1

Introduction

1.1 What is dynamo theory?

Dynamo theory is concerned with the manner in which magnetic fields are generated and maintained in planets, stars and galaxies. The Earth, Sun and Milky Way provide examples of most immediate interest, for which a huge quantity of observational detail is now available; and yet the fundamental theory applies equally to any sufficiently large mass of electrically conducting fluid, either liquid metal or ionised gas ('plasma' when fully ionised), under the combined effects of global rotation and convective motion, this usually having a turbulent character. This turbulence may be either 'strong turbulence' of a type familiar in aerodynamics and meteorology, or 'weak turbulence' – a field of weakly interacting random waves internal to the fluid. Either way, it is the combination of rotation and convection that turns out to be particularly conducive to the spontaneous growth of magnetic fields in fluid systems of sufficient spatial extent.

It is this latter requirement that has made laboratory realisation of the self-exciting dynamo process such a great challenge for experimentalists. The triple requirements of sufficient conductivity, scale and turbulent intensity have placed huge demands on the design of experiments, and it is only over the last decade that the necessary conditions have been achieved and that self-exciting dynamo action has been convincingly demonstrated. These experimental achievements have run in parallel with great computational achievements in modelling the dynamo process both in planetary liquid cores and in stellar convection zones. Theoretical progress, so essential for a full understanding of the dynamo process, has been much stimulated by the great advances on observational, experimental and numerical fronts.

In this introductory chapter, we first set out some of the historical background, with reference to subsequent chapters where specific issues are treated in detail.

1.2 Historical background

1.2.1 The geodynamo

A very complete history of magnetism over the past millennium may be found in Stern (2002). We content ourselves here with some of the highlights of a fascinating story.[1]

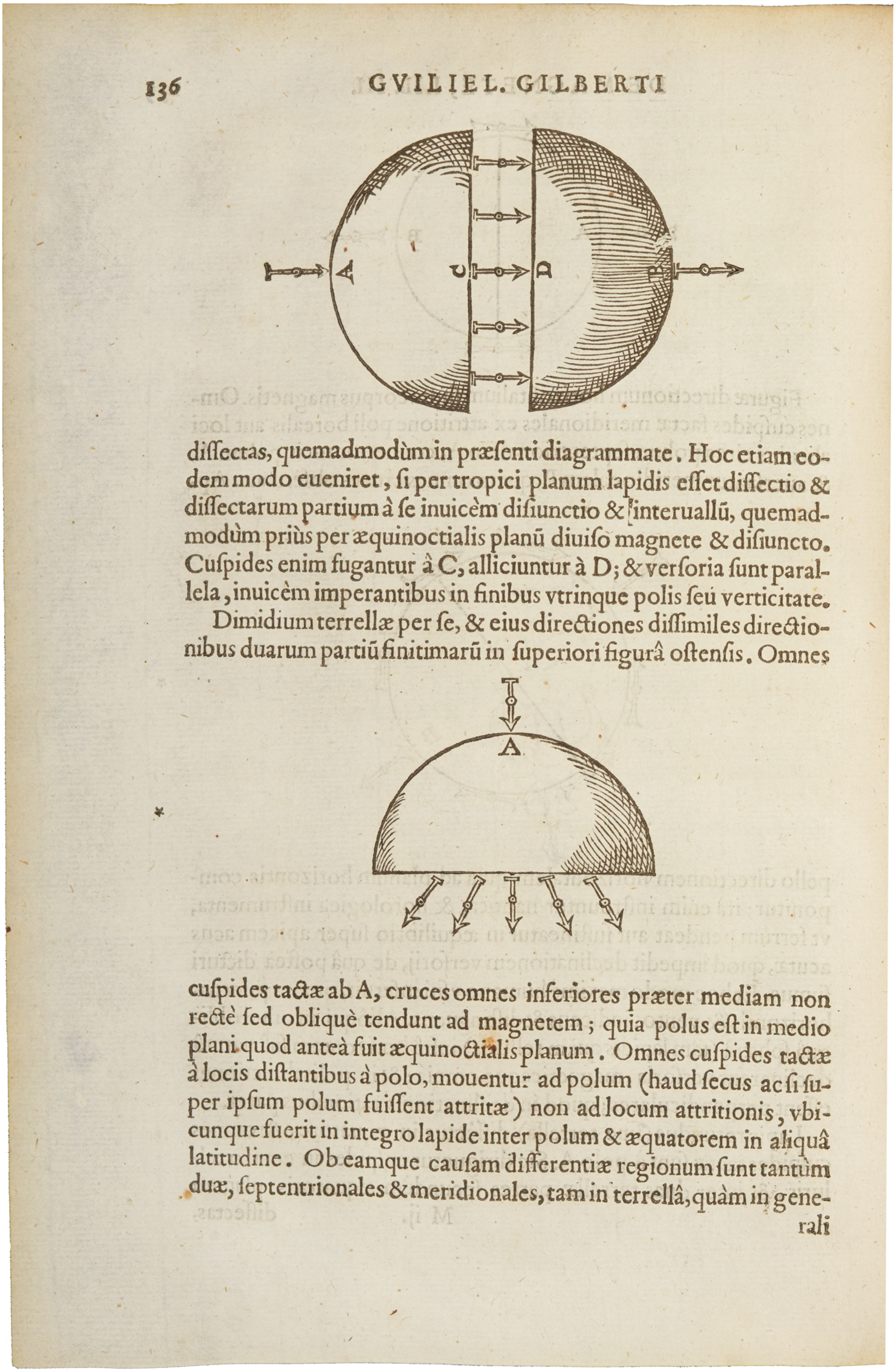
Every child who has played with a magnetic compass knows that the compass needle points North; but he learns as she grows older that magnetic North is not quite the same as 'true' North defined by the Pole star; or to put it differently, that the magnetic dipole axis is slightly inclined to the axis of rotation of the Earth. This worrying mismatch was already known to the Chinese of the Sung dynasty. In his great work *Science and Civilisation in China*, Joseph Needham (1962) quotes the *Mêng Chhi Pi Than* of Shen Kua (c.1088), which he translates thus: "Magicians rub the point of a needle with the lodestone; then it is able to point to the south. But it always inclines slightly to the east, and does not point directly at the south". So the 'declination' of the field was known, at least to the Chinese, 930 years ago. It was rediscovered and charted out by the early navigators of the fifteenth and sixteenth centuries and in particular by Christopher Columbus whose great voyage of discovery in 1492 opened new windows in the Western World. We recognise this declination now as a manifestation of a crucial departure from axisymmetry which is essential for the Earth's internal dynamo to operate.

In his seminal work *De Magnete*, William Gilbert (1600) recognised that 'magnet Earth' could be modelled by a spherical lodestone – his 'terrella' – over whose surface he was able to measure the magnetic field and plot its direction. Figure 1.1 shows a page from the second (1613) edition of this book, describing the sort of measurement that Gilbert was able to make. He spoke scathingly of earlier fanciful speculations concerning magnetism and was a pioneer of the 'scientific approach' based on careful observation and experiment.

The distinction between local magnetic north and 'true' north is often indicated on large-scale maps. The small print usually warns that the angle between the two directions changes irregularly by up to 1° in 6 years. This is the 'secular variation' of the magnetic field which was known to navigators of the seventeenth century and was no doubt a considerable nuisance to them. Edmund Halley considered the possible causes of this secular variation (Halley 1692) and concluded that

> the external parts of the globe may well be reckoned as the shell, and the internal as a nucleus or inner globe included within ours, with a fluid medium between ... only this outer Sphere having its turbinating motion some small matter either swifter or slower than the inner Ball.

[1] Parts of this section are an edited version of the introduction to a Union Lecture (Moffatt 1992) delivered at the IUGG General Assembly (Vienna 1991).



136 GVILIEL. GILBERTI

diſſectas, quemadmodùm in præſenti diagrammate. Hoc etiam eodem modo eueniret, ſi per tropici planum lapidis eſſet diſſectio & diſſectarum partium à ſe inuicèm diſiunctio & interuallũ, quemadmodùm priùs per æquinoctialis planũ diuiſo magnete & diſiuncto. Cuſpides enim fugantur à C, alliciuntur à D; & verſoria ſunt parallela, inuicèm imperantibus in finibus vtrinque polis ſeù verticitate.

Dimidium terrellæ per ſe, & eius directiones diſſimiles directionibus duarum partiũ finitimarũ in ſuperiori figurâ oſtenſis. Omnes

cuſpides tactæ ab A, cruces omnes inferiores præter mediam non rectè ſed obliquè tendunt ad magnetem; quia polus eſt in medio plani quod anteà fuit æquinoctialis planum. Omnes cuſpides tactæ à locis diſtantibus à polo, mouentur ad polum (haud ſecus ac ſi ſuper ipſum polum fuiſſent attritæ) non ad locum attritionis, vbicunque fuerit in integro lapide inter polum & æquatorem in aliquâ latitudine. Ob eamque cauſam differentiæ regionum ſunt tantùm duæ, ſeptentrionales & meridionales, tam in terrellâ, quàm in generali

Figure 1.1 A page from the 1613 edition of Gilbert's *De Magnete* showing the result of an experiment conducted with his 'terrella', modelling the Earth's magnetic field. [Courtesy of the Wren Library, Trinity College, Cambridge.]

https://doi.org/10.1017/9781107588691.003 Published online by Cambridge University Press

This was a prophetic vision as far as the inner structure of the Earth is concerned, but also remarkable in its perception of the need for *differential* rotation, now recognised as a further key element in the dynamo process.

A discovery of great importance was made by Oersted (1820), namely that current in a wire produces a magnetic field whose field lines embrace the wire. This led Ampère (1822) to propose that an east/west current must flow within the Earth. He wrote[2]

> L'idée la plus simple, et celle qui se présenterait immédiatement à celui qui voudrait expliquer cette direction constante de l'aiguille, ne serait-elle pas d'admettre dans la terre un courant électrique, dans une direction telle que le nord se trouvât à gauche d'un homme qui, couché sur sa surface pour avoir la face tournée du côté de l'aiguille, recevrait ce courant dans la direction de ses pieds à sa tête, et d'en conclure qu'il a lieu, de l'est à l'ouest, dans une direction perpendiculaire au méridien magnétique ?

A modern understanding of the origin of these currents is based both on Ampère's law, essentially that electric current is the source of magnetic field, and on Faraday's law of induction (Faraday 1832). By painstaking experiments, Faraday discovered that if a conductor moves across a magnetic field, and if a path is available for the completion of a current circuit, then in general current will flow in that circuit. For this achievement, Faraday was awarded the Copley Medal of the Royal Society of London. The citation records that

> he gives indisputable evidence of electric action due to terrestrial magnetism alone. An important addition is thus made to the facts which have long been accumulating for the solution of that most interesting problem, the magnetism of the Earth.

It was in fact more than an important addition; it was the key ingredient of the dynamo process, although this was not recognised till much later.

At about the same time, in two great papers Carl Friedrich Gauss (1832, 1838) established the spherical harmonic decomposition of the Earth's magnetic field and the technique by which secular variation of the field could be quantified. The traditional unit of field intensity in geomagnetism, and equally in astrophysics, is of course the Gauss (G), and it is arguably regrettable that the Système International of units now favours the tesla ($1\mathrm{T} = 10^4$ G). Gauss' spherical harmonic decomposition allows us to extrapolate the Earth's field (assumed potential) down to the core–mantle boundary (CMB), to map the contours of constant radial field at the CMB, and to do so at different epochs using all available data (Bloxham et al. 1989, see Chapter 4). In these maps, the dipole ingredient of the field is still quite evident at

[2] This translates, somewhat freely, as follows: "The simplest idea that must occur immediately to anyone attempting to explain the constant direction of the compass needle is this: there must exist an electric current in the Earth, such that, if a man were to lie on the surface of the Earth with the north to his left and his face turned in the direction of the needle, he would sense this current in the direction from his feet to his head; and should we not therefore conclude that this current flows from east to west perpendicular to the magnetic meridian?"

the CMB, but there is also a strong presence of quadrupole, octupole and higher-order ingredients, as is to be expected from the nature of downward extrapolation towards the region where the 'source' currents are confined. The slow evolution of the pattern (i.e. its secular variation) is also evident.

The high point and climax of electromagnetic theory in the nineteenth century came with the publication of James Clerk Maxwell's *Treatise on Electricity and Magnetism* (Maxwell 1873). Maxwell built on Faraday's discoveries and completed the system of equations that bear his name. It is interesting to note however that in a late chapter of the treatise, devoted to *Terrestrial Magnetism*, Maxwell comes nowhere near to any explanation of the real nature of the phenomenon. He confines himself to a description of Gauss' techniques for the determination of the Earth's field and its time variation, and his demonstration that the dominant source for the field is of internal rather than external origin; but as to the root cause of the phenomenon, he writes in sonorous tones:

> The field of investigation into which we are introduced by the study of terrestrial magnetism is as profound as it is extensive. . . . What cause [is it], whether exterior to the Earth or in its inner depths, [that] produces such enormous changes in the Earth's magnetism, that its magnetic poles move slowly from one part of the globe to another? . . . These immense changes in so large a body force us to conclude that we are not yet acquainted with one of the most powerful agents in nature, the scene of whose activity lies in those inner depths of the Earth, to the knowledge of which we have so few means of access.

It was the science of seismology that was to provide the vital means of access, establishing the existence first of a liquid outer core (Jeffreys 1926, who concluded that "the central core is probably fluid, but its viscosity is uncertain"), and secondly of an inner solid core (Lehmann 1936, Bullen 1946); both inner and outer cores are believed to be important for the operation of the geodynamo.[3]

One of the earliest discussions of possible causes of terrestrial magnetism was given by Arthur Schuster (1911) in his Presidential Address to the Physical Society of London. Schuster discussed the arguments for and against a system of electric currents in the Earth's interior and concluded that "the difficulties which stand in the way of basing terrestrial magnetism on electric currents inside the Earth are insurmountable" – strong words, which have since been invalidated with the passage of time and the birth and advance of magnetohydrodynamics. Nevertheless, even as late as 1940 in their great treatise on *Geomagnetism*, Chapman & Bartels (1940) came to the same defeatist conclusion as Schuster. They discussed Larmor's (1919) suggestion concerning the possibility of self-exciting dynamo action (see below) but stated that "Cowling, however, has shown that such self-excitation

[3] An illuminating discussion of the developments leading to these discoveries is given by Brush (1980).

is not possible. Consequently, Schuster's view still holds, that the difficulties . . . are insuperable". Cowling (1934) had not in fact shown that such self-excitation is not possible: he had merely shown that it was not possible for axisymmetric systems (for Cowling's anti-dynamo theorem, see §§6.4 and 6.5), and yet the tilt of the magnetic dipole which had been known for centuries shows that we are dealing with an emphatically non-axisymmetric system. Nevertheless the fact that Chapman & Bartel could be so easily persuaded that Cowling's theorem closed the matter is an indication of the powerful influence that this theorem then had – this no doubt because it was one of the few exact results of the subject. The year 1940 marked a high point in the collection and systematisation of geomagnetic data, but it also marks the nadir as regards real understanding of the origins of terrestrial magnetism,

The post-war years saw a profound transformation in the situation, to the point at which a dynamo theory of the origin of the Earth's magnetic field is now universally accepted among geophysicists. The progress in dynamo theory has been dramatic, and the theory applies with equal force to planets other than the Earth. Statements in textbooks since the 1980s are as vigorously positive as Schuster's (1912) statement was negative. Thus, for example, Jacobs (1994) writes, "There has been much speculation on the origin of the Earth's magnetic field. . . . The only possible means seems to be some form of electromagnetic induction, electric currents flowing in the Earth's core"; and Cook (2009) writes, "There is no theory other than a dynamo theory that shows any signs of accounting for the magnetic fields of the planets". It is a dynamo theory based on the principles of magnetohydrodynamics, and ultimately on a suitable exploitation of Faraday's law of induction, that has led to this remarkable revolution in our understanding of Nature.

1.2.2 The solar dynamo

Galileo's celebrated discovery of sunspots dates back to the MDCXIII publication of his *Istoria e Dimostrazioni*. Figure 1.2 shows Galileo's representation of the sunspots that he had by then observed. This apparent 'imperfection' in God's creation caused consternation in the powerful Catholic Church of that epoch; but paradoxically, it is this very imperfection and the manner in which it has evolved over the last four centuries that has provided a prime source of information concerning the physics of the surface layers of the Sun. This will be discussed in detail in Chapter 5; for the moment, we need only note Maunder's discovery of the 11-year sunspot cycle (Maunder 1904), and Hale's discovery of the relatively strong magnetic field in sunspots (Hale 1908) and the polarity laws that govern their behaviour.

Since the 1990s, the science of 'helioseismology' (analysis of the spectrum of solar oscillations, Christensen-Dalsgaard et al. 1996) has provided a wealth of

https://doi.org/10.1017/9781107588691.003 Published online by Cambridge University Press

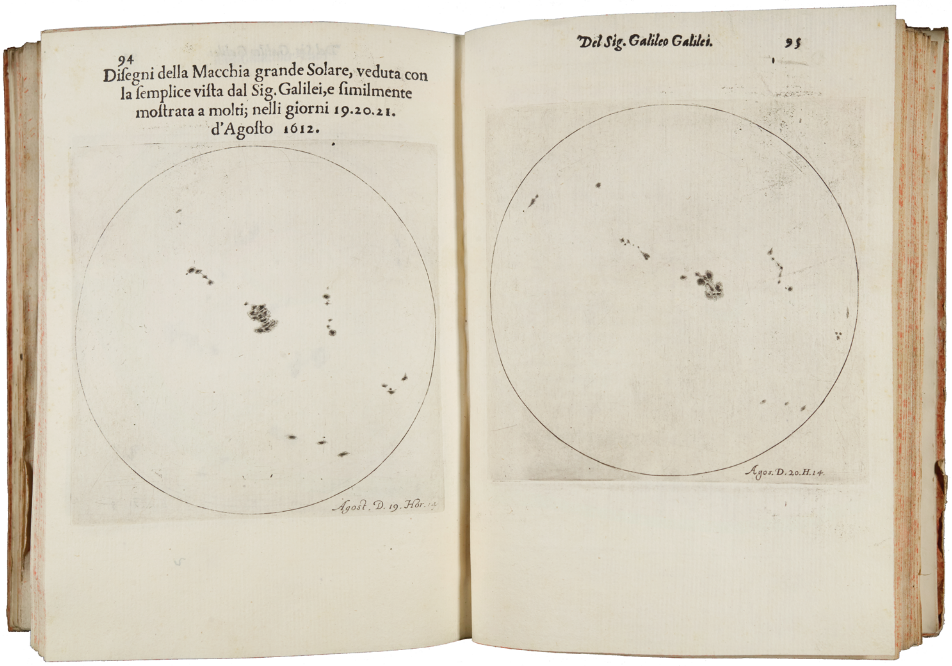


Figure 1.2 Galileo's volume *Istoria e Dimostrazioni*, showing here the record of his observations of sunspots on successive days in August 1612 (Galileo 1613). [Courtesy of the Wren Library, Trinity College, Cambridge.]

information concerning the flow field within the solar interior. In particular, through helioseismology, the differential rotation throughout most of the solar convection zone has been determined, and the presence of the 'tachocline', a layer of rapid shear at the base of the convection zone postulated by Spiegel & Zahn (1992), has been confirmed (Charbonneau et al. 1999). This in turn has led to renewed debate concerning the 'mean-field electrodynamics' applicable to the Sun through the '$\alpha\omega$-mechanism', matters that will be discussed in detail in later chapters.

The birth of solar dynamo theory proper is generally attributed to Joseph Larmor, Lucasian Professor at the University of Cambridge, who, exactly 100 years before publication of this book, posed the question "How could a rotating body such as the Sun become a magnet?" (Larmor 1919); and the question was certainly a natural one since the origin of the magnetic field of the Sun was at that time a total mystery.

And not only the Sun! We now know that a magnetic field is a normal accompaniment of any cosmic body that is both fluid (wholly or in part) and rotating. There appears to be a universal validity about this statement which applies quite irrespective of the length-scales considered. For example, on the planetary length-scale, Jupiter shares with the Earth the property of strong rotation (its rotation period

https://doi.org/10.1017/9781107588691.003 Published online by Cambridge University Press

being approximately 10 hours) and it is believed to have a fluid interior composed of an alloy of liquid metallic hydrogen and helium (Hide 1974); it exhibits a surface magnetic field of order 10 G in magnitude (as compared with the Earth's field of order 1 G). On the stellar length-scale, magnetic fields as weak as 1 G are hard to detect in general; there are however numerous examples of stars which rotate with periods ranging from several days to several months, and with detectable surface magnetic fields in the range 10^2 to 3×10^4 G (Preston 1967); and on the galactic length-scale, our own galaxy rotates about the normal to the plane of its disc with a period of order 3×10^8 years and exhibits a galactic-scale magnetic field roughly confined to the plane of the disc whose typical magnitude is of order 3 or 4×10^{-6} G.

The detailed character of these naturally occurring magnetic fields and the manner in which they evolve in time will be described in subsequent chapters; for the moment it is enough to note that it is the mere existence of these fields (irrespective of their detailed properties) which provides the initial motivation for the various investigations which will be described in this book.

Larmor put forward three alternative and very tentative suggestions concerning the origin of the Sun's magnetic field, only one of which has in any sense stood the test of time. This suggestion, which is fundamental to hydromagnetic dynamo theory, was that, just as for the Earth, motion of the electrically conducting fluid within the rotating body, might by its inductive action in flowing across the magnetic field generate just those currents $\mathbf{J}(\mathbf{x})$ required to provide the self-same field $\mathbf{B}(\mathbf{x})$.

1.3 The homopolar disc dynamo

This type of 'bootstrap' effect is most simply illustrated with reference to a system consisting entirely of solid (rather than fluid) conductors. This is the 'homopolar' disc dynamo (Bullard 1955) illustrated in Figure 1.3. A solid copper disc rotates about its axis with angular velocity Ω, and a current path between its rim and its axle is provided by the wire twisted as shown in a loop round the axle. This system can be unstable to the growth of magnetic perturbations. For suppose that a current $I(t)$ flows in the loop; this generates a magnetic flux Φ across the disc, and, provided the conductivity of the disc is not too high,[4] this flux is given by $\Phi = M_0 I$ where M_0 is the mutual inductance between the loop and the rim of the disc. Rotation of the disc leads to an electromotive force $\mathcal{E} = \Omega\Phi/2\pi$ which drives the current I, and the equation for $I(t)$ is then

[4] This proviso is necessary as is evident from the consideration that a superconducting disc would not allow any flux to cross its rim; a highly conducting disc in a time-dependent magnetic field tends to behave in the

https://doi.org/10.1017/9781107588691.003 Published online by Cambridge University Press

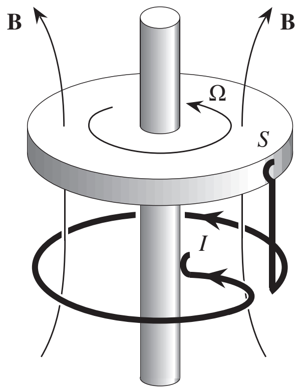


Figure 1.3 The homopolar disc dynamo. Note that the twist in the wire which carries the current $I(t)$ must be in the same sense as the sense of rotation Ω.

$$L\frac{dI}{dt} + RI = \mathcal{E} = M\Omega I\,, \qquad (1.1)$$

where $M = M_0/2\pi$ and L and R are the self-inductance and resistance of the complete current circuit. The device is evidently unstable to the growth of I (and so of Φ) from an infinitesimal level if

$$\Omega > R/M. \qquad (1.2)$$

Under this condition, the current grows exponentially, as does the retarding torque associated with the Lorentz force distribution in the disc. Ultimately the disc angular velocity slows down, and tends to equilibrium at the critical level $\Omega_0 = R/M$ at which the driving torque G just balances the sum of this retarding torque and any frictional torque that may be present. (The system may overshoot this equilibrium state and then oscillate about it, such oscillations being damped by frictional resistance.)

This type of example is certainly suggestive, but it differs from the conducting fluid situation in that the current is constrained by the twisted geometry to follow a very special path that is particularly conducive to dynamo action (i.e. to the conversion of mechanical energy into magnetic energy). No such geometrical constraints are apparent in, say, a spherical body of fluid of uniform electrical conductivity, and the question arises whether fluid motion within such a sphere, or other simply-connected region, can drive a suitably contorted current flow to provide the same sort of homopolar (self-excited) dynamo effect.

There are however two properties of the disc dynamo which reappear in some of

same way; a corrected theory allowing for the associated azimuthal current in the disc will be presented in §10.4.

https://doi.org/10.1017/9781107588691.003 Published online by Cambridge University Press

the hydromagnetic situations to be considered later, and which deserve particular emphasis at this stage. Firstly, there is a discontinuity in angular velocity at the sliding contact S between the rotating disc and the stationary wire, i.e. the system exhibits *differential rotation*. The concentration of this differential rotation at the single point S is by no means essential for the working of the dynamo; we could in principle distribute the differential rotation arbitrarily by dividing the disc into a number of rings, each kept in electrical contact with its neighbours by means of lubricating films of, say, mercury, and by rotating the rings with different angular velocities. If the outermost ring is held fixed (so that there is no longer any sliding at the contact S), then the velocity field is entirely axisymmetric, the differential rotation being distributed across the plane of the disc. The system will still generally work as a dynamo provided the angular velocity of the inner rings is in the sense indicated in Figure 1.3 and sufficiently large.

Secondly, the device *lacks reflectional symmetry*: in Figure 1.3 the disc must rotate in the same sense as the twist in the wire if dynamo action is to occur. Indeed it is clear from equation (1.1) that if $\Omega < 0$ the rotation leads only to an *accelerated decay* of any current that may initially flow in the circuit. Recognition of this essential lack of reflectional symmetry provides the key to understanding the nature of dynamo action as it occurs in conducting fluids undergoing complex motions.

1.4 Axisymmetric and non-axisymmetric systems

It was natural however for early investigators to analyse systems having a maximum degree of symmetry in order to limit the analytical difficulties of the problem. The most natural 'primitive' system to consider in the context of rotating bodies such as the Earth or the Sun is one in which both the velocity field and magnetic field are axisymmetric. As already mentioned, Cowling (1934) considered this idealisation in an investigation of the origin of the much more local and intense magnetic fields of sunspots, but concluded that a steady axisymmetric field could not be maintained by axisymmetric motions. This first 'anti-dynamo' theorem was reinforced by later investigations (Backus & Chandrasekhar 1956; Cowling 1975b) and it was finally shown by Backus (1957) that axisymmetric motions could at most extend the natural decay time of an axisymmetric field in a spherical system by a factor of about 4. In the context of the Earth's magnetic field, whose natural decay time is of the order of 10^4–10^5 years (see Chapter 4), this modest delaying action is totally inadequate to explain the continued existence of the main dipole field for a period of the same order as the age of the Earth itself (3×10^9 years) (the evidence being from studies of rock magnetism), and its relative stability over periods of order 10^6 years and greater (Bullard 1968). It was clear that non-axisymmetric configurations had to be considered if any real progress in dynamo theory were to

https://doi.org/10.1017/9781107588691.003 Published online by Cambridge University Press

be made. It is in fact the essentially three-dimensional character of 'the dynamo problem' (as the problem of explaining the origin of the magnetic field of the Earth or of any other cosmic body has come to be called) that provides both its particular difficulty and its peculiar fascination.

Recognition of the three-dimensional nature of the problem led Elsasser (1946) to initiate the study of the interaction of a prescribed non-axisymmetric velocity field with a general non-axisymmetric magnetic field in a conducting fluid contained with a rigid spherical boundary, the medium outside this boundary being assumed non-conducting. Elsasser advocated the technique of expansion of both fields in spherical harmonics, a technique that was greatly developed and extended in the pioneering study of Bullard & Gellman (1954). The discussion of §7(e) of this remarkable paper shows clear recognition of the desirability of two ingredients in the velocity field for effective dynamo action: (i) a differential rotation which would draw out the lines of force of the poloidal magnetic field to generate a toroidal field (for the definition of these terms, see Chapter 2), and (ii) a non-axisymmetric motion capable of distorting a toroidal line of force by an upwelling followed by a twist in such a way as to provide a feedback to the poloidal field.

Interaction of the velocity field $\mathbf{u}(\mathbf{x})$ and the magnetic field $\mathbf{B}(\mathbf{x})$ (through the $\mathbf{u} \wedge \mathbf{B}$ term in Ohm's law) leads to an infinite set of coupled ordinary differential equations for the determination of the various spherical-harmonic ingredients of possible steady magnetic field patterns, and numerical solution of these equations naturally involves truncation of the system and discretisation of radial derivatives. These procedures are of course legitimate in a numerical search for a solution that is known to exist, but they can lead to erroneous conclusions when the existence of an exact steady solution to the problem is in doubt. The dangers were recognised and accepted by Bullard & Gellman, but it was in fact later demonstrated that the velocity field $\mathbf{u}(\mathbf{x})$ that they proposed most forcibly as a candidate for steady dynamo action in a sphere is a failure in this respect under the more searching scrutiny of higher-speed computers (Gibson & Roberts 1969).

The inadequacy of purely computational approaches to the problem intensified the need for theoretical approaches that do not, at the fundamental level, require recourse to the computer. In this respect a breakthrough in understanding was provided by Parker (1955b) who argued that the effect of the non-axisymmetric upwellings (his 'cyclonic events') referred to above might be incorporated by an averaging procedure in equations for the components of the *mean magnetic field* (i.e. the field averaged over the azimuth angle φ about the axis of rotation of the system). Parker's arguments were heuristic rather than deductive, and it was perhaps for this reason that some years elapsed before the power of the approach was generally appreciated. The theory is referred to briefly in Cowling's monograph *Magnetohydrodynamics* (Cowling 1957) with the following conclusions:

The argument is not altogether satisfactory; a more detailed analysis is really needed. Parker does not attempt such an analysis; his mathematical discussion is limited to elucidating the consequences if his picture of what occurs is accepted. But clearly his suggestion deserves a good deal of attention.[5]

This attention was not provided for some years, however, and was finally stimulated by two rather different approaches to the problem, one by Braginskii (1964b) and the other by Steenbeck et al. (1966). The essential idea behind Braginskii's approach was that, while steady axisymmetric solutions to the dynamo problem are ruled out by Cowling's theorem, nevertheless weak departures from axisymmetry might provide a means of regeneration of the mean magnetic field. This approach can succeed only if the fluid conductivity σ is very high (or equivalently if the magnetic diffusivity $\eta = (\mu_0\sigma)^{-1}$ is very weak), and the theory was developed in terms of power series in a small parameter proportional to $\eta^{1/2}$. By this means, Braginskii demonstrated that, as Parker had argued, non-axisymmetric motions could indeed provide an effective mean toroidal electromotive force (emf) in the presence of a predominantly toroidal magnetic field. This emf drives a toroidal current thus generating a poloidal field, and the dynamo cycle anticipated by Bullard & Gellman can be completed. An account of Braginsky's theory, as reformulated by Soward (1972), is presented in Chapter 8.

The approach advocated by Steenbeck, Krause & Rädler is potentially more general, and is applicable when the velocity field consists of a mean and a turbulent (or random) ingredient having widely different length-scales L and ℓ, say ($L \gg \ell$). Attention is then focussed on the evolution of the mean magnetic field on scales large compared with ℓ. The mean-field approach is of course highly developed in the theory of shear-flow turbulence in non-conducting fluids (see, for example, Townsend, 1975) and it had previously been advocated in the hydromagnetic context by, for example, Kovasznay (1960). The power of the approach of Steenbeck et al. (1966) however lay in recognition of the fact that the turbulence can give rise to a mean electromotive force having a component parallel to the prevailing local mean magnetic field (as in Braginskii's model); and these authors succeeded in showing that this effect would certainly occur whenever the statistical properties of the background turbulence *lack reflectional symmetry*. This property of 'chirality' is the random counterpart of the purely geometrical property of the simple disc dynamo discussed above. The theory of 'mean-field electrodynamics' will be presented in Chapter 7, where the central role of chirality will become apparent.

Since 1966, there has been a growing flood of papers developing different aspects of these theories and their applications to the Earth and Sun and other

[5] It is only fair to note that this somewhat guarded assessment is eliminated in the later edition of the book (Cowling 1975a).

celestial systems. It is the aim of this book to provide a coherent account of the most significant of these developments, and reference to specific papers published since 1970 will for the most part be delayed till the appropriate point in the text.

Several other earlier papers are, however, historical landmarks and deserve mention at this stage. The fact that turbulence could be of crucial importance for dynamo action was recognised independently by Batchelor (1950) and Schlüter & Biermann (1950), who considered the effect of a random velocity field on a random magnetic field, both having zero mean. Batchelor recognised that random stretching of magnetic lines of force would lead to exponential increase of magnetic energy in a fluid of infinite conductivity; and, on the basis of the analogy with vorticity (see §3.6), he obtained a criterion for just how large the conductivity must be for this conclusion to remain valid, and an estimate for the ultimate equilibrium level of magnetic energy density that might be expected when Lorentz forces react back upon the velocity field. Schlüter & Biermann, by arguments based on the concept of equipartition of energy, obtained a different criterion for growth and a much greater estimate for the ultimate level of magnetic energy density. Yet a third possibility was advanced by Saffman (1963) who came to the conclusion that, although the magnetic energy might increase for a while from a very weak initial level, ultimately it would always decay to zero due to accelerated ohmic decay associated with persistent decrease in the characteristic length-scale of the magnetic field. It is now known from consideration of the effect of turbulence that lacks reflectional symmetry (see Chapter 7) that none of the conclusions of the above papers can have any general validity, although the question of what happens when the turbulence is reflectionally symmetric remains to some extent open (see §15.4).

The problem as posed by Batchelor has to some extent been bypassed through recognition of the fact that it is the ensemble-average magnetic field that is of real interest and that if this average vanishes, as in the model conceived by Batchelor, the model can have little direct relevance for the Earth and Sun, both of which certainly exhibit a non-zero dipole moment. It is fortunate that the problem has been bypassed, because in a rather pessimistic diagnosis of the various conflicting theories, Kraichnan & Nagarajan (1967) concluded that

> equipartition arguments, the vorticity analogy, and the known turbulence approximations are all found inadequate for predicting whether the magnetic energy eventually dies away or grows exponentially. Lack of bounds on errors makes it impossible to predict reliably the sign of the eventual net growth rate of magnetic energy.

Kraichnan & Nagarajan have not yet been proved wrong as regards the basic problem with homogeneous isotropic reflectionally symmetric turbulence. Parker (1970) comments on the situation in the following terms:

https://doi.org/10.1017/9781107588691.003 Published online by Cambridge University Press

Cyclonic turbulence,[6] together with large-scale shear, generates magnetic field at a very high rate. Therefore we ask whether the possible growth of fields in random turbulence without cyclonic ordering ... is really of paramount physical interest. We suggest that, even if random turbulence could be shown to enhance magnetic field densities, the effect in most astrophysical objects would be obscured by the more rapid generation of fields by the cyclonic turbulence and non-uniform rotation.

The crucial importance of a lack of reflectional symmetry in fluid motions conducive to dynamo action is apparent also in the papers of Herzenberg (1958) and Backus (1958) who provided the first examples of laminar velocity fields inside a sphere which could be shown by rigorous procedures to be capable of sustained dynamo action. Herzenberg's model involved two spherical rotors rotating with angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ and separated by vector distance $\mathbf{R}$ inside the conducting sphere. The configuration can be described as right-handed or left-handed according as the triple scalar product $[\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{R}]$ is positive or negative. A necessary condition for dynamo action (see §6.9) was that this triple scalar product should be non-zero, and the configuration then certainly lacks reflectional symmetry.

The Backus dynamo followed the pattern of the Bullard & Gellman dynamo, but decomposed temporally into mathematically tractable units. The velocity field considered consisted of three active phases separated by long periods of rest (or 'stasis') to allow unwanted high harmonics of the magnetic field to decay to a negligibly low level. The three phases were: (i) a vigorous differential rotation which generated strong toroidal field from pre-existing poloidal field; (ii) a non-axisymmetric poloidal convection which regenerated poloidal field from toroidal; (iii) a rigid rotation through an angle of $\pi/2$ to bring the newly generated dipole moment into alignment with the direction of the original dipole moment. The lack of reflectional symmetry lies here in the mutual relationship between the phase (i) and phase (ii) velocity fields (see §6.15). These theories, and others relating to laminar dynamo action, will be presented in detail in Chapter 6.

The viewpoint adopted in this book is that random fluctuations in the velocity field and the magnetic field are almost certainly present both in the Earth's core and in the Sun's convection zone, and that a realistic theory of dynamo action should incorporate effects of such fluctuations at the outset. Laminar theories are of course not without value, particularly for the mathematical insight that they provide; but anyone who has conscientiously worked through such papers as those of Bullard & Gellman (1954), Herzenberg (1958) and Backus (1958) will readily admit the enormous complexity of the laminar problem. It is a remarkable fact that acceptance of turbulence (or possibly random wave motions) and appropriate averaging procedures actually leads to a dramatic simplification of the problem. The reason

[6] This is Parker's terminology for turbulence whose statistical properties lack reflectional symmetry.

is that the mean fields satisfy equations to which Cowling's anti-dynamo theorem does not apply, and which are therefore amenable to an axisymmetric analysis with distinctly positive and encouraging results. The equations admit both steady solutions modelling the Earth's quasi-steady dipole field, and, in other circumstances, time-periodic solutions which behave in many respects like the magnetic field of the Sun with its 22-year periodic cycle.

A further crucial advantage of an approach involving random fluctuations is that dynamic, as opposed to purely kinematic, considerations become to some extent amenable to analysis. A kinematic theory is one in which a kinematically possible velocity field $\mathbf{u}(\mathbf{x}, t)$ is assumed known, either in detail or at least statistically when random fluctuations are involved, and its effect on magnetic field evolution is studied. A dynamic theory is one in which $\mathbf{u}(\mathbf{x}, t)$ is constrained to satisfy the relevant equations of motion (generally the Navier–Stokes equations with buoyancy forces, Coriolis forces and Lorentz forces included according to the context); and again the effect of this velocity field on magnetic field evolution is studied. It is only since the advent of the mean-field electrodynamics of Steenbeck, Krause & Rädler that progress on the dynamic aspects of dynamo theory has become possible. These dynamic aspects, which have been increasingly explored over the last 30 years, will be treated in Part III of this book.

The general pattern of the book will therefore be as follows. Chapter 2 will be devoted to simple preliminaries concerning magnetic field structure and diffusion in a stationary conductor. Chapter 3 will be concerned with the interplay of convection and diffusion effects insofar as these influence magnetic field evolution in a moving fluid. In Chapters 4 and 5 we shall digress from the purely mathematical development to provide a necessarily brief survey of the observed properties of the Earth's magnetic field (and other planetary fields) and of the Sun's magnetic field (and other astrophysical fields) and of the relevant physical properties of these systems. This is designed to provide more detailed motivation for the material of subsequent chapters. Some readers may find this motivation superfluous; but it is necessary, particularly when it comes to the study of specific dynamic models, to consider limiting processes in which the various dimensionless numbers characterising the system are either very small or very large; and it is clearly desirable that such limiting processes should at the least be not in contradiction with observation in the particular sphere of relevance claimed for the theory.

Part II (Chapters 6–10) is concerned with the foundations of kinematic dynamo theory, with Chapter 6 focussing on laminar flows and Chapter 7 on both weak and strong turbulence in mean-field electrodynamics, and the theory of the famous α-effect. Chapter 8 will treat nearly axisymmetric systems, and Chapter 9 will survey solutions of the mean-field equations of α^2- or $\alpha\omega$-type. Chapter 10 will treat the concept of the 'fast dynamo' as introduced by Vainshtein & Zel'dovich (1972), i.e. dynamo action for which the growth rate of the magnetic field is independent of resistivity η in the limit $\eta \to 0$, and the pathological structure of magnetic fields that emerge in this situation.

Part III (Chapters 11–17) is concerned with dynamic aspects of the theory. We start in Chapter 11 with low-dimensional models that incorporate dynamic effects, with either mechanical or thermal forcing; such models exhibit the manner in which a dynamo-generated field saturates due to the back reaction of the Lorentz force, frequently in a chaotic state. Chapter 12 treats the phenomenon that has come to be known as α-quenching, which again induces magnetic energy saturation. Chapters 13 and 14 focus respectively on the geodynamo and the solar dynamo, with consideration of the full equations of magnetohydrodynamics in a rotating medium, including the back-reaction of the Lorentz force. Chapter 15 treats specific aspects of turbulence with and without 'helicity' (the simplest measure of chirality) in the statistics of the turbulence.

Finally, Chapters 16 and 17 treat the problem of magnetic relaxation under the topological contraint of a 'frozen-in' magnetic field. In a statistically steady state, the magnetic energy will continue to grow by dynamo action in some regions of the flow, but this growth will be compensated by relaxation of magnetic energy in other regions, a statistically steady state being thereby maintained. Magnetic relaxation is in some respects therefore the counterpart of dynamo action, the two process having comparable 'weight' when statistical equilibrium is attained.

Given the present state of knowledge, it is inevitable that the kinematic theory will occupy a rather greater proportion of the book than it would ideally deserve. It must be remembered however that any results that can be obtained in kinematic theory on the minimal assumption that $\mathbf{u}(\mathbf{x}, t)$ is a kinematically possible but otherwise *arbitrary* velocity field will have a generality that transcends any dynamical model that is subsequently adopted for the determination of $\mathbf{u}$. It is important to seek this generality because, although there is little uncertainty regarding the equations governing magnetic field evolution (i.e. Maxwell's equations and Ohm's law), there are wide areas of uncertainty concerning the relevance of different dynamical models in both terrestrial and solar contexts; for example, it is not yet known what the ultimate source of energy is for core motions that drive the Earth's dynamo. In this situation, any results that do not depend on the details of the governing dynamical equations (whatever these may be) are of particular value. For this reason, the postponement of dynamical considerations to Part III of this work should perhaps be welcomed rather than lamented.

It will be found that the concept of helicity – the spatial average of the scalar product of velocity and vorticity – plays a very central role in dynamo theory and equally in the theory of magnetic relaxation. Figure 1.4 shows a reproduction of Leonardo da Vinci's drawing *The Deluge* held in the Royal Archive at Windsor

Figure 1.4 Leonardo's drawing *The Deluge*, c.1517/18 (pen and black ink with wash): an artist's conception of turbulence with helicity. [RCIN 912380; Royal Collection Trust / © Her Majesty Queen Elizabeth II 2018; reproduced by permission.]

Castle. It would appear that Leonardo was well aware of the helicity of the flows that he imagined, predominantly left-handed in this drawing. The year 2019 is the 500th anniversary of Leonardo's death, when the royal collection of his works will be on open display to the public, an opportunity for helicity to be more widely appreciated! We shall in fact find in Chapter 7 that turbulence with non-zero mean helicity is always capable of generating a large-scale magnetic field in a conducting fluid of sufficient spatial extent, a result that may be conveniently summarised in the couplet

Convection and diffusion in turb'lence with helicity
Yields order from confusion in cosmic electricity,

a fitting note on which to terminate this introductory chapter.