligned}
\frac{1}{N_1(r_{\vec{n}+\vec{e}_j})} - \frac{1}{N_1(r_{\vec{n}+\vec{e}_j}+1)} &\leq \frac{1}{N_1(|\vec{n}+\vec{e}_j-\vec{p}|_1)} - \frac{1}{N_1(|\vec{n}+\vec{e}_j-\vec{p}|_1+1)} \\
&= \frac{1}{N_1(|\vec{n}-\vec{p}|_1+1)} - \frac{1}{N_1(|\vec{n}-\vec{p}|_1+2)} \\
&< \frac{1}{N_1(|\vec{n}-\vec{p}|_1)} - \frac{1}{N_1(|\vec{n}-\vec{p}|_1+1)} \\
&= \frac{1}{N_1(|\vec{n}-\vec{p}|_1)} - \frac{1}{N(|\vec{n}+\vec{e}_j-\vec{p}|_1)}.
\end{aligned}$$

Case 3: $\vec{n} \in X_j^-$ and $n_j < p_j$. If $|\vec{n} - \vec{p}|_1 > r_{\vec{n}}$, the contribution of $f(\vec{p})$ to $A_{r_{\vec{n}}} f(\vec{n}) - A_{r_{\vec{n}}+1} f(\vec{n} + \vec{e}_j)$ is nonpositive; if $|\vec{n} - \vec{p}|_1 \leq r_{\vec{n}}$, the contribution is

$$\begin{aligned}
\frac{1}{N_1(r_{\vec{n}})} - \frac{1}{N_1(r_{\vec{n}}+1)} &\leq \frac{1}{N_1(|\vec{p}-\vec{n}|_1)} - \frac{1}{N_1(|\vec{p}-\vec{n}|_1+1)} \\
&< \frac{1}{N_1(|\vec{p}-\vec{n}-\vec{e}_j|_1)} - \frac{1}{N_1(|\vec{p}-\vec{n}|_1)}.
\end{aligned}$$

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Case 4: $\vec{n} \in X_j^+$ and $n_j < p_j$. If $|\vec{p} - \vec{n} - \vec{e}_j|_1 > r_{\vec{n}+\vec{e}_j}$, the contribution of $f(\vec{p})$ to $A_{r_{\vec{n}+\vec{e}_j}} f(\vec{n} + \vec{e}_j) - A_{r_{\vec{n}+\vec{e}_j}+1} f(\vec{n})$ is nonpositive; if $|\vec{p} - \vec{n} - \vec{e}_j|_1 \leq r_{\vec{n}+\vec{e}_j}$, the contribution is

$$\begin{aligned}\frac{1}{N_1(r_{\vec{n}+\vec{e}_j})} - \frac{1}{N_1(r_{\vec{n}+\vec{e}_j}+1)} &\leq \frac{1}{N_1(|\vec{p}-\vec{n}-\vec{e}_j|_1)} - \frac{1}{N_1(|\vec{p}-\vec{n}-\vec{e}_j|_1+1)} \\ &= \frac{1}{N_1(|\vec{p}-\vec{n}-\vec{e}_j|_1)} - \frac{1}{N_1(|\vec{p}-\vec{n}|_1)}.\end{aligned}$$

The equality is achieved if and only if $r_{\vec{n}+\vec{e}_j} = |\vec{p} - \vec{n} - \vec{e}_j|_1$.

Conclusion. Given a line l in the lattice, we define the distance from $\vec{p}$ to l by

$$d(l, \vec{p}) = \min\{|\vec{m} - \vec{p}|_1 \mid \vec{m} \in l\}.$$

If the direction of l is the same as the direction of $\vec{e}_j$, by intersecting l with the hyperplane $H_j = \{\vec{z} \in \mathbb{Z}^d : z_j = p_j\}$ we obtain the point that realises the distance from p to l. By the previous analysis, the contribution of $f(\vec{p})$ to

$$\sum_{\vec{n}\in l\cap X_j^-} (A_{r_{\vec{n}}} f(\vec{n}) - A_{r_{\vec{n}}+1} f(\vec{n}+\vec{e}_j)) + \sum_{\vec{n}\in l\cap X_j^+} (A_{r_{\vec{n}+\vec{e}_j}} f(\vec{n}+\vec{e}_j) - A_{r_{\vec{n}+\vec{e}_j}+1} f(\vec{n}))$$

is less than or equal to

$$\frac{2}{N_{1,d}(d(l,\vec{p}))}. \tag{3.6}$$

As p belongs to d lines of the lattice, there are $d(N_{1,d-1}(k) - N_{1,d-1}(k-1))$ lines such that $d(l, \vec{p}) = k$ for a given $k \in \mathbb{N}$. Thus, the contribution of $f(\vec{p})$ to the right-hand side of (3.1) is less than or equal to

$$2d + \sum_{k\geq 1} \frac{2d(N_{1,d-1}(k) - N_{1,d-1}(k-1))}{N_{1,d}(k)},$$

and this yields the desired inequality.

If f is a delta function, then there exist $\vec{y} \in \mathbb{Z}^d$ and $k \in \mathbb{R}$ such that

$$f(\vec{y}) = k \quad \text{and} \quad f(\vec{x}) = 0 \quad \text{for all } \vec{x} \in \mathbb{Z}^d \setminus \{y\}.$$

Considering the contribution of $|f(\vec{y})|$ to a line l in the lattice $\mathbb{Z}^d$, we have equality in (3.6) and hence in (1.10). On the other hand, suppose that $f : \mathbb{Z}^d \to \mathbb{R}$ is a nonnegative function that gives equality in (1.10). Define $P = \{\vec{t} \in \mathbb{Z}^d : f(\vec{t}) \neq 0\}$. Then

$$\operatorname{Var} M_1 f = \left(2d + \sum_{k\geq 1} \frac{2d(N_{1,d-1}(k) - N_{1,d-1}(k-1))}{N_{1,d}(k)}\right) \sum_{\vec{t}\in P} f(\vec{t}).$$

By the previous analysis, given $\vec{s} = (s_1, s_2, \dots, s_d) \in P$ and a line l in the lattice, the contribution of $f(\vec{s})$ to l in (3.6) must be $2/N_{1,d}(d(l, \vec{s}))$. If there exists $\vec{u} \in P \setminus \{\vec{s}\}$, the contribution of $f(\vec{u})$ to l in (3.1) must also be $2/N_{1,d}(d(l, \vec{u}))$. Assume without loss of generality that $s_d > u_d$ and consider the line $l = \{(s_1, s_2, \dots, s_{n-1}, x) : x \in \mathbb{Z}\}$. Since we have equality in (1.10), given $\vec{n} \in l$ such that $n_d \geq s_d$, we must have $\vec{n} \in X_j^-$ and $|\vec{n} - \vec{s}|_1 = r_{\vec{n}} = |\vec{n} - \vec{u}|_1$, which gives a contradiction. Thus, f must be a delta function.

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4. Proof of Theorem 1.5

4.1. Preliminaries. As before, since $f \in \ell^1(\mathbb{Z}^d)$, for each $\vec{n} \in \mathbb{Z}^d$ there exist $r_{\vec{n}} \in \mathbb{R}^+$ and $c_{\vec{n}} \in \mathbb{R}^d$ such that $\vec{n} \in c_{\vec{n}} + Q(r_{\vec{n}})$ and $\widetilde{M}f(\vec{n}) = A_{r_{\vec{n}}}f(c_{\vec{n}})$, where $Q_{r_{\vec{n}}}$ is the cube $Q_{r_{\vec{n}}} = \{m \in \mathbb{Z}^d : |m|_\infty \le r_{\vec{n}}\} = \{m \in \mathbb{Z}^d : \max\{|m_1|, \dots, |m_d|\} \le r_{\vec{n}}\}$. We now introduce the local maxima and minima of a discrete function $g : \mathbb{Z} \to \mathbb{R}$. We say that an interval $[n, m]$ is a *string of local maxima* of g if

$$g(n-1) < g(n) = \cdots = g(m) > g(m+1).$$

If $n = -\infty$ or $m = \infty$ (but not both simultaneously), we modify the definition accordingly, eliminating one of the inequalities. The rightmost point m of such a string is a *right local maximum* of g, while the leftmost point n is a *left local maximum* of g. We define *string of local minima*, *right local minimum* and *left local minimum* analogously.

Given a line l in the lattice $\mathbb{Z}^d$ parallel to $\vec{e}_d$, there exists $n' \in \mathbb{Z}^{d-1}$ such that $l = \{(n', m) : m \in \mathbb{Z}\}$. Let us assume that $\widetilde{M}f(n', x)$ is not constant as a function of x (otherwise the variation of the maximal function over this line will be zero). Let $\{[a_j^-, a_j^+]\}_{j\in\mathbb{Z}}$ and $\{[b_j^-, b_j^+]\}_{j\in\mathbb{Z}}$ be the ordered strings of local maxima and local minima of $\widetilde{M}f(n', x)$ (we allow the possibilities of a_j^- or $b_j^- = -\infty$ and a_j^+ or $b_j^+ = \infty$), that is,

$$\cdots < a_{-1}^- \le a_{-1}^+ < b_{-1}^- \le b_{-1}^+ < a_0^- \le a_0^+ < b_0^- \le b_0^+ < a_1^- \le a_1^+ < b_1^- \le b_1^+ < \cdots .$$

This sequence may terminate on one or both sides and we adjust the notation and the proof below accordingly. We have at least one string of local maxima since $\widetilde{M}f(\vec{n}) \to 0$ as $|\vec{n}|_\infty \to \infty$. Therefore, if the sequence terminates on one or both sides, it must terminate in a string of local maxima. The variation of the maximal function on l is

$$2\sum_{j\in\mathbb{Z}}(\widetilde{M}f(n', a_j^+) - \widetilde{M}f(n', b_j^-)) \le 2\sum_{j\in\mathbb{Z}}(A_{r_{(n',a_j^+)}}f(c_{(n',a_j^+)}) - A_{r_{(n',a_j^+)}+|a_j^+-b_j^-|}f(c_{(n',a_j^+)})). \tag{4.1}$$

Lemma 4.1. *Given $\vec{q} \in \mathbb{Z}^d$ and a line l in the lattice $\mathbb{Z}^d$, there exists at most one string of local maxima of $\widetilde{M}f$ on l such that there exists $\vec{n}$ in the string for which the contribution of $f(\vec{q})$ to $A_{r_{\vec{n}}}f(c_{\vec{n}})$ is positive.*

Proof. Assume, without loss of generality, that $l = \{(m_1, m_2, \dots, m_{d-1}, x) : x \in \mathbb{Z}\} = \{(m', x) : x \in \mathbb{Z}\}$. Consider a string of local maxima of $\widetilde{M}f$ on l, say

$$\widetilde{M}f(m', a-1) < \widetilde{M}f(m', a) = \cdots = \widetilde{M}f(m', a+n) > \widetilde{M}f(m', a+n+1). \tag{4.2}$$

Let

$$\widetilde{M}f(m', a+i) = A_{r_{(m',a+i)}}f(c_{(m',a+i)}) \quad \text{for } 0 \le i \le n.$$

Given $\vec{q} = (q_1, q_2, \dots, q_d) \in \mathbb{Z}^d$, a necessary condition for the contribution of $f(\vec{q})$ to $A_{r_{(m',a+i)}}f(c_{(m',a+i)})$ to be positive for some i is that $a - 1 < q_d < a + n + 1$ (otherwise this would violate one of the end-point inequalities in (4.2)). This gives the result. □

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4.2. Proof of Theorem 1.5. Given $\vec{p} \in \mathbb{Z}^d$ and a line l in the lattice $\mathbb{Z}^d$, we define $d(l, \vec{p}) = \min\{|\vec{p} - \vec{m}|_\infty : \vec{m} \in l\}$ and $d(l, \vec{p})_+ = \max\{1, d(l, \vec{p})\}$. By Lemma 4.1, given $\vec{p} = (p_1, p_2, \ldots, p_{d-1}, p_d) \in \mathbb{Z}^d$ and a line $l = \{(n_1, n_2, \ldots, n_{d-1}, x) \in \mathbb{Z}^d : x \in \mathbb{Z}\}$ such that $|\{i \in \{1, 2, \ldots, d-1\} : |n_i - p_i| = d(l, \vec{p})\}| = j$, the contribution of $f(\vec{p})$ to the right-hand side of (4.1) is less than or equal to

$$\frac{2}{(d(l, \vec{p}) + 1)^j (d(l, \vec{p}))_+^{d-j}}. \tag{4.3}$$

In fact, if an ℓ^∞-cube contains $\vec{p}$ and a point in l then it must have side at least $d(l, \vec{p})$, and it must contain $d(l, \vec{p}) + 1$ lattice points in each direction $\vec{e}_i$ for i such that $|n_i - p_i| = d(l, \vec{p})$. In the other $d - j$ directions the cube contains at least $d(l, \vec{p})$ lattice points. This leads to (4.3).

If equality in (4.3) is attained for a point $\vec{p}$ and a line l, then there is a point $\vec{q} \in l$ that realises the distance to $\vec{p}$, belongs to a string of local maxima of l and for which $\vec{p} \in c_{\vec{q}} + Q(r_{\vec{q}})$. Moreover, this string of local maxima must be unique, otherwise $f(\vec{p})$ would also have a negative contribution coming from a string of minima in (4.1). In particular, this implies that $\widetilde{M} f(\vec{p}) \geq \widetilde{M} f(\vec{n})$ for all $\vec{n} \in l$. If we fix a point $\vec{p}$ and assume that equality in (4.3) is attained *for all lines* l in our lattice, then $\widetilde{M} f(\vec{p}) \geq \widetilde{M} f(\vec{n})$ for all $\vec{n} \in \mathbb{Z}^d$.

Now $\vec{p}$ belongs to d lines of the lattice $\mathbb{Z}^d$ and, given $k \in \mathbb{N}$ and $j \in \{1, 2, \ldots, d-1\}$, there exist $2^j \binom{d-1}{j} (2(k-1)+1)^{d-1-j}$ lines $l = \{(n_1, n_2, \ldots, n_{d-1}, x) \mid x \in \mathbb{Z}\}$ such that $d(l, \vec{p}) = k$ and $|\{i \in \{1, 2, \ldots, d-1\} : |n_i - p_i| = k\}| = j$. Thus, the contribution of $f(\vec{p})$ to the variation of the maximal function in $\mathbb{Z}^d$ is less than or equal to

$$\begin{aligned}
2d &+ d \sum_{k \geq 1} \sum_{j=1}^{d-1} 2^j \binom{d-1}{j} (2k-1)^{d-1-j} \frac{2}{(k+1)^j k^{d-j}} \\
&= 2d + \sum_{k \geq 1} \frac{2d}{k} \sum_{j=1}^{d-1} \binom{d-1}{j} \left(\frac{2}{k+1}\right)^j \left(\frac{2k-1}{k}\right)^{d-1-j} \\
&= 2d + \sum_{k \geq 1} \frac{2d}{k} \left(\left(\frac{2}{k+1} + \frac{2k-1}{k} \right)^{d-1} - \left(\frac{2k-1}{k} \right)^{d-1} \right).
\end{aligned}$$

This concludes the proof of (1.11).

If f is a delta function, with $f(\vec{n}) = 0$ for all $n \in \mathbb{Z}^d \setminus \{\vec{p}\}$ for some $p \in \mathbb{Z}^d$, it is easy to see that we have equality in (4.3) for the contribution of $|f(\vec{p})|$ to all lines l, which implies equality in (1.11). On the other hand, assume that $f : \mathbb{Z}^d \to \mathbb{R}$ is a nonnegative function that gives equality in (1.11). Define $P = \{\vec{t} \in \mathbb{Z}^d : f(\vec{t}) \neq 0\}$. Then

$$\operatorname{Var} \widetilde{M} f = \left(2d + \sum_{k \geq 1} \frac{2d}{k} \left(\left(\frac{2}{k+1} + \frac{2k-1}{k} \right)^{d-1} - \left(\frac{2k-1}{k} \right)^{d-1} \right) \right) \sum_{t \in P} f(t).$$

Then, given $\vec{s} \in P$, if there exists $\vec{u} \in P \setminus \{\vec{s}\}$, we consider a line l in the lattice $\mathbb{Z}^d$ such that $\vec{s} \in l$ and $\vec{u} \notin l$. The contribution of $f(\vec{s})$ to l must be 2, $\widetilde{M}f(\vec{s}) = f(\vec{s})$ belongs to the unique string of local maxima of $\widetilde{M}f$ on l and the right-hand side of (4.1) must be $2f(\vec{s})$, by the previous analysis. Therefore, the contribution of $f(\vec{u})$ to the line l is 0 and $f(\vec{u})$ does not provide the maximum contribution as predicted in (4.3). Thus, equality in (1.11) cannot be attained. We conclude that f must be a delta function.

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