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E-IDEALS IN BARIC ALGEBRAS: BASIC PROPERTIES

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In this work we introduce the notion of E-ideal, generalizing I. M. H. Etherington's idea. We study the general characteristics of the lattice of E-ideals in baric algebras, and some properties inherited from an arithmetic of train polynomials.

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1. Preliminaries

The concept of commutative train algebra over the real or complex field was introduced by I. M. H. Etherington in [5]. In the same paper the author shows how to obtain train algebras of rank two and three from an arbitrary baric algebra A. He defines the ideals P and Q_λ as the ideals generated respectively by all elements of the form $a^2 - w(a)a$ and $a^3 - (1+\lambda)w(a)a^2 + \lambda w(a)^2 a$, where $\lambda \in F$ and a runs over A. Then A/P satisfies the train equation $x^2 - w(x)x = 0$ and A/Q_λ satisfies $x^3 - (1+\lambda)w(x)x^2 + \lambda w(x)^2 x = 0$. He also shows that $N^2 \subseteq P \subseteq N$, $Q_\lambda \subseteq N$ and $Q_\lambda + Q_\mu = N$ $(\lambda \neq \mu)$, where N is the kernel of the weight function of A. In this paper we generalize these constructions and introduce the class of the E-ideals of an arbitrary commutative baric algebra A, with the purpose of obtaining train algebras as homomorphic images of A. Before this, we recall some concepts of genetic algebra theory.

Let F be a field of characteristic not 2, A be a commutative nonassociative algebra over F of arbitrary dimension. If $w : A \to F$ is a nonzero homomorphism, the ordered pair (A, w) is called a baric algebra over F, w is its weight function. For any $a \in A$, $w(a)$ is the weight of a. The set $N = \{a \in A : w(a) = 0\}$ is an ideal of codimension 1 in A. Several sub-classes of the class of all baric algebras have been defined along the time by imposing some finiteness conditions on the baric algebra (A, w), usually with a background in Population Genetics. As a rule, the ideal N has some property related to nilpotency. In this direction, we have the Bernstein algebras, which satisfy the equation $(x^2)^2 = w(x)^2 x^2$. There exists an extensive bibliography on this subject. Another class is formed by those baric algebras satisfying the equation

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$x^n + \gamma_1 w(x)x^{n-1} + \ldots + \gamma_{n-1}w(x)^{n-1}x = 0$, where $\gamma_1, \ldots, \gamma_{n-1}$ are elements in the field F satisfying $1 + \gamma_1 + \ldots + \gamma_{n-1} = 0$. These are called train algebras of rank n (if the above equation is minimal). The study of these two sub-classes has been the major concern of the researchers in this field in the last 20 years. More recently algebras satisfying $(x^2)^2 = w(x)^3x$ have been studied. Some generalizations of Bernstein algebras also have been studied. For general results about these classes (and others not cited here) the reader can consult **[8]** and **[14]**.

We review some facts about the ideals P and Q_λ introduced by Etherington. If A is a Bernstein algebra with Peirce decomposition $A = Fe \oplus U_e \oplus Z_e$ relative to an idempotent e, then $P = (U_eZ_e + Z_e^2) \oplus Z_e$. For general train algebras, the situation is much more difficult. It is known that train algebras A of rank 3, satisfying $x^3 - (1+\lambda)w(x)x^2 + \lambda w(x)^2x = 0$, where $\lambda \neq \frac{1}{2}$, have a Peirce decomposition relative to an idempotent $e : A = Fe \oplus U_e \oplus V_e$. In this case $P = U_eV_e \oplus V_e$ and $Q_\lambda = 0$ but $Q_\mu = P$ if $\mu \neq \lambda$. We have some results for algebras of rank 4. If A satisfies $x^4 - (1+r+s)w(x)x^3 + (r+s+rs)w(x)^2x^2 - rsw(x)^3x = 0$ and $r, s, \frac{1}{2}$ are distinct elements in the field F, then A has a Peirce decomposition $A = Fe \oplus A_{\frac{1}{2}} \oplus A_r \oplus A_s$ (see **[7**, Theorem 3.1**]**) provided A has an idempotent e. If $A_{\frac{1}{2}} = 0$ then $Q_\lambda = P$ if $\lambda \neq r, s$ and $Q_s = A_r \oplus A_r^2$, $Q_r = A_s \oplus A_s^2$. Baric algebras satisfying $(x^2)^2 = w(x)^3x$ also have a Peirce decomposition $A = Fe \oplus A_{\frac{1}{2}} \oplus A_{-\frac{1}{2}}$. Then $P = A_{\frac{1}{2}}A_{-\frac{1}{2}} + A_{-\frac{1}{2}}$, $Q_\lambda = P$ if $\lambda \neq -\frac{1}{2}$ and $Q_{-\frac{1}{2}} = A_{\frac{1}{2}} \oplus A_{-\frac{1}{2}}^2$. See **[13]** for basic facts about these algebras.

2. E-ideals

Let (A, w) be a commutative baric algebra of arbitrary dimension over the field F and let $\gamma_1, \ldots, \gamma_{n-1}$ be arbitrary elements of F subject to the relation $1 + \gamma_1 + \ldots + \gamma_{n-1} = 0$. The generalized Etherington's ideal of A (in short, E-ideal) corresponding to these scalars, is the ideal of A generated by all elements of the form

$$a^n + \gamma_1 w(a)a^{n-1} + \ldots + \gamma_{n-1}w(a)^{n-1}a \tag{1}$$

where a runs over A. It is usual to call the formal expression

$$p(x) = x^n + \gamma_1 w(x)x^{n-1} + \ldots + \gamma_{n-1}w(x)^{n-1}x \tag{2}$$

a train polynomial of degree n and coefficients $\gamma_1, \ldots, \gamma_{n-1}$. Then (1) is the value of this polynomial on a. The ideal defined above will be denoted by $E_A(1, \gamma_1, \ldots, \gamma_{n-1})$ or $E_A(p)$. We denote by $\mathcal{E}_A$ the class of all these ideals of A when $p(x)$ runs over the set of all train polynomials. The ideals P and Q_λ are denoted now by $E_A(1, -1)$ and $E_A(1, -(1+\lambda), \lambda)$ respectively. Note that the powers a^j in (1) are the principal powers defined by $a^1 = a$, $a^2 = aa$, $a^3 = (aa)a$, $\ldots$, $a^n = a^{n-1}a$. Observe also that $w(a^n + \gamma_1 w(a)a^{n-1} + \ldots + \gamma_{n-1}w(a)^{n-1}a) = (1 + \gamma_1 + \ldots + \gamma_{n-1})w(a)^n = 0$ so that $E_A(p) \subseteq N$ for all $p(x)$. This allows us to form the quotient algebra $A/E_A(p)$ and to consider the induced homomorphism $\bar{w}: A/E_A(p) \to F$ given by $\bar{w}(a + E_A(p)) = w(a)$. Then $\bar{w} \neq 0$ and the baric algebra $(A/E_A(p), \bar{w})$ satisfies the equation $\bar{a}^n + \gamma_1 \bar{w}(\bar{a})\bar{a}^{n-1} + \dots + \gamma_{n-1}\bar{w}(\bar{a})^{n-1}\bar{a} = 0$ for $\bar{a} = a + E_A(p)$. Clearly $E_A(p)$ is the smallest ideal of A contained in N, such that $A/E_A(p)$ is train satisfying $p(x) = 0$. Then (A, w) is already a train algebra if and only if $\{0\} \in \mathcal{E}_A$. For future use, we introduce the following definition.

Definition. Let Ω be a fixed class of baric algebras over F. Two train polynomials $p(x)$ and $q(x)$ are *equivalent* modulo Ω if $E_A(p) = E_A(q)$ for all $A \in \Omega$. (In particular, when $\Omega = \{A\}$, a unitary class, the equivalence classes, modulo Ω, of train polynomials correspond bijectively with the ideals in $\mathcal{E}_A$.)

3. An example

In general, given the baric algebra A, the determination of $\mathcal{E}_A$ is a hard task. We present now an example of this calculation for infinite dimensional algebras, showing that $\mathcal{E}_A$ may be a small or large set. In the forthcoming sections, we deal with algebras satisfying some finiteness condition, which will imply that $\mathcal{E}_A$ is finite. We shall use here the fact (to be proved in the next section) that $E_A(p)$ is generated by all $p(a)$ such that $w(a) = 1$.

Suppose A is an infinite dimensional vector space over an infinite field F with a countable basis $\{c_0, c_1, c_2, \dots\}$. Let $w: A \to F$ be the linear form defined by $w(c_0) = 1$ and $w(c_i) = 0$ $(i \geq 1)$. Fix a train polynomial $p(x)$ as given in (2). In each case, w will be a homomorphism of F-algebras.

(a) Define in A the following commutative multiplication:

$$c_i^2 = c_i \ (i \geq 0); \text{ other products are zero.} \tag{3}$$

We know already that $E_A(p) \subseteq N$, for every A and $p(x)$. In our case, $E_A(p) = N$. In fact, for $i \geq 1$, $p(c_i) = c_i^n = c_i \in E_A(p)$, thus $N = E_A(p)$ and $\mathcal{E}_A$ is a unitary set.

(b) Define in A the following commutative multiplication:

$$c_i^2 = c_i \ (i = 0, 1, \dots, r); \text{ other products are zero.} \tag{4}$$

We have $ker\ w = N = \langle c_1, c_2, \dots \rangle$ and $N^2 = \langle c_1, \dots, c_r \rangle$. If $a = c_0 + \sum_{i\geq 1} \alpha_i c_i$ (finite sum) then, by induction, $a^j = c_0 + \sum_{i=1}^{r} \alpha_i^j c_i$ for $j \geq 2$. From this, $p(a) = \sum_{i=1}^{r} p(\alpha_i)c_i + \gamma_{n-1}(\sum_{i\geq 1} \alpha_i^j c_i)$. If $\gamma_{n-1} = 0$ then $p(a) \in N^2$ which implies $E_A(p) \subseteq N^2$. For the converse inclusion take c_i, $1 \leq i \leq r$. Then $p(c_i) = c_i \in E_A(p)$ so that $E_A(p) = N^2$. Suppose now $\gamma_{n-1} \neq 0$. We have already proved that $E_A(p)$ contains $c_1, \dots, c_r$. Suppose now $r + 1 \leq k < \infty$ and let $b = c_0 + c_k$. Then $b^j = c_0$ for $j \geq 2$, thus $p(b) = \gamma_{n-1}c_k$, which implies that $c_k \in E_A(p)$ and

$E_A(p) = N$. We have proved that $\mathcal{E}_A = \{N, N^2\}$.

(c) Fix a sequence $t_0, t_1, t_2, \ldots$ of elements in the field F, with $t_0 = 1$. Define a commutative multiplication by

$$c_0 c_i = t_i c_i \ (i \geq 0); \text{ other products are zero.} \tag{5}$$

As $N^2 = 0$ every subspace of N is an ideal of N. Every subset of the basis $c_1, c_2, \ldots$ of N generates additively an ideal of A. If $a = c_0 + \sum_{i\geq 1} \alpha_i c_i$ (finite sum) then $a^2 - a = \sum_{i\geq 1}(2t_i - 1)\alpha_i c_i$. We distinguish some cases.

c(i) All $t_i = \frac{1}{2}$ for $i \geq 1$. Then $a^2 = a$ for every a such that $w(a) = 1$. Then $E_A(1, -1) = 0$ and so $\mathcal{E}_A = \{0\}$ and so A is a train algebra of rank 2.

c(ii) There is only a finite number of t_i's distinct from $\frac{1}{2}$. Let us call them $t_{i_1}, \ldots, t_{i_r}$. Then for $a = c_0 + \sum_{i\geq 1} \alpha_i c_i$, $a^2 - a = \sum_{k=1}^{r}(2t_{i_k} - 1)\alpha_{i_k} c_{i_k}$ and it is easily seen that $E_A(1, -1)$ is the subspace generated by $c_{i_1}, \ldots, c_{i_r}$. The remaining E-ideals are obtained as follows. We have $(a^2 - a)a - (a^2 - a)t_{i_l} = \sum_{k=1}^{r}(2t_{i_k} - 1)(t_{i_k} - t_{i_l})\alpha_{i_k} c_{i_k}$. If we suppose that $t_{i_1}, \ldots, t_{i_r}$ are all distinct, then it is easily seen that the E-ideal associated to the train polynomial $x(x-1)(x - t_{i_l})$, $1 \leq l \leq r$, is the subspace $\langle c_{i_1}, \ldots, c_{i_{l-1}}, \hat{c}_{i_l}, c_{i_{l+1}}, \ldots, c_{i_r} \rangle$ where "$\wedge$" denotes absence. In a similar way the E-ideal associated to the train polynomial $x(x-1)(x - t_{i_l})(x - t_{i_m})$ is the subspace $\langle c_{i_1}, \ldots, \hat{c}_{i_l}, \ldots, \hat{c}_{i_m}, \ldots, c_{i_r} \rangle$. The reader can see easily that this can be extended to three, $\ldots$, r factors. The E-ideals are the subspaces generated by all the subsets of $\{c_{i_1}, \ldots, c_{i_r}\}$ so that $\mathcal{E}_A$ is anti-isomorphic, as a partially ordered set (by inclusion), to the set of divisors of the polynomial $(x - t_{i_1}) \ldots (x - t_{i_r})$.

c(iii) There is an infinite number of scalars t_i's distinct from $\frac{1}{2}$. A similar argument will show that $\mathcal{E}_A$ is infinite. Let $t_{i_1}, t_{i_2}, \ldots$ be the scalars distinct from $\frac{1}{2}$. It follows that $E_A(1, -1)$ is the subspace generated by all $c_{i_1}, c_{i_2}, \ldots$ because, for $a = \sum_i \alpha_i c_i$, $a^2 - a = \sum_{k\geq 1} \alpha_{i_k} c_{i_k}$. The remaining E-ideals are the subspaces generated by finite subsets of the complementary set of $\{c_{i_1}, c_{i_2}, \ldots\}$.

4. Properties of E-ideals

In this section we establish some properties of the E-ideals of an arbitrary commutative baric algebra. Our naive definition of train polynomial will suffice for the moment. But we are obliged to give a more precise construction of train polynomials in Section 5 to derive some other properties of $\mathcal{E}_A$.

Proposition 1. *For every baric algebra (A, w) and every train polynomial $p(x)$ as in (2), we have $E_A(p) \subseteq E_A(1, -1)$, that is, $E_A(1, -1)$ is the maximum element of $\mathcal{E}_A$ (ordered by inclusion).*

Proof. Denote by $p_{n,k}(x)$ the train polynomial $p_{n,k}(x) = x^n - w(x)^{n-k}x^k$ for $1 \le k \le n-1$. Given the train polynomial $p(x)$ as in (2) and $a \in A$, we have $p(a) = a^n + \gamma_1 w(a)a^{n-1} + \ldots + \gamma_{n-1}w(a)^{n-1}a = a^n - (1+\gamma_2+\ldots+\gamma_{n-1})w(a)a^{n-1} + \ldots + \gamma_{n-1}w(a)^{n-1}a = (a^n - w(a)a^{n-1}) - \gamma_2 w(a)(a^{n-1} - w(a)a^{n-2}) - \ldots - \gamma_{n-1}w(a)(a^{n-1} - w(a)^{n-2}a) \in E_A(p_{n,n-1}) + E_A(p_{n-1,n-2}) + \ldots + E_A(p_{n-1,1})$ and this implies that $E_A(p) \subseteq E_A(p_{n,n-1}) + \sum_{k=1}^{n-2} E_A(p_{n-1,k})$ as the elements $p(a)$, $a \in A$, generate $E_A(p)$. Similarly, for each $k = 1, \ldots, n-2$ we have $E_A(p_{n-1,k}) \subseteq E_A(p_{n-1,n-2}) + \sum_{r=1}^{n-3} E_A(p_{n-2,r})$ and by repeated application of this method we get $E_A(p) \subseteq \sum_{k=2}^{n} E_A(p_{k,k-1})$. But each of the ideals $E_A(p_{k,k-1})$ is contained in $E_A(1,-1)$ because each generator $a^k - w(a)a^{k-1}$ can be put in the form $a^k - w(a)a^{k-1} = (\ldots((a^2 - w(a)a)a)\ldots)a$, which belongs to $E_A(1,-1)$. □

Denote by $E_A^1(p)$ the ideal of A generated by the elements $p(a)$ such that $w(a) = 1$ and by $E_A^0(p)$ the ideal of A generated by the elements $p(a)$ such that $w(a) = 0$. Clearly both $E_A^1(p)$ and $E_A^0(p)$ are contained in $E_A(p)$ so that $E_A^0(p) + E_A^1(p) \subseteq E_A(p)$.

Proposition 2. *With the above notation we have* $E_A(p) = E_A^1(p) + E_A^0(p)$.

Proof. Suppose $a \in A$ and let $p(a)$ be one of the generators of $E_A(p)$. If $w(a) = 0$ then $p(a)$ is a generator of $E_A^0(p)$. If $w(a) \neq 0$ then for $b = (w(a))^{-1}a$ we have $p(a) = w(a)^n p(b) \in E_A^1(p)$. □

For the proof of the next proposition, we need to introduce some notation and quote a result from [**6**]. Let A be an arbitrary algebra over F and let $f_i : A \to K_i$, where $K_i = A$ or F, $i = 1, \ldots, m$, be arbitrary functions. We define recursively the function $(f_1, \ldots, f_m) : A \oplus \ldots \oplus A \to K$, where $K = F$ if all f_i takes values in F and $K = A$ otherwise, by:

(1) $(f_1) = f_1$

(2) $(f_1, \ldots, f_m)(a_1, \ldots, a_m) = \frac{1}{m}\sum_{i=0}^{m-1}(f_1, \ldots, f_{m-1})(a_{\tau^i(1)}, \ldots, a_{\tau^i(m-1)})f_m(a_{\tau^i(m)})$, where $(a_1, \ldots, a_m) \in A \oplus \ldots \oplus A$ and τ is the cycle $(1, 2, \ldots, m)$ in the symmetric group S_m.

Lemma ([**6**, Lemma 2.1]). *If all f_i are linear functions then $(f_1, \ldots, f_m)$ is a m-linear symmetric function of m variables defined in $A \oplus \ldots \oplus A$ with values in A* (or F).

Let now $f, g : A \to K$ be as above, $a, b \in A$. We introduce the notation: $f^{(1)} = f$; $f^m = (f, \ldots, f)$; $(f^{(m)}, g^{(p)}) = (f, \ldots, f, g, \ldots g)$; $(f^0, g^{(p)}) = g^{(p)}$; $(f^{(m)}, g^{(0)}) = f^{(m)}$; $a^{(1)} = a$; $a^{(m)} = (a, \ldots, a)$; $(a^{(m)}, b^{(p)}) = (a, \ldots, a, b, \ldots, b)$; $(a^{(m)}, b^{(0)}) = a^{(m)}$; $(a^{(0)}, b^{(p)}) = b^{(p)}$.

Lemma ([6, Lemma 2.2]). *If A is an arbitrary nonassociative algebra over F, $a, b \in A$ and $m \geq 1$, then $(a+b)^m = \sum_{i=0}^{m} \binom{m}{i} id_A^{(m)}(a^{(m-i)}, b^{(i)})$ where id_A is the identity function on A.*

Proposition 3. *If F is an infinite field, (A, w) an arbitrary baric algebra and $p(x)$ a train polynomial then $E_A^0(p) \subseteq E_A^1(p)$.*

Proof. A typical generator of $E_A^0(p)$ is $p(b) = b^n$ where $b \in N = ker\, w$. Fix one of these generators and consider the one parameter family $(a_\lambda)_{\lambda \in F}$ of elements of A defined by $a_\lambda = e + \lambda b$ where e is an element of weight 1 in A. Then $w(a_\lambda) = 1$ so that $p(a_\lambda) \in E_A^1(p)$. But $p(a_\lambda) = \sum_{i=0}^{n-1} \gamma_i (e + \lambda b)^{n-i} = \sum_{i=0}^{n-1} \gamma_i \sum_{k=0}^{n-i} \binom{n-i}{k} id_A^{(n-i)}(e^{(n-i-k)}, (\lambda b)^{(k)}) = \sum_{k=0}^{n} \lambda^k \binom{n}{k} id_A^{(n)}(e^{(n-k)}, b^{(k)}) + \gamma_1 \sum_{k=0}^{n-1} \lambda^k \binom{n-1}{k} id_A^{(n-1)}(e^{(n-k-1)}, b^{(k)}) + \ldots + \gamma_{n-1} \sum_{k=0}^{1} \lambda^k \binom{1}{k} id_A^{(1)}(e^{(1-k)}, b^{(k)})$, which we may write as $p(a_\lambda) = c_0 + \lambda c_1 + \ldots + \lambda^n c_n$, with $c_k = \sum_{j=0}^{n-k} \gamma_j \binom{n-j}{k} id_A^{(n-j)}(e^{(n-j-k)}, b^{(k)})$, for $k = 0, 1, \ldots, n$. In particular, $c_n = b^n$. As $p(a_\lambda) \in E_A^1(p)$ for all $\lambda \in F$, we can choose $\lambda_1, \ldots, \lambda_n \in F$, mutually distinct, thus obtaining a Vandermonde system of equations

$$c_0 + \lambda_i c_1 + \lambda_i^2 c_2 \ldots + \lambda_i^n c_n \in E_A^1(p) \quad (i = 1, \ldots, n)$$

The solution of this system shows that each $c_i \in E_A(p)$ and so also $c_n = b^n = p(b) \in E_A^1(p)$ and this proves the proposition. □

Corollary. *Under the same conditions of Proposition 3, we have $E_A(p) = E_A^1(p)$.*

From now on, we shall assume that F is an infinite field so that each $E_A(p)$ is generated by the elements $p(a)$ where $w(a) = 1$.

Suppose (A, w) is a baric algebra over the field F and consider the n-linear functions $(w^{(j)}, id_A^{(n-j)}) : A \oplus \ldots \oplus A \to A$, for $0 \leq j \leq n-1$. Take a linear combination $\mu = id_A^{(n)} + \gamma_1(w, id_A^{(n-1)}) + \ldots + \gamma_{n-1}(w^{(n-1)}, id_A)$, where $1 + \gamma_1 + \ldots + \gamma_{n-1} = 0$. This n-linear symmetric function μ is called the *complete linearisation* of the train polynomial $p(x) = x^n + \gamma_1 w(x) x^{n-1} + \ldots + \gamma_{n-1} w(x)^{n-1} x$.

Theorem 1. ([6, Theorem 2.1]) *For any baric algebra (A, w) the following conditions are equivalent:*

(i) (A, w) *satisfies identically* $a^n + \gamma_1 w(a)a^{n-1} + \ldots + \gamma_{n-1} w(a)^{n-1} a = 0$.

(ii) $\mu = id_A^{(n)} + \gamma_1(w, id_A^{(n-1)}) + \ldots + \gamma_{n-1}(w^{(n-1)}, id_A)$ *is identically zero on* $A \oplus \ldots \oplus A$.

Proposition 4. *For each train polynomial* $p(x)$*, the ideal* $E_A(p)$ *is generated by the values* $\mu(a_1, \ldots, a_n)$*, where* a_i *runs over* A.

Proof. For each $a \in A$, $\mu(a^{(n)}) = id_A^{(n)}(a^{(n)}) + \gamma_1(w, id_A^{(n-1)})(a^{(n)}) + \ldots + \gamma_{n-1}(w^{(n-1)}, id_A)(a^{(n)}) = a^n + \gamma_1 w(a)a^{n-1} + \ldots + \gamma_{n-1} w(a)^{n-1} a = p(a)$, showing that each generator of $E_A(p)$ belongs to the ideal generated by the values of μ on $A \oplus \ldots \oplus A$. For the converse, consider the quotient algebra $\bar{A} = A/E_A(p)$ with its weight function $\bar{w}$ defined by $\bar{w}(a + E_A(p)) = w(a)$. It satisfies the train polynomial $\bar{p}(x) = x^n + \gamma_1 \bar{w}(x)x^{n-1} + \ldots + \gamma_{n-1}\bar{w}(x)^{n-1}x$ and according to Theorem 1 above, we must have $\bar{\mu}(\bar{a}_1, \ldots, \bar{a}_n) = 0$ on $\bar{A} \oplus \ldots \oplus \bar{A}$. This shows that $\mu(a_1, \ldots, a_n) \in E_A(p)$ and this is the end of the proof. □

Corollary. *If* A *is finite dimensional,* $E_A(p)$ *is generated by the finite set* $\mu(c_{i_1}, c_{i_2}, \ldots, c_{i_n})$ *where* $i_1 \le i_2 \le \ldots \le i_n$ *and* $c_{i_1}, \ldots, c_{i_n}$ *are chosen among the vectors of a basis of* A.

We recall now the Krull–Schmidt theorem for baric algebras [3, Theorem 2] and the concept of closed class [4]. Our aim is to reduce the determination of $\mathcal{E}_A$ to the case where A is indecomposable.

Suppose (A_1, w_1) and (A_2, w_2) are commutative baric algebras over the field F both having idempotents of weight 1. Choose $e_1 \in A_1$, $e_1^2 = e_1$ and $w(e_1) = 1$ and similarly $e_2 \in A_2$. Then $A_1 = Fe_1 \oplus N_1$ and $A_2 = Fe_2 \oplus N_2$, $N_i = ker\, w_i (i = 1, 2)$. Let $A = Fe \oplus N$ where $N = N_1 \oplus N_2$ and e satisfies $en_1 = e_1 n_1$, $en_2 = e_2 n_2$, where $n_1 \in N_1$, $n_2 \in N_2$. This baric algebra, which is called the join of A_1 and A_2, is denoted by $A_1 \vee A_2$. A baric algebra (A, w) is called *decomposable* in the case that it can be put in the form $A_1 \vee A_2$, where both A_1 and A_2 have dimensions at least two. Otherwise, it is indecomposable. For further details, see [3] and [4]. A class Ω of baric algebras with idempotent of weight 1 is closed when $A_1 \vee A_2$ belongs to Ω if and only if both A_1 and A_2 belong to Ω. We assume now that our baric algebra A satisfies the following conditions: every strictly ascending (resp. descending) chain of ideals of A, contained in N, is finite. We refer, as usual, as the a.c.c. and d.c.c. conditions.

Theorem 2. ([3, Theorem 2]) *If* (A, w) *satisfies both a.c.c. and d.c.c. then* (A, w) *can be uniquely decomposed* (*up to isomorphisms and permutations*) *as the join of a finite number of indecomposable baric algebras.*

Proposition 5. *Let* (A_1, w_1) *and* (A_2, w_2) *be baric algebras,* $\varphi : A_1 \to A_2$ *a baric homomorphism and* $p(x)$ *a train polynomial. Then* $\varphi(E_{A_1}(p)) = E_{\varphi(A_1)}(p) \subseteq E_{A_2}(p)$.

Proof. It is enough to show that φ takes generators of $E_{A_1}(p)$ into generators of $E_{\varphi(A_1)}(p)$ and conversely. As φ is baric, we have $w_2 \circ \varphi = w_1$ and so for $a \in A_1$ with $w_1(a) = 1$, we have $\varphi(p(a)) = \varphi(a^n + \gamma_1 a^{n-1} + \ldots + \gamma_{n-1}a) = \varphi(a)^n + \gamma_1 \varphi(a)^{n-1} + \ldots + \gamma_{n-1}\varphi(a)$, which proves what is wanted. □

Corollary 1. *Under the same conditions, if φ is an epimorphism, $\varphi(E_{A_1}(p)) = E_{A_2}(p)$ and so φ induces a surjective function from $\mathcal{E}_{A_1}$ to $\mathcal{E}_{A_2}$.*

Corollary 2. *Under the same conditions, if A_1 is a train algebra satisfying the train equation $p(x) = 0$ of degree n, then A_2 is also train, satisfying some train equation of degree $\leq n$.*

Proof. Follows from $\{0\} \in \mathcal{E}_{A_1}$. □

For the next corollary, we recall that the commutative duplicate of a commutative baric algebra (A, w) is the baric algebra (A^D, w_D), where $A^D = S^2(A)$, the second symmetric power of A, endowed with the product $(a.b)(c.d) = ab.cd$ and where $w_D(a.b) = w(a)w(b)$. The function $\mu : A^D \to A^2$ given by $\mu(a.b) = ab$ gives rise to the exact sequence

$$0 \to ker\, \mu \to A^D \xrightarrow{\mu} A^2 \to 0$$

where $\mu : A^D \to A^2$ is defined by $\mu(a.b) = ab$.

For further details, see **[11]**.

Corollary 3. *If (A, w) is a baric algebra with commutative duplicate (A^D, w_D) and $p(x)$ is a train polynomial, then $\mu(E_{A^D}(p)) = E_{A^2}(p)$.*

Corollary 4. *Let (A, w) be a baric algebra, I an ideal of A contained in $N = ker\, w$. Then $(A/I, \bar{w})$, where $\bar{w}(a + I) = w(a)$, is a baric algebra and for every train polynomial $p(x)$, we have $\pi(E_A(p)) = E_{A/I}(p) = (E_A(p) + I)/I$ where $\pi : A \to A/I$ is the canonical projection.*

Corollary 5. *Let (A, w) be a baric algebra, B a baric sub-algebra of A. Then $E_B(p) \subseteq E_A(p)$.*

Proposition 6. *Let (A_1, w_1) and (A_2, w_2) be baric algebras with idempotents of weight 1 and $p(x)$ a train polynomial. Then $E_{A_1 \vee A_2}(p) = E_{A_1}(p) \oplus E_{A_2}(p)$.*

Proof. Consider the canonical projections $\varphi_0 : A_1 \vee A_2 \to Fe$, $\varphi_1 : A_1 \vee A_2 \to A_1$ and $\varphi_2 : A_1 \vee A_2 \to A_2$ given by $\varphi_0(\alpha e + n_1 + n_2) = \alpha e$, $\varphi_1(\alpha e + n_1 + n_2) = \alpha e_1 + n_1$ and $\varphi_2(\alpha e + n_1 + n_2) = \alpha e_2 + n_2$, where $\alpha \in F$, $n_1 \in N_1$ and $n_2 \in N_2$. They are clearly baric homomorphisms. Moreover $a = -\alpha e + (\alpha e + n_1) + (\alpha e + n_2)$ so that $-\varphi_0 + \varphi_1 + \varphi_2 = id_A$.

Let now $p(a)$ be one of the generators of $E_{A_1 \vee A_2}(p)$. Then $p(a) = (-\varphi_0 + \varphi_1 + \varphi_2)(p(a)) = -\varphi_0(p(a)) + \varphi_1(p(a)) + \varphi_2(p(a)) = \varphi_1(p(a)) + \varphi_2(p(a)) = p(\varphi_1(a)) + p(\varphi_2(a))$, which belongs to $E_{A_1}(p) \oplus E_{A_2}(p)$ so that $E_{A_1 \vee A_2}(p) \subseteq E_{A_1}(p) \oplus E_{A_2}(p)$. The other inclusion follows from Corollary 5. □

Corollary 6. *If (A, w) satisfies a.c.c. and d.c.c. then $E_A(p)$ is the direct sum of the E-ideals $E_{A_1}(p), \ldots, E_{A_r}(p)$ where each A_i is indecomposable and $A = A_1 \vee A_2 \vee \ldots \vee A_r$.*

We close this section with a brief discussion of the equality $N = E_A(1, -1)$. We have seen before that for every baric algebra (A, w) and every train polynomial $p(x)$, $E_A(p) \subseteq E_A(1, -1) \subseteq N$, $N = ker\, w$. As stated by Etherington, we have $N^2 \subseteq E_A(1, -1) \subseteq N$ so a sufficient condition for the equality $N = E_A(1, -1)$ is $N = N^2$. But this condition, which will occur rarely in genetic algebra theory, is not necessary, as shown by the following example. Let A be the Bernstein algebra with basis $\{e, u, z\}$ and multiplication table $e^2 = e$, $eu = \frac{1}{2}u$, $uz = u$, all other products are zero. Then $E_A(1, -1) = N$ but $N^2 = Fu$. In general, for a Bernstein algebra $A = Fe \oplus U_e \oplus Z_e$, we have $E_A(1, -1) = N$ if and only if $U_eZ_e + Z_e^2 = U_e$. Similar statements for other classes of baric algebras are easy consequences of the equations appearing in Section 1. For general baric algebras, we have the following sufficient condition, which, unfortunately, is not efficient in the context of genetic algebra theory, where $\frac{1}{2}$ is, in many situations, a proper value of L_e.

Proposition 7. *Let (A, w) be a baric algebra (possibly infinite dimensional) and suppose that, for some idempotent e of weight 1, the linear operator $L_e : N \to N$ satisfies a polynomial identity $Q(L_e) = 0$ where the roots of Q are in F, are simple and all distinct from $\frac{1}{2}$. Then $N = E_A(1, -1)$.*

Proof. It is enough to prove that $N \subseteq E_A(1, -1)$. By elementary linear algebra, we have $N = N_1 \oplus \ldots \oplus N_s$, where $N_i = ker(L_e - \alpha_i I)$ $(i = 1, \ldots, s)$ are the proper subspaces corresponding to the proper values $\alpha_1, \ldots, \alpha_s$ of L_e. If $a = e + n_i$, $n_i \in N_i$ then $a^2 = e + 2en_i + n_i^2 = e + 2\alpha_i n_i + n_i^2$ so $a^2 - w(a)a = (2\alpha_i - 1)n_i + n_i^2 \in E_A(1, -1)$. As $n_i^2 \in E_A(1, -1)$ we have $(2\alpha_i - 1)n_i \in E_A(1, -1)$ so $n_i \in E_A(1, -1)$. Then $N_i \subseteq E_A(1, -1)$ and so $N = \oplus_{i=1}^{s} N_i \subseteq E_A(1, -1)$. □

5. Train polynomials

The above "definition" of a train polynomial as a formal expression must be reformulated in order to prove some properties of $\mathcal{E}_A$. We describe how to do this but proofs are only sketched.

Let F be a field and (A, w) a commutative baric algebra over F. Consider the vector space A^A of all functions from A to A and, in this space, take the infinite family of functions f_{ij}, where $i = 0, 1, 2, \ldots$ and $j = 1, 2, \ldots$, defined by $f_{ij}(a) = w(a)^i a^j$ for all $a \in A$. These functions generate $GTF(A)$: the subspace of generalized train functions of A^A. Every element f in this subspace has at least one representation of the form

$$f = \sum \gamma_{ij} f_{ij}, \quad \gamma_{ij} \in F, \quad i \geq 0, \quad j \geq 1 \tag{6}$$

so that for $a \in A$

$$f(a) = \sum \gamma_{ij}\, w(a)^i a^j \quad i \geq 0, \quad j \geq 1 \tag{7}$$

With this definition, (A, w) is a train algebra if and only if for some elements $\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{n-1} \in F$, with $\gamma_0 = 1, f = \sum_{i=0}^{n-1} \gamma_i f_{i,n-i}$ is identically zero on A.

For a given representation of f in the form (6), we consider the integer $\max\{j : \gamma_{ij} \neq 0\}$ and define the degree of f as the minimum of these numbers, when we allow all possible representations of f in the form (6). The corresponding representation will be referred to as the *minimal representation* of f.

Proposition 8. *The following conditions are equivalent:*

(i) *(A, w) is a train algebra.*

(ii) *The family (f_{ij}) for $0 \leq i$ and $1 \leq j$, is linearly dependent in $GTF(A)$.*

Proof. (i) $\Rightarrow$ (ii): Suppose $\gamma_1, \ldots, \gamma_{n-1}$ are elements in F such that for all $a \in A$, $a^n + \gamma_1 w(a) a^{n-1} + \ldots + \gamma_{n-1} w(a)^{n-1} a = 0$. Then $f_{0n} + \gamma_1 f_{1,n-1} + \ldots + \gamma_{n-1} f_{n-1,1} = 0$ which implies that the whole family is linearly dependent.

(ii) $\Rightarrow$ (i): Suppose the whole family is linearly dependent. Choose a finite subfamily which is linearly dependent. We may suppose this subfamily has the form (f_{ij}), $0 \leq i \leq m$; $1 \leq j \leq n$, for some integers m, n. Let $\gamma_{ij} \in F$, not all zero, such that $\sum \gamma_{ij} f_{ij} = 0$ $(i = 0, 1, \ldots, m; j = 1, \ldots, n)$, so that, for every $a \in A$, $\sum \gamma_{ij} w(a)^i a^j = 0$. If we choose a such that $w(a) = 1$, then $0 = \sum_{i,j} \gamma_{ij} a^j = \sum_j (\sum_i \gamma_{ij}) a^j = \sum_j \alpha_j a^j$. For every $y \in A$ such that $w(y) \neq 0$ the element $y' = \frac{y}{w(y)}$ has weight 1 so that $0 = \sum_j \alpha_j (y')^j$. Then $w(y)^{-n} \left(\sum_{j=1}^n \alpha_j w(y)^{n-j} y^j \right) = 0$ which means that $\sum_{j=1}^n \alpha_j w(y)^{n-j} y^j = 0$ for all y such that $w(y) \neq 0$. As this set of vectors is dense in A (Zariski's topology) and $\sum_{j-1}^n \alpha_j f_{n-j,j}$ is continuous, we must have $\sum_{j=1}^n \alpha_j f_{n-j,j} = 0$ on A. This proves that (A, w) is a train algebra. $\square$

Remark. Using the same argument, we can see easily that the rank of the train algebra (A, w) is n if and only if the rank of the set of functions $f_{n-j,j} (i \geq 0, j \geq 1)$ is also n.

Proposition 9. *Suppose (A, w) is a train algebra of rank n. Then:*

(i) *there exists, up to a scalar factor, a unique function $\sum_{i=0}^{n-1} \gamma_i f_{i,n-i}$, with $\gamma_0 = 1$, which is zero on A;*

(ii) *every function $f \in GTF(A)$ can be represented as a linear combination of $f_{0,n}$, $f_{1,n-1}, \ldots, f_{n-1,1}$.*

Proof. (i) Suppose we have $\sum_{i=0}^{n-1} \gamma_i w(a)^i a^{n-i} = \sum_{i=0}^{n-1} \delta_i w(a)^i a^{n-i}$ for all $a \in A$, with $\gamma_0 = \delta_0 = 1$. Then $\sum_{i=0}^{n-1} (\gamma_i - \delta_i) w(a)^i a^{n-1} = 0$ which implies that $\gamma_i = \delta_i$.

(ii) By hypothesis, we have $f_{0,n} = -\sum_{i=1}^{n-1} \gamma_i f_{i,n-i}$ and $f_{0,n+1} = f_{0,n} f_{0,1}$ so that, by recurrence, every $f_{0,n+k} (k \geq 1)$ will be a linear combination of $f_{n-1,1}, \ldots, f_{1,n-1}$.

Corollary. *If (A, w) is a train algebra, every element of $GTF(A)$ has a unique minimal representation.*

The elements $f = \sum_i \gamma_{i,n-i} f_{i,n-1}$ of $GTF(A)$ are called *train functions* of (A, w) and the set of all train functions is denoted $TF(A)$. In this subset of $GTF(A)$, we select the elements of the above form such that $\sum_i \gamma_{i,n-i} = 0$ and call them *train polynomials* over A, denoted by $TP(A)$.

Consider now the usual algebra of polynomials over F in the indeterminate x and its ideal $\langle x \rangle$, generated by x. We define $t : \langle x \rangle \to TF(A)$ by $t : \sum_{i=0}^{n-1} \gamma_i x^{n-i} \mapsto \sum_{i=0}^{n-1} \gamma_i f_{i,n-i}$. We denote by p_t the image of $p \in \langle x \rangle$ under this map, so that $p_t(a) = \sum_{i=0}^{n-1} \gamma_i w(a)^i a^{n-i}$ for every $a \in A$. We always have $p_t \in TF(A)$ and p_t is a train polynomial if 1 is a root of p. If $f = \sum \gamma_i f_{i,n-i}$ is the minimal representation of f then $p = \sum \gamma_i x^{n-i}$ is called the *canonical antecedent* of f. The following equalities are easily proved:

$$p_t + q_t = (p + q)_t$$

$$p_t q_t = (pq)_t$$

$$\lambda p_t = (\lambda p)_t$$

where $p_t + q_t$ is defined by $(p_t + q_t)(a) = p_t(a) + w(a)^{m-n} q_t(a)$, $a \in A$, if degree $p(x) = m$, degree $q(x) = n$, with $m \geq n$. Observe that for elements a of weight 1, $(p_t + q_t)(a) = p_t(a) + q_t(a)$. Moreover the product $p_t q_t$ must be understood to obey the rule $a^j a^k = a^{j+k}$. That is, for $p_t(x) = \sum_{i=0}^{n-1} \gamma_i w(x)^i x^{n-i}$ and $q_t(x) = \sum_{j=0}^{m-1} \lambda_j w(x)^j x^{m-j}$ then

$$p_t(x)q_t(x) = (pq)_t(x) = \sum_{j=0}^{m-1} \lambda_j w(x)^j R_x^{m-j} p_t(x)$$

where R_x is the operator *multiplication by* x. In this way, $TF(A)$ is an associative algebra and t is an epimorphism. The restriction of the mapping t to $\langle x(x-1)\rangle$ is an epimorphism over $TP(A)$. We see that (A, w) is a train algebra if and only if $ker\ t \neq 0$. If A satisfies some equation (which we take minimal) $p_t(x) = x^n + \gamma_1 w(x) x^{n-1} + \ldots + \gamma_{n-1} w(x)^{n-1} x = 0$, then $ker\ t = \langle p(x)\rangle$ and t is an isomorphism when A is not a train algebra.

The concepts of divisibility, greatest common divisor and least common multiple of two train polynomials are defined by imposing the same conditions on their canonical antecedents.

Proposition 10. *If (A, w) is a baric algebra, $p(x)$ and $q(x)$ are train polynomials such that $p(x)$ is a divisor of $q(x)$ then $E_A(q) \subseteq E_A(p)$.*

Proof. By hypothesis there exists a polynomial $f(x)$ such that $q_t(x) = (pf)_t(x)$ for all $x \in A$, which belongs to $E_A(p)$. □

Corollary. *For every baric algebra (A, w) and $\gamma_1, \ldots, \gamma_{n-1} \in F$, $E_A(1, \gamma_1, \ldots, \gamma_{n-1}, 0) \subseteq E_A(1, \gamma_1, \ldots, \gamma_{n-1})$.*

Proposition 11. *Let (A, w) be a baric algebra, $p(x)$ and $q(x)$ train polynomials and $r(x)$ the least common multiple of $p(x)$ and $q(x)$. Then $E_A(r) \subseteq E_A(p) \cap E_A(q)$.*

Proof. The proof follows from Proposition 10. □

Remark. The intersection of two E-ideals is not, in general, an E-ideal.

Proposition 12. *Let (A, w) be a baric algebra, $p(x)$ and $q(x)$ train polynomials and $r(x)$ the greatest common divisor of $p(x)$ and $q(x)$. Then $E_A(r) = E_A(p) + E_A(q)$.*

Proof. The proof follows from the Bézout identity. □

Corollary (Etherington). *For every baric algebra (A, w) and train polynomials $p(x)$, $q(x)$ which are relatively prime, we have $E_A(1, -1) = E_A(p) + E_A(q)$.*

Final remark. In a forthcoming paper, the authors study properties of E-ideals for train algebras and Bernstein algebras. An alternative method for calculating E-ideals, based on recurrent sequences, will be introduced. This method works when there is a Peirce decomposition and $ker\omega$ is nilpotent.

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