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EXISTENCE THEOREM FOR PROXIMATE TYPE OF ENTIRE FUNCTIONS WITH INDEX-PAIR (P,Q)

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R.S.L. Srivastava and O.P. Juneja (1967) proved an existence theorem for the proximate type $T(r)$ of an entire function with classical growth. For an interesting generalization of this theorem for an entire function with index-pair (p,q), which is due essentially to H.S. Kasana and S.K. Vaish (1984), a remarkably simple (and markedly different) construction of $T(r)$ is presented here. The main theorem established here applies to a much larger class of entire functions with index-pair (p,q) than that considered earlier.

Introduction

Let $f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}$ $(0 \le \lambda_1 < \ldots < \lambda_n < \lambda_{n+1} \to \infty$ as $n \to \infty)$ be a nonconstant entire function. Set $M(r) = \max_{|z|=r} |f(z)|$; $M(r)$ is called the maximum modulus of $f(z)$.

The concept of (p,q)-order and lower (p,q)-order of an entire function with index-pair (p,q), $p \ge q \ge 1$, was introduced by Juneja

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et al. [1]. Thus $f(z)$ is said to be of (p,q)-order ρ and lower (p,q)-order λ if it is of index-pair (p,q) and

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p]} M(r)}{\log^{[q]} r} = \begin{matrix}\rho(p,q) \equiv \rho\\ \lambda(p,q) \equiv \lambda\end{matrix}$$

DEFINITION 1. An entire function $f(z)$ for which (p,q)-order and lower (p,q)-order are the same is said to be of regular (p,q)-growth. Functions which are not of regular (p,q)-growth are called of irregular (p,q)-growth.

Juneja et al. [2] also defined (p,q)-type and lower (p,q)-type as:

DEFINITION 2. An entire function $f(z)$ having (p,q)-order $\rho(b<\rho<\infty)$ is said to be of (p,q)-type T and lower (p,q)-type t if

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} M(r)}{(\log^{[q-1]} r)^{\rho}} = \begin{matrix}T(p,q) \equiv T\\ t(p,q) \equiv t ,\end{matrix}$$

where $b = 1$ if $p = q$ and $b = 0$ if $p>q$.

Considering a natural extension of the notion of Lindelöf's proximate order (see Valiron [5, pp.64-67]) of an entire function, Srivastava and Juneja [4,p.8, Theorem 1] proved the corresponding existence theorem for proximate type $T(r)$. Recently, Kasana and Vaish [3] generalized this theorem by establishing the existence of a real valued function, called (p,q)-proximate type or simply proximate type of an entire function $f(z)$ with index-pair (p,q) , (p,q)-order $\rho(b<\rho<\infty)$ and (p,q)-type $T(0<T<\infty)$. The object of the present note is to give an elementry proof for the existence of $T(r)$ in this general case. Our proof presented in Section 3 below, is applicable to a much wider class of entire functions with index-pair (p,q) ; indeed, for the entire functions of (p,q)-type zero or infinity it provides a considerable improvement on Kasana and Vaish's theorem.

2. The main existence theorem

The following result of existence of proximate type of an entire function with index-pair (p,q) is an interesting generalization of Kasana and Vaish's theorem [3, p.334, Theorem 1]:

THEOREM. *For every entire function* $f(z)$ *with index-pair* (p,q), (p,q)*-order* ρ $(b<\rho<\infty)$ *and* (p,q)*-type* $T(0\le T\le\infty)$, *there exists a positive continuous function* $T(r)$, *called the proximate type of* $f(z)$, *which for a given number* $\alpha(0<\alpha<\infty)$, *satisfies each of the following conditions:*

(i) $T(r)$ *is piecewise differentiable for* $r \in [r_0,\infty)$, $r_0 > \exp[q-1]0$;

(ii) $T(r) \to T$ *as* $r \to \infty$;

(iii) $\dfrac{\Lambda_{[q-1]}(r)T'(r)}{T(r)} \to 0$ *as* $r\to\infty$, *where* $T'(r)$ *can be interpreted as either* $T'(r-)$ *or* $T'(r+)$ *when these are unequal; and*

(iv) $$\limsup_{r\to\infty} \frac{\log^{[p-2]} M(r)}{\exp\{(\log^{[q-1]}r)^{\rho}\ T(r)\}} = \alpha ,$$

where, for convenience, $\Lambda_{[q]}(x) = \prod_{i=0}^{q} \log^{[i]} x$.

Remark. For an entire function of finite positive (p,q)-type , the assertion III is equivalent to the corresponding assertion due to Kasana and Vaish [3, p.333, equation (1.3)].

Our proof of this general theorem, given in the next section, is rather simple and markedly different from the proof presented earlier by Kasana and Vaish [3, pp. 334-337]. Moreover, it includes a larger class of entire functions with index-pair (p,q) .

The usefulness of the approach developed in this paper lies in the fact that while defining the lower proximate type (a real valued function which takes into account the lower (p,q)-type of the function and is closely linked with its maximum modulus $M(r)$) for entire functions with index-pair (p,q) , we need not to consider separately λ-proximate type [3, p.337, Definition] for studying the growth of entire functions of irregular (p,q)-growth.

3. Construction of proximate type

The proof consists of verifying that the definition of $T(r)$ corresponding to each of the alternatives given below posseses the properties required by the theorem.

If there exists r_0 such that $\log^{[p]} M(r) < \rho \log^{[q]} r$ for $r > r_0$, then for a given number $\alpha (0<\alpha<\infty)$, we define

$$(3.1)\qquad \mu_q^p(r) = \begin{cases} \sup\limits_{r_0 \le x \le r} \left\{ \dfrac{\log^{[2]}(\alpha^{-1} \log^{[p-2]} M(x))}{\log^{[q]} x} \right\} - \rho \ , & for\ r > r_0 \\ \mu_q^p(r_0) \ , & for\ 0 \le r \le r_0 \ , \end{cases}$$

otherwise we define

$$(3.2)\qquad \mu_q^p(r) = \sup_{x \ge r} \left\{ \frac{\log^{[2]}(\alpha^{-1} \log^{[p-2]} M(x))}{\log^{[q]} x} \right\} - \rho \ .$$

If $\limsup\limits_{r\to\infty} \{\mu_q^p(r) \log^{[q]} r\} = -\infty$, when $\mu_q^p(r)$ must be defined by (3.1), we set

$$T(r) = \sup_{x \ge r} \{exp(\mu_q^p(x) \log^{[q]} x)\} \ .$$

If $\limsup\limits_{r\to\infty} \{\mu_q^p(r) \log^{[q]} r\} = \infty$, we set

$$T(r) = \sup_{x \le r} \{exp(\mu_q^p(x) \log^{[q]} x)\} \ .$$

If $\limsup\limits_{r\to\infty} \{\mu_q^p(r) \log^{[q]} r\} = \log \delta$, $0 < \delta < \infty$, we set

$$\text{(a)}\quad T(r) = \delta + \sup_{r_1 \le x \le r} \left\{ \frac{\log \left[\sup\limits_{t \ge x} \left\{ \dfrac{\alpha^{-1} \log^{[p-2]} M(t)}{exp\{\delta (\log^{[q-1]} t)^{\rho}\}} \right\} \right]}{(\log^{[q-1]} x)^{\rho}} \right\}, \qquad \gamma = 0$$

where r_1 is such that $\log^{[p-2]} M(r) < \alpha\, exp\{\delta (\log^{[q-1]} r)^{\rho}\}$ for all $r > r_1$;

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(b) $T(r) = \delta + \dfrac{\log \gamma}{\log^{[q-1]} r}$, if $0 < \gamma < \infty$;

and

$$\text{(c)}\quad T(r) = \delta + \sup_{x \geq r} \left\{ \frac{\log \left[\sup\limits_{t \leq x} \left\{ \dfrac{\alpha^{-1} \log^{[p-2]} M(t)}{exp\{\delta(\log^{[q-1]} t)^{\rho}\}} \right\} \right]}{(\log^{[q-1]} x)} \right\}, \text{ if } \gamma = \infty ,$$

where, for convenience,

$$\gamma = \limsup_{r \to \infty} \left\{ \frac{\log^{[p-2]} M(r)}{exp\{\delta(\log^{[q-1]} r)^{\rho}\}} \right\}$$

We note that if $f(z)$ is of finite positive (p,q)-type, then $\delta = T$.

Verification. We illustrate the general approach by establishing the result for the case corresponding to (c), above. For this case, let

$$\phi(r) = \sup_{r_0 \leq x \leq r} \frac{\alpha^{-1} \log^{[p-2]} M(x)}{exp\{\delta(\log^{[q-1]} x)^{\rho}\}}, \quad (r_0 > exp^{[q-2]} 1) \ ;$$

then $\phi(r)$ is continuous, nondecreasing and tends to infinity with r, r_0 is fixed so that $\log \phi(r)$ is positive. Let A be the set of $r \in [r_0, \infty)$ for which $\phi(r) = \phi(t)$ for some $t < r$, then A is the set of bounded semi-open intervals $\{(a_n, b_n]\}$ and may be empty, bounded and unbounded. In the complement of the set of open intervals $\{(a_n, b_n)\}$ we have

$$\phi(r) = \frac{\alpha^{-1} \log^{[p-2]} M(r)}{exp\{\delta(\log^{[q-1]} r)^{\rho}\}} .$$

If $r \in (a_n, b_n)$, *then*

$$0 < \frac{\log \phi(r)}{(\log^{[q-1]} r)^{\rho}} = \frac{\log^{[p-1]} M(a_n) - \delta(\log^{[q-1]} a_n)^{\rho} - \log \alpha}{(\log^{[q-1]} r)^{\rho}}$$

$$\leq \frac{\log(\alpha^{-1} \log^{[p-2]} M(a_n))}{(\log^{[q-1]} a_n)^{\rho}} - \delta \ ,$$

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and outside these intervals

$$0 < \frac{\log \phi(r)}{(\log^{[q-1]} r)^{\rho}} = \frac{\log(\alpha^{-1} \log^{[p-2]} M(r))}{(\log^{[q-1]} r)^{\rho}} - \delta .$$

It follows that $\log \phi(r)/(\log^{[q-1]} r)^{\rho} \to 0$ as $r \to \infty$, since otherwise it would contradict the definition of δ. Therefore

$$\psi(r) = \sup_{x \geq r} \frac{\log \phi(x)}{(\log^{[q-1]} x)^{\rho}} , \quad r > r_0$$

is a positive and nonincreasing function with limit zero as $r \to \infty$.

If, for some $r_n \in (a_n, b_n)$, we have

$$\psi(r_n) = \frac{\log \phi(r_n)}{(\log^{[q-1]} r_n)^{\rho}} = \frac{\log \phi(a_n)}{(\log^{[q-1]} r_n)^{\rho}} ;$$

we must also have

$$\psi(r) = \frac{\log \phi(a_n)}{(\log^{[q-1]} r)^{\rho}} \quad \text{for} \quad a_n \leq r \leq r_n ;$$

in particular this holds for $r = a_n$. Now the set of r for which $\psi(r) = \log \phi(r)/(\log^{[q-1]} r)^{\rho}$ is necessarily unbounded and we have shown that this set cannot be a subset of the intervals $\{(a_n, b_n)\}$. It follows that we have, simultaneously,

$$\psi(r) = \frac{\log \phi(r)}{(\log^{[q-1]} r)^{\rho}} = \frac{\log(\alpha^{-1} \log^{[p-2]} M(r))}{(\log^{[q-1]} r)^{\rho}} - \delta$$

for an unbounded set of r. For the case under discussion $T(r)$ is equal to $\psi(r) + \delta$ and hence for an unbounded set of r we have

$$T(r) = \frac{\log(\alpha^{-1} \log^{[p-2]} M(r))}{(\log^{[q-1]} r)^{\rho}} .$$

This readily implies the assertions (i) and (ii) of the theorem. Since, by definition,

$$T(r) \geq \frac{\log(\alpha^{-1} \log^{[p-2]} M(r))}{(\log^{[q-1]} r)^{\rho}} \quad \text{for all} \quad r > r_0$$

the assertion (iv) of the theorem also follows at once.

The set $\{r : r_0 < r < \infty\}$ can be divided into a sequence of intervals in which

$$\psi(r) = \text{a constant},$$

or

$$\psi(r)(\log^{[q-1]} r)^{\rho} = \text{a constant},$$

or

$$\psi(r) = \frac{\log(\alpha^{-1} \log^{[p-2]} M(r))}{(\log^{[q-1]} r)^{\rho}} - \delta$$

and since $M(r)$ is differentiable in adjacent intervals, the same is true for $\psi(r)$ and hence also for $T(r)$. For such intervals we note that $T'(r) = \psi'(r)$. Now

$$\psi'(r) = 0,$$

or

$$\psi'(r) = -\frac{\rho\psi(r)}{\Lambda_{[q-1]}(r)},$$

or

$$\psi'(r) = -\frac{\rho\psi(r)}{\Lambda_{[q-1]}(r)} + \frac{\phi'(r)}{\phi(r)(\log^{[q-1]} r)^{\rho}}$$

and since $\psi'(r) \le 0$ and $\phi'(r) \ge 0$, we have

$$0 \ge \frac{\Lambda_{[q-1]}(r)T'(r)}{T(r)} \ge -\frac{\rho\psi(r)}{\delta+\psi(r)} \to 0 \quad \text{as} \quad r \to \infty .$$

This gives part (iii) of the theorem; the results for $T'(r-)$ and $T'(r+)$ follow similarly.

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