STABILITY OF PERIODIC PLANETARY-TYPE ORBITS OF THE GENERAL PLANAR N-BODY PROBLEM

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1. INTRODUCTION

It is known that families of periodic orbits in the general N-body problem ($N>3$) exist, in a rotating frame of reference (Hadjidemetriou 1975, 1977). A special case of the above families of periodic orbits are the periodic orbits of the planetary type. In this latter case only one body, which we shall call sun, is the more massive one and the rest N-1 bodies, which we shall call planets, have small but not negligible masses. The aim of this paper is to study the properties of the families of periodic planetary-type orbits, with particular attention to stability. To make the presentation clearer, we shall start first with the case N=3 and we shall extend the results to N>3. We shall discuss planar orbits only.

2. PERIODIC PLANETARY-TYPE ORBITS INVOLVING THE SUN AND TWO PLANETS

(a) Families of Periodic orbits

In order to compute periodic orbits of the planetary-type involving 3 bodies, we may start from a degenerate family of periodic orbits where the massless bodies P_1 and P_3 revolve around P_2, whose mass is finite, in circular orbits in the same plane. We shall also assume that they revolve in the same direction. This motion is periodic with respect to a rotating frame of reference Oxy whose x axis contains always P_2 and P_1 (the positive direction being from P_2 to P_1). In fact, if we normalize the units so that the gravitational constant is equal to 1, the total mass is equal to 1 and the radius of P_1 around P_2 is equal to 1, the motion is periodic in the Oxy frame for any value R of the radius of P_3 around P_2, with a period equal to

$$T = 2\pi/(1-T_1/T_3), \qquad (1)$$

where T_1, T_3 are the periods of the two planets, respectively, in the inertial frame and $T_1/T_3=R^{-3/2}$, according to the normalization mentioned

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above. Thus, we have a monoparametric family of periodic orbits, with R as the parameter, which can be considered as a particular case of the restricted circular 3-body problem. The continuation of the orbits of this family to periodic orbits of the general 3-body problem ($m_1,m_3>0$) is possible for all values of R except for those where $T=2\pi n$, $n=1,2,3,\ldots$ In this latter case the continuation theorem is not applicable (Hadjidemetriou 1975). These orbits correspond to the resonances for the periods of the two planets, given by

$$T_1/T_3 = n/(n+1), \qquad (2)$$

as obtained from (1). The same result was obtained by Griffin (1920), by a different proceedure.

The numerical computations have revealed that the orbits of the above degenerate family, for $m_1=m_3=0$, can be extended to the general case (i.e. $m_1,m_3>0$) even for large values of m_1 and m_3 (Hadjidemetriou 1976). The motion obtained by this continuation process in a symmetric periodic motion of the general 3-body problem in a rotating frame Oxy whose origin coincides with the center of mass of P_2 and P_1 and the x axis contains always these two bodies. The two planets P_1 and P_3 describe (in the inertial frame) nearly circular or elliptic orbits around the Sun (P_2), as we shall describe below.

Let us see now how we can represent the above planetary-type periodic orbits. The position of the three bodies, in the rotating frame Oxy can be determined by the coordinates x_1 of P_1 and (x_3,y_3) of P_3. Consequently, a symmetric periodic motion is specified by the initial conditions $x_{10},x_{30},\dot{y}_{30}$ (for the rest variables we have $\dot{x}_{10}=y_{30}=\dot{x}_{30}=0$). Thus, a monoparametric family of periodic orbits is represented by a smooth curve in the space $x_{10}\ x_{30}\ \dot{y}_{30}$. In this paper, we shall use the projection of this curve to the plane $x_{10}\ x_{30}$, which suffices for illustration purposes.

We consider the space of initial conditions $x_{10}\ x_{30}$ (Fig.1). Evidently, the family of degenerate orbits, where P_1 and P_3 have zero masses and describe circular orbits around P_2, is given by the straight line $x_{10}=1$.

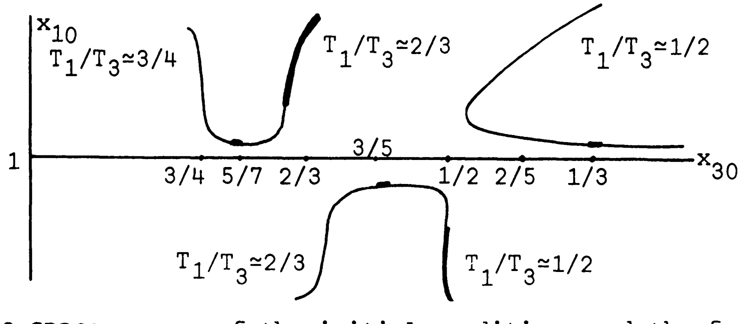

Fig.1: The space $x_{10}x_{30}$ of the initial conditions and the families of planetary-type periodic orbits (graphically).

On this line we mark the resonant orbits of the form $T_1/T_3=n/(n+1)$ for which the continuation theorem is not applicable. These orbits have an accumulation point at $x_{30}=1$. When the degenerate family $x_{10}=1$ is extended to $m_1,m_3>0$, we obtain the set of families of periodic orbits shown in Fig.1 (Hadjidemetriou 1976, Delibaltas 1977). The numerical computations were made for the cases $m_1=m_3=0.001$ and also for the cases m_1=mass of Jupiter, m_3=mass of Saturn and vice-versa. (The total mass was normalized to 1). The results are qualitatively the same for all cases. We note that the degenerate family breaks to (presumably) an infinite number of families of periodic orbits, for $m_1,m_3>0$, which are "separated" by the resonant orbits $T_1/T_3=1/2$, 2/3, 3/4,... The part of the families for $m_1,m_3>0$ which is near the line $x_{10}=1$ corresponds to nearly circular orbits of the two planets. The rest part corresponds to nearly elliptic orbits of the two planets, with eccentricities which increase as we go outwards. Along each elliptic branch the ratio T_1/T_3 of the osculating periods (at t=0) is almost constant for all members of the family. Note that there are two separate branches of nearly elliptic orbits of the two planets for each value of $T_1/T_3=1/2$, 2/3... The difference between them is that at t=0 the two planets are in a different initial situation (at pericenter or apocenter). These elliptic branches can be associated with families of asteroids with the corresponding ratio T_1/T_3, with Jupiter as one of the planets and the asteroid as the other. Also, one member of the first family, in the circular part corresponds to the resonance 2/5 and approximates the Sun-Jupiter-Saturn case.

(b) Stability

We shall study first the stability of the periodic planetary-type orbits where the two planets describe nearly circular orbits. We note that the degenerate periodic orbits of the family $x_{10}=1$ (Fig.1) are, evidently, all stable. It can be proved (e.g. Hadjidemetriou 1978) that there are for each degenerate periodic orbit two stability indices, corresponding to the nonzero characteristic exponents, which are equal to each other and are given, for the normalization used here, by

$$K=-2\cos T, \tag{3}$$

where T is given by (1). These stability indices are not, in general, critical (equal to $K=\pm 2$) and consequently the orbits obtained by the continuation procces, by increasing the masses m_1 and m_3 are stable, for continuation reasons. (The two zero characteristic exponents of the degenerate orbit in the rotating frame are preserved when m_1, $m_3>0$, due to the existence of the energy integral). However, there are degenerate periodic orbits with critical stability indices, (K=2), corresponding to $T=(2\nu+1)\pi$, i.e. to the resonant orbits $T_1/T_3=(2\nu-1)/(2\nu+1)$. The resonant orbits $T_1/T_3=1/3$, 3/5, 5/7, of this kind are shown in Fig.1. These orbits can become unstable when extended to $m_1,m_3>0$, i.e. $|K|>2$. And in fact, the numerical computations have revealed that in each of the families for $m_1,m_3>0$, in their part of circular orbits, there is a small unstable region, generated from the above critical degenerate periodic orbits.

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We come now to the branches of the elliptic orbits of the families for $m_1,m_3>0$. Such an orbit can be considered to be obtained by the continuation of a degenerate periodic orbit where one planet describes an elliptic orbit. The stability indices in this latter case are all critical and may become stable or unstable when the masses are increased. The numerical computations have revealed that some of the branches of elliptic orbits are stable and others are unstable (Hadjidemetriou 1976, Delibaltas 1977). The unstable parts of the families are shown by bold lines in Fig.1. It was also revealed by the numerical computations that the stability of the same branch of elliptic orbits depends on the ratio m_1/m_3. For example, the resonant branch 1/2 of the first family is stable when the inner planet P_1 has the mass of Saturn and the outer planet P_3 has the mass of Jupiter, but it is unstable when the inner planet P_1 has the mass of Jupiter and the outer planet P_3 the mass of Saturn.

Another question concerning the stability is how the stability evolves when the masses of the planets increase. A general result is that the unstable regions in the families of periodic planetary-type orbits extend when the masses of the planets increase. For example, the unstable region in the first family, at the resonance 1/3, extends and when the masses of the planets become about 38 times larger than the masses of Jupiter and Saturn, (the ratio m_1/m_3 being kept fixed) the unstable region covers the resonant orbit 2/5 which represents the Jupiter-Saturn system (Hadjidemetriou and Michalodimitrakis, 1978a).

3. PERIODIC PLANETARY-TYPE ORBITS INVOLVING THE SUN AND THREE PLANETS

The method used to compute periodic orbits of the planetary type involving two planets can be extended to any number of planets. We shall study here the case N=4, i.e. the Sun (or planet) and three planets (or satellites). We call P_1, P_3 and P_4 the three planets, respectively and P_2 the Sun and consider the degenerate system with $m_1=m_3=m_4=0$, $m_2=1$, where the three planets revolve around the sun in circular orbits in the same plane. We shall also assume that they revolve in the same direction. The motion is periodic with respect to a rotating frame Oxy whose x axis contains the bodies P_2 and P_1, and the origin is at P_2, if

$$(\omega_3-\omega_1)/(\omega_4-\omega_1) = p/q, \qquad (4)$$

where p,q are integers and $\omega_i=2\pi/T_i$, i=1,3,4, T_i being the periods of the planets in the inertial frame. Using the same normalization as in the case N=3 (i.e. G=1, $m_1+m_2+m_3+m_4=1$, radius of P_1=1) we have for the period of the above degenerate periodic orbit in the rotating frame,

$$T = 2\pi q/(1-T_1/T_4). \qquad (5)$$

Note that when p/q is fixed, for each radius R_3 of P_3 there corresponds a certain radius R_4 of P_4, as can be seen from (4). Thus, for fixed p,q, we have a monoparametric family of degenerate periodic orbits with R_3 as the parameter, in which the three planets describe circular orbits. All these orbits can be continued to the general case m_1, m_3, $m_4>0$ with the exception of those orbits where $T=2\pi n$, $n=1,2,3,...$ (Hadjidemetriou, 1977). As a consequence, the degenerate family of periodic orbits mentioned above breaks to an infinite number of families of periodic orbits when the masses are increased, in the same way as in the case N=3. These families, for m_1, m_3, $m_4>0$, contain symmetric periodic orbits with respect to a rotating frame Oxy whose origin is at the center of mass of P_2,P_1 and its x axis contains always these bodies. As in the N=3 case, these families contain branches along which the ratios T_1/T_3 and T_1/T_4 are nearly constant, but the eccentricities of the osculating orbits of P_1, P_3, P_4 around P_2 vary. A detailed analysis of this type of planetary-type orbits, corresponding to the case p/q=2/3 will be given elsewhere (Hadjidemetriou and Michalodimitrakis, 1978b). We describe here four distinct branches, A,B,C,D, of the above monoparametric families, corresponding to the ratios $T_1/T_3=1/2$, $T_1/T_4=1/4$. This case represents the motion of the three inner Galilean satellites of Jupiter. The computations were made for the actual masses of Jupiter and its three satellites. Branch A corresponds to the actual case. The main characteristics of these branches are shown below. a stands for apojove and p for perijove at t=0.

Branch	I	II	III	Initial situation	Stability
A	p	p	p	I, III in conjunction, II in opposition	stable
B	a	a	a	I, III " " , II " "	unstable
C	p	p	p	I, II " " , III " "	unstable
D	a	a	a	I, II " " , III " "	unstable

De Sitter (1908, 1909, 1918, 1928) (see Hagihara 1961) has obtained one periodic orbit with the properties of the branch A and one with the properties of B. It is clear however from this analysis that there exists an inifinite number of such orbits, with the same resonance but different eccentricities.

4. THE RESTRICTED PLANETARY 4-BODY PROBLEM.

As an approximation to the above families of periodic orbits we consider now the case where the mass of one planet is negligible. This is justified in the Solar System where the main planets are Jupiter and Saturn. Thus, we may consider a four-body system with P_2 as the Sun, P_1 as Jupiter, P_3 as Saturn and P_4 as a massless planet, and take the case $T_1/T_3=2/5$, as in the actual motion of the Sun-Jupiter-Saturn system. Since the motion of P_4 does not affect the motion of P_1,P_2 and P_3, its motion is determined from a system with two degrees of freedom (for planar motion), in the rotating frame of reference Oxy defined in section 3. This case can be considered as the generalization of the restricted 3-body problem. The conditions under which the motion of P_4 is periodic are similar to the general case N=4, discussed in section 3. A complete description of this problem will be given elsewhere (Hadjidemetriou 1978).

We shall present here one example of a periodic motion, corresponding to the case p/q=-1/9. This motion approximates the system Sun-Mars-Jupiter-Saturn, as the "radius" of the nearly circular orbit of P_4 is equal to 1.51 A.U. There are two such periodic orbits with the same resonance, differing in phase. One of them corresponds to $x_{10}>0$, $x_{30}<0$, $x_{40}>0$, and the other to $x_{10}>0$, $x_{30}<0$, $x_{40}<0$, at t=0. The numerical computations revealed that the first periodic orbit is stable and the second is unstable.

Let us draw now our attention to the unstable case. The orbits of all the planets are nearly circular and this periodic motion can be considered to be obtained from a degenerate periodic motion where all the planets have zero masses and describe circular orbits, by continuing through the masses. Evidently, if only the mass of Jupiter is increased, and its motion is kept circular, the motion of P_4 is determined from the well-known restricted circular 3-body problem. The numerical integrations reveal in this case that the motion of P_4 is stable. Let us introduce now the planet Saturn with its actual mass. The above results show that the orbit of P_4 may become unstable due to the effect of Saturn! This indicates that the stability analysis of planetary systems based on the study of the restricted 3-body problem may not be always correct, because the effect of other planets than Jupiter is not taken into account.

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