29 Large and warped extra dimensions

Considerations of the sort we encountered in the previous chapter have inspired two approaches to Beyond the Standard Model physics: large extra dimensions (LED or ADD) and warped spaces (Randall–Sundrum). In this chapter we will provide a brief introduction to each.

29.1 Large extra dimensions: the ADD proposal

In string theory it is natural to imagine that the compactification scale is not much different from the Planck scale. The size of the compact space is typically a modulus, and if it is stabilized then one might this to happen at a value not much different from one, in string (and therefore Planck) units. In terms of our general discussion of moduli stabilization we have seen that, once the radius becomes very large, any potential, perturbative or non-perturbative, tends to zero.

But if we are willing to discard this natural prejudice, an extraordinary possibility opens up. Perhaps the extra dimensions are not Planck size but much larger, even macroscopic? Arkani-Hamed, Dimopoulos and Dvali (ADD) realized that, from an experimental point of view, the limits on the size of such large compact dimensions are surprisingly weak. Allowing the extra dimensions to be large totally reorients our thinking about the nature of couplings and scales in string theory (or any underlying fundamental theory). Such a viewpoint places the hierarchy problem in a whole different light, perhaps allowing solutions entirely different from technicolor or supersymmetry.

Branes are crucial to this picture. The observed gauge couplings are small, but not extremely small. In Kaluza–Klein theory and in weakly coupled string theories, however, they are related to the underlying scales in a clear way. For example, in the heterotic string,

$$g_4^{-2} \cong g_s^{-2} M_s^6 R^6. \tag{29.1}$$

So, if g_4 is fixed then as $R \to \infty$, $g_s \to \infty$, but even in a compactified theory the gauge coupling on $D3$-branes is insensitive to the large volume. With more general branes one has more intricate possibilities, depending on how the branes wrap the internal space. However, gravity becomes weak as R becomes large:

$$G_N = \frac{1}{M_p^2} = \frac{1}{M_p^8 R^6} = \frac{g_s^2}{\ell_s^8 R^6}. \tag{29.2}$$

https://doi.org/10.1017/9781009290883.036 Published online by Cambridge University Press

Here M_p is the Planck mass. Now, if g_s is fixed and of order one, as $R \to \infty$, the Planck length tends to zero.

How large might we imagine R could be? If we assume that R is macroscopic, or nearly so, then on distance scales smaller than R the force of gravity will be that appropriate to a higher-dimensional theory. In d space–time dimensions,

$$\text{force}_g \sim \frac{1}{r^{d-2}}. \tag{29.3}$$

If there are a large extra dimensions, any others will be comparable in size with the fundamental scale, $M_{\text{p}}^2 = M_{\text{fund}}^{2+a} R^a$, or

$$M_{\text{fund}} = \left(M_{\text{p}}^2 R^{-a}\right)^{1/(2+a)}, \quad R = M_{\text{p}}^{-1} (M_{\text{p}}/M_{\text{fund}})^{-(2+a)/a}. \tag{29.4}$$

A new viewpoint on the hierarchy problem arises by supposing that M_{fund} is close to the scale of weak interactions, say $M_{\text{fund}} \sim 1$ TeV.Then we can use Eq. (29.4) to relate R to the Planck scale and the weak scale. For example, if $a = 2$, $R \approx 0.01$ cm! For larger a, R is smaller, but still dramatically large; for $a = 3$, for example, it is about 10^{-7} cm. But the value of R for $a = 1$ would be, quite literally, astronomical in size and is clearly ruled out by observations.

Subsequently to the ADD proposal there has been a serious campaign to improve the experimental limits on gravity at mm and smaller scales. With

$$V(r) = -G_N \frac{m_1 m_2}{r} \left(1 + \alpha e^{-r/R}\right), \tag{29.5}$$

one now knows that $R < 37$ μm.

The possibility of large extra dimensions offers a different perspective on the hierarchy problem. The weak scale is fundamental; the issue is to understand why the radius of the large dimensions is so large. One possibility which has been seriously considered is that there are some very large fluxes. For example, if H_{MN} is a two-form associated with a $U(1)$ gauge field and Σ is some closed two-dimensional surface, we could have

$$\int_\Sigma H_{MN} dx^M \wedge dx^N = N. \tag{29.6}$$

If the radius of the dimensions associated with Σ were large then

$$H \sim \frac{N}{R^2}. \tag{29.7}$$

The potential, in turn, would receive a contribution behaving as N^2/R^2. If there were also a (positive) cosmological constant then

$$V = \Lambda R^2 + \frac{N^2}{R^2} \tag{29.8}$$

and, assuming that Λ were of order the fundamental scale,

$$R^4 \sim N^2 \ell_{\text{fund}}^4. \tag{29.9}$$

https://doi.org/10.1017/9781009290883.036 Published online by Cambridge University Press

To obtain a sufficiently large radius in this way, then, requires an extremely large flux. There are some circumstances where such large pure numbers may not be required; supersymmetry and low dimensionality ($a = 2$) would help.

For now we will assume that somehow a large radius arises, for dynamical reasons, and consider some other questions which, ultimately, such a picture raises.

1. *Proton decay* With no further assumptions about the theory we would expect that baryon-number-violating operators would arise, suppressed only by the TeV scale. It would then be necessary to suppress operators of very high dimension. One possible resolution of this problem is elaborate discrete symmetries. Another suggestion has been that the modes responsible for the different low-energy fermions might be very nearly orthogonal.
2. *Other flavor-changing processes* For the same reason, flavor changing processes in weak interactions, processes such as $\mu \to e + \gamma$ and the like pose a danger. One possible solution is that there is a fundamental scale a few orders of magnitude larger than the weak scale. This raises the question of why the weak scale is small – the hierarchy problem again. The orthogonality of fermions, again, can help with many of these difficulties.

We turn, finally, to the phenomenology of large extra dimensions. Here there are exciting possibilities. If R is large then the Kaluza–Klein modes are very light. They are very weakly coupled, but there are many of them and little energy is required for their production. So, let us consider the inclusive production of Kaluza–Klein particles in an accelerator. In terms of $G_{\rm N} = \kappa^2/(8\pi)$, the amplitude for the emission of a Kaluza–Klein particle is proportional to κ. For any given mode, then, the cross section behaves as $\sigma_n \sim G_{\rm N}E^2$, where the E^2 factor follows from dimensional analysis. We need to sum over n or, equivalently, to integrate over a-dimensional phase space. As a crude estimate that we can treat the amplitude as constant and cut off the integration at E, so

$$\sigma_{\rm tot} = R^a \int d^a k\, \sigma_k = G_{\rm N} R^a E^{2+a}. \tag{29.10}$$

Recalling that $G_{\rm N} = G_{\rm fund} R^{-a}$, we see that the tower of Kaluza–Klein particles couples like a $(4 + a)$-dimensional particle: at high energies the extra dimensions are manifest! The cross section exhibits exactly the behavior with energy that one expects in $4 + a$ dimensions.

The actual processes which might be observed in accelerators are quite distinctive. One would expect to see, for example, the production of high-energy photons accompanied by missing energy, with the cross section showing a dramatic rise with energy. Such signatures have already been used (as of the time of writing) to set limits on such couplings.

The production of Kaluza–Klein particles in astrophysical environments can be used to set limits on extra dimensions as well. For example, in the case of two large dimensions and a fundamental scale of order 1 TeV, we saw that the scale of the Kaluza–Klein excitations – the inverse of the radius of the extra dimensions – is of order 10^{-12} GeV, so such particles are easy to produce. Like axions, they might be readily produced in stars.

29.2 Warped spaces: the Randall–Sundrum proposal

Having entertained the possibility that some compact dimensions of space might be very large, one might wonder why the extra dimensions should be flat. In fact, in the Horava–Witten theory the extra dimensions are not that. Taking the formulas of this theory literally, we have seen that if it describes nature then the eleventh dimension is quite large in fundamental units. The metric of this dimension is significantly distorted; we might say that it is warped. This is not surprising; the geometry is essentially one-dimensional. The Green's functions for the fields grow linearly with distance. One of the appealing features of the Horava–Witten proposal is that the dimensions are just large enough that the distortion of the geometry is of order one.

Randall and Sundrum made a more radical proposal: they argued that the warping might be enormous and might account for the large hierarchy between the weak scale and the Planck scale. In the simplest version of their model there is again one extra dimension; call its coordinate ϕ, $0 < \phi < \pi$. The model contains two branes, one at $\phi = 0$, one at $\phi = \pi$. The tensions of the two branes are taken to be equal and opposite. One imagines that the Standard Model fields propagate on one brane, the "visible sector" brane, while some other, hidden, sector fields propagate on the other. The action is then

$$S = S_{\rm grav} + S_{\rm vis} + S_{\rm hid}. \tag{29.11}$$

The bulk gravitational action $S_{\rm grav}$ includes a cosmological constant term:

$$S_{\rm grav} = \int d^4x \int d\phi \, \sqrt{-G}(-\Lambda + 2M^3\mathcal{R}), \tag{29.12}$$

where M is the five-dimensional Planck mass. The brane actions are

$$S_{\rm vis} = \int d^4x \sqrt{-g_{\rm vis}}(\mathcal{L}_{\rm vis} - \Lambda_{\rm vis}), \quad S_{\rm hid} = \int d^4x \sqrt{-g_{\rm hid}}(\mathcal{L}_{\rm hid} - \Lambda_{\rm hid}). \tag{29.13}$$

Here we have separated off a brane tension term on each brane; we have also distinguished the bulk five-dimensional metric G_{MN} from the metrics on each of the branes, $g_{\mu\nu}$. This has the structure of a gravitational problem in five dimensions, with δ-function sources at $\phi = 0, \pi$. Einstein's equations are

$$\sqrt{-G}\left(R_{MN} - \frac{1}{2}G_{MN}R\right) = -\frac{1}{4M^3}\Big[\Lambda\sqrt{-G}G_{MN} + \Lambda_{\rm vis}\sqrt{-g_{\rm vis}}g^{\rm vis}_{\mu\nu}\delta^\mu_M\delta^\nu_N\delta(\phi-\pi) + \Lambda_{\rm hid}\sqrt{-g_{\rm hid}}g^{\rm hid}_{\mu\nu}\delta^\mu_M\delta^\nu_N\delta(\phi)\Big]. \tag{29.14}$$

Now one makes an ansatz for the metric which leads to warping:

$$ds^2 = e^{-2\sigma(\phi)}\eta_{\mu\nu}dx^\mu dx^\nu + r_{\rm c}^2 d\phi^2. \tag{29.15}$$

Here r_c is the radius of the compact dimension. Substituting the ansatz Eq. (29.15) into the five-dimensional Einstein equation (29.14) one obtains equations for σ:

$$\frac{6\sigma'^2}{r_{\rm c}^2} = \frac{-\Lambda}{4M^3}, \quad \frac{3\sigma''}{r_{\rm c}^2} = \frac{\Lambda_{\rm hid}}{4M^3 r_{\rm c}}\delta(\phi) + \frac{\Lambda_{\rm vis}}{4M^3 r_{\rm c}}\delta(\phi - \pi). \tag{29.16}$$

https://doi.org/10.1017/9781009290883.036 Published online by Cambridge University Press

This is solved by

$$\sigma = r_c|\phi|\sqrt{-\frac{\Lambda}{24M^3}}, \tag{29.17}$$

provided that the following conditions on the Λs hold:

$$\Lambda_{\text{hid}} = \Lambda_{\text{vis}} = 24M^3k, \quad \Lambda = -24M^3k^3. \tag{29.18}$$

In this case the metric varies exponentially rapidly. Note that r_c does not need to be extremely large in order that one obtain an enormous hierarchy. One might worry, though, about the identification of the graviton. It turns out that the metric has zero modes:

$$ds^2 = e^{-2kr_c|\phi|}[\eta_{\mu\nu} + \tilde{h}_{\mu\nu}(x)dx^\mu dx^\nu + T^2(x)d\phi^2], \tag{29.19}$$

where T^2 represents a variation on r_c, usually referred to as the *radion*, and $\tilde{h}_{\mu\nu}$ is the four-dimensional metric. If one substitutes into the action, one finds

$$S = \int d^4x \int d\phi\, 2M^3 r_c e^{-2kr_c|\phi|}\sqrt{-\tilde{g}}\tilde{R}. \tag{29.20}$$

From this we can read off the effective Planck mass:

$$M_p^2 = M^3 r_c \int d\phi\, e^{-2kr_c|\phi|} = \frac{M^3}{k}\left(1 - e^{-2kr_c}\right). \tag{29.21}$$

So, the four-dimensional Planck scale is comparable with the fundamental five-dimensional scale.

To see that the physical masses on the visible brane are small, consider the visible sector action for a scalar particle:

$$S_{\text{vis}} = \int d^4x\, \sqrt{-g}e^{-4kr_c\pi}\left[\tilde{g}_{\mu\nu}e^{2kr_c\pi}|D_\mu\phi|^2 - \lambda\left(|\phi|^2 - v_0^2\right)^2\right]. \tag{29.22}$$

Rescaling ϕ to $e^{kr_c\pi}\phi$, we have

$$S_{\text{vis}} = \int d^4x\, \sqrt{-g}\left[\tilde{g}_{\mu\nu}|D_\mu\phi|^2 - \lambda\left(|\phi|^2 - e^{-2kr_c\pi}v_0^2\right)^2\right], \tag{29.23}$$

so the scale is indeed exponentially smaller than the scale on the other brane.

There are many questions one can ask about this structure.

1. How robust is this type of localization of gravity?
2. How do higher excitations, e.g. bulk fields, interact with the fields on the brane? Is the hierarchy stable? (The answer is yes.)
3. Does this sort of warping arise in string theory? Again, the answer is yes, though the details look different.
4. As in the case of large extra dimensions, if this picture makes sense then there are many excitations on the branes; higher-dimension operators are suppressed only by the TeV scale. As there, one has to ask: how does one understand the conservation of baryon number? Other flavor-changing processes? Neutrino masses? Precision electroweak physics? Answers have been put forward to all these questions, but they remain suitable subjects for research. Precision electroweak corrections typically require that the lightest Kaluza–Klein K modes be more massive than 3 TeV.

5. Again as in the case of large extra dimensions, for experimental searches one wants to focus on the additional degrees of freedom associated with bulk fields and the brane. In this case, *unlike* the case of large extra dimensions, the Kaluza–Klein states are not dense. Instead, the low-lying states have masses and spacings of order the TeV scale. Their couplings are not of gravitational strength but, rathers, scaled by inverse powers of the scale of the visible sector brane. The limits are model-dependent (e.g. they depend on which are the bulk fields residing on one brane) but, from LHC result, are in many cases larger than 2 TeV.
6. Given the relatively large scales, how does one understand a comparatively light Higgs? Obtaining a custodial $SU(2)$ symmetry (see Section 8.1) tends to require a large gauge group in the bulk. One might suspect that tuning, similar to that of supersymmetric theories, is also required to obtain a light Higgs.

Finally, there are other variants of the Randall–Sundrum proposal which have been put forward. Perhaps the most interesting is one in which space is not compactified at all but simply warped, with gravity localized on the visible brane. These ideas suggest a rich set of possibilities for what might underlie a quantum theory of gravity. Some features – the exponential warping of the metric, in particular – have been observed in string theory but many, at least to date, have not. This is a potentially important area for further research.

Suggested reading

The original paper of Arkani-Hamed *et al*. (1999) is quite clear and comprehensive, as is the paper of Randall and Sundrum (1999). Good reviews of the Randall–Sundrum proposal are provided by the lecture notes of Sundrum (2005), Csaki *et al.* (2005) and Kribs (2006). The Particle Data Group website provides an up-to-date summary of experimental limits on both large and warped extra dimensions.

Exercise

(1) Verify the Randall–Sundrum solution of Eq. (29.14).