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ALGEBRAS OF GENERALIZED FUNCTIONS WITH SMOOTH PARAMETER DEPENDENCE

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Abstract We show that spaces of Colombeau generalized functions with smooth parameter dependence are isomorphic to those with continuous parametrization. Based on this result we initiate a systematic study of algebraic properties of the ring $\tilde{\mathbb{K}}_{\mathrm{sm}}$ of generalized numbers in this unified setting. In particular, we investigate the ring and order structure of $\tilde{\mathbb{K}}_{\mathrm{sm}}$ and establish some properties of its ideals.

Keywords: algebras of generalized functions; Colombeau algebras; smooth parametrization; algebraic properties

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1. Introduction

Algebras of generalized functions, in particular Colombeau algebras, are a versatile tool for studying singular problems in analysis, geometry and mathematical physics (see, for example, [**8**, **9**, **14**, **24**]). Over the past decade there has been increased interest in the structural theory of such algebras, in particular concerning topological and functional analytic aspects of the theory (see, for example, [**10**–**12**, **27**, **28**]). Furthermore, starting with the fundamental paper [**1**], algebraic properties, both of the ring of Colombeau generalized functions and of Colombeau algebras, have become a main line of research [**1**, **2**, **29**, **30**].

From the very outset, certain questions of an algebraic nature have played an important role in Colombeau theory. Among them is the solution of algebraic equations in generalized functions. In the standard (special or full) version of the theory, polynomials have additional roots when considered as generalized functions. These roots are obtained by mixing classical roots. For example, apart from its classical solutions ± 1, the equation $x^2 = 1$ additionally has the generalized root given by the equivalence class of $(x_\varepsilon)_\varepsilon$ with $x_\varepsilon = 1$ for $\varepsilon \in \mathbb{Q}$ and $x_\varepsilon = -1$ for $\varepsilon \notin \mathbb{Q}$. Usually, such additional roots are an unwanted phenomenon (cf. the discussion in [**4**, Chapter 1.10]). They can be avoided by demanding continuous dependence of representatives on the regularization parameter ε (see [**25**, Proposition 12.2]). More generally, one can show that algebraic equations only possess classical solutions in a setting with continuous parameter dependence [**22**].

Apart from avoiding pathological solutions of algebraic equations, there are a number of intrinsic reasons for studying Colombeau spaces with continuous or smooth parameter dependence. To begin with, when considering full versions of the construction smooth in the test function variable, as done, for example, in [**8**, **14**], smooth dependence on all variables is automatic. This is inherited by special Colombeau algebras when these are considered as subspaces of such full algebras [**25**, p. 111]. Smooth dependence on the regularization parameter is, in fact, built into the image of the space of distributions within the Colombeau algebra. Indeed, regularization via convolution yields as the embedded image of a (say, compactly supported) distribution w the net $(w * \rho_\varepsilon)_\varepsilon$, where $\rho_\varepsilon = 1/\varepsilon^n \rho(\cdot/\varepsilon)$ and ρ is an $\mathcal{S}$-mollifier with all higher moments vanishing, which is obviously smooth in ε. Thus, it is natural to require the same regularity for all elements of the Colombeau algebra (or its ring of constants, respectively).

Moreover, certain geometrical constructions in special Colombeau algebras require smooth parameter dependence. We mention, in particular, the notion of generalized vector fields along a generalized curve (which is needed to model geodesics in singular space-times in general relativity; [**17**, **18**]) and sheaf properties in spaces of manifold-valued generalized functions [**19**].

Finally, we point out the important characterization result on isomorphisms of Colombeau algebras on differentiable manifolds due to Vernaeve. He proved that, up to multiplication by an idempotent-generalized number, multiplicative linear functionals on a Colombeau algebra are given precisely as evaluation maps in generalized points (see [**30**, Theorem 4.5]) and algebra isomorphisms are realized as pullbacks under invertible manifold-valued generalized functions [**30**, Theorem 5.1]. When transferring these results to the case of smooth parameter dependence, due to the fact that there are no non-trivial idempotents in this setting (see Proposition 3.3), both characterizations hold without restriction [**6**, **7**].

The purpose of the present paper is to initiate a systematic study of special Colombeau algebras with continuous or smooth parameter dependence. It is structured as follows: after fixing some notation in § 2, the main result of the first part of the paper is given in § 3, namely that Colombeau spaces with continuous or smooth parameter dependence are in fact isomorphic. Based on this identity, in § 4 we study algebraic properties of the space $\tilde{\mathbb{K}}_{\mathrm{sm}}$ of smoothly parametrized generalized numbers. In particular, we analyse the ring structure of $\tilde{\mathbb{K}}_{\mathrm{sm}}$ (zero divisors, exchange ring, Gelfand ring, and partial order) and conclude by establishing some fundamental properties of ideals in $\tilde{\mathbb{K}}_{\mathrm{sm}}$.

2. Notation

Throughout this paper we will write I for the interval $(0, 1]$. The manifolds M and N are assumed to be smooth, Hausdorff and second countable. For any two sets A and B the relation $A \subset\subset B$ denotes that $A \subseteq \bar{A} \subseteq B$ with $\bar{A}$ compact. Whenever we do not have to distinguish between $\mathbb{R}$ and $\mathbb{C}$ we will denote either of the fields by $\mathbb{K}$.

The prototypical special Colombeau algebra of generalized functions over some smooth manifold M is given as the quotient $\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$, where the algebra $\mathcal{E}_M(M)$ and the ideal $\mathcal{N}(M)$ of $\mathcal{E}_M(M)$ are defined by (with $\mathcal{P}(M)$ the space of linear differential operators on M)

$$\mathcal{E}_M(M) := \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M)^I \mid \forall K \subset\subset X,\ \forall P \in \mathcal{P}(M),\ \exists N \in \mathbb{N}: \sup_{x\in K} |Pu_\varepsilon(x)| = O(\varepsilon^{-N})\Big\},$$

$$\mathcal{N}(M) := \Big\{(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(M)^I \mid \forall K \subset\subset X,\ \forall P \in \mathcal{P}(M),\ \forall m \in \mathbb{N}: \sup_{x\in K} |Pu_\varepsilon(x)| = O(\varepsilon^{m})\Big\}.$$

The corresponding ring of constants in $\mathcal{G}(M)$ is given as $\tilde{\mathbb{K}} := \mathcal{E}_M/\mathcal{N}$, where

$$\mathcal{E}_M = \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^I \mid \exists N \in \mathbb{N}\colon |r_\varepsilon| = O(\varepsilon^{-N})\},$$
$$\mathcal{N} = \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^I \mid \forall m \in \mathbb{N}\colon |r_\varepsilon| = O(\varepsilon^{m})\}.$$

The equivalence class of some representative $(u_\varepsilon)_\varepsilon$ is denoted by $[(u_\varepsilon)_\varepsilon]$. In the above definitions, the representatives $(u_\varepsilon)_\varepsilon$ and $(r_\varepsilon)_\varepsilon$ are allowed to depend arbitrarily on the regularization parameter ε. If instead we consider representatives that depend continuously or smoothly on ε (i.e. $(\varepsilon, x) \mapsto u_\varepsilon(x)$ is continuous in ε and smooth in x, or smooth in both variables, respectively, and analogously for $\varepsilon \mapsto r_\varepsilon$), we denote this by the following subscripts: none (any parametrization, which is the standard definition); 'co' (continuous parametrization); 'sm' (smooth parametrization). Moderateness and negligibility are denoted by $\mathcal{E}_M$, $\mathcal{E}_{M,\mathrm{co}}$, $\mathcal{E}_{M,\mathrm{sm}}$ and $\mathcal{N}$, $\mathcal{N}_{\mathrm{co}}$, $\mathcal{N}_{\mathrm{sm}}$, respectively. The rings of *generalized numbers* are $\tilde{\mathbb{K}}$, $\tilde{\mathbb{K}}_{\mathrm{co}}$ and $\tilde{\mathbb{K}}_{\mathrm{sm}}$. Given two manifolds M and N, we write $\mathcal{G}(M)$, $\mathcal{G}_{\mathrm{co}}(M)$ and $\mathcal{G}_{\mathrm{sm}}(M)$ for the special Colombeau algebras and $\mathcal{G}[M,N]$, $\mathcal{G}_{\mathrm{co}}[M,N]$ and $\mathcal{G}_{\mathrm{sm}}[M,N]$ for the spaces of manifold-valued generalized functions. We refer the reader to [**14**,**16**,**18**] for details on these spaces.

By τ_{co} and τ_{sm} we denote the natural homomorphisms between spaces of generalized numbers and functions with continuous, smooth or arbitrary dependence on ε. For simplicity, we do not distinguish notationally between these homomorphisms on different domains: τ_{co} will always denote maps from spaces with continuous parametrization to those with general parametrization, and τ_{sm} maps from spaces with smooth parametrization to the corresponding spaces with continuous parametrization, e.g. $\tau_{\mathrm{co}}\colon \tilde{\mathbb{K}}_{\mathrm{co}} \to \tilde{\mathbb{K}}$ and $\tau_{\mathrm{sm}}\colon \tilde{\mathbb{K}}_{\mathrm{sm}} \to \tilde{\mathbb{K}}_{\mathrm{co}}$, etc. We will sometimes use τ if a distinction is not necessary.

3. Smooth, continuous and arbitrary parametrization

In this section we examine the interrelation between the various versions of spaces of generalized functions and generalized numbers introduced in § 2. In particular, we shall prove that $\tilde{\mathbb{K}}_{\mathrm{sm}} \cong \tilde{\mathbb{K}}_{\mathrm{co}} \subsetneq \tilde{\mathbb{K}}$ and $\mathcal{G}_{\mathrm{sm}}(M) \cong \mathcal{G}_{\mathrm{co}}(M) \subsetneq \mathcal{G}(M)$.

To begin with we note that $\tilde{\mathbb{K}}_{\mathrm{sm}} \subseteq \tilde{\mathbb{K}}_{\mathrm{co}} \subseteq \tilde{\mathbb{K}}$ via the canonical embeddings τ_{co} and τ_{sm}, defined by $[(r_\varepsilon)_\varepsilon] \mapsto [(r_\varepsilon)_\varepsilon]$:

$$\begin{tikzcd} \tilde{\mathbb{K}}_{\mathrm{sm}} \arrow[r, hook, "\tau_{\mathrm{sm}}"] \arrow[rr, hook, bend right, "\tau_{\mathrm{co}}\circ\tau_{\mathrm{sm}}"'] & \tilde{\mathbb{K}}_{\mathrm{co}} \arrow[r, hook, "\tau_{\mathrm{co}}"] & \tilde{\mathbb{K}} \end{tikzcd}$$

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These maps are well defined as $\mathcal{E}_{M,\mathrm{sm}} \subseteq \mathcal{E}_{M,\mathrm{co}} \subseteq \mathcal{E}_M$ and $\mathcal{N}_{\mathrm{sm}} \subseteq \mathcal{N}_{\mathrm{co}} \subseteq \mathcal{N}$: if $(s_\varepsilon)_\varepsilon$ is another representative of r, then $\tau([(s_\varepsilon)_\varepsilon]) = \tau([(r_\varepsilon)_\varepsilon])$. Moreover, τ_{co}, τ_{sm} and therefore also $\tau_{\mathrm{co}} \circ \tau_{\mathrm{sm}}$ are ring homomorphisms. They are injective because $\mathcal{E}_{M,\mathrm{co}} \cap \mathcal{N} \subseteq \mathcal{N}_{\mathrm{co}}$ and $\mathcal{E}_{M,\mathrm{sm}} \cap \mathcal{N}_{\mathrm{co}} \subseteq \mathcal{N}_{\mathrm{sm}}$. Thus, we obtain the following.

Lemma 3.1. *The maps $\tau_{\mathrm{co}} \colon \tilde{\mathbb{K}}_{\mathrm{co}} \to \tilde{\mathbb{K}}$, $\tau_{\mathrm{sm}} \colon \tilde{\mathbb{K}}_{\mathrm{sm}} \to \tilde{\mathbb{K}}_{\mathrm{co}}$ and $\tau_{\mathrm{co}} \circ \tau_{\mathrm{sm}} \colon \tilde{\mathbb{K}}_{\mathrm{sm}} \to \tilde{\mathbb{K}}$, defined by $[(r_\varepsilon)_\varepsilon] \mapsto [(r_\varepsilon)_\varepsilon]$, are injective and unital ring homomorphisms.*

Let M be a smooth, Hausdorff and second-countable manifold. As for generalized numbers, we consider the following maps between the different versions of algebras of generalized functions:

$$\begin{tikzcd} \mathcal{G}_{\mathrm{sm}}(M) \arrow[r, hook, "\tau_{\mathrm{sm}}"] \arrow[rr, hook, bend right, "\tau_{\mathrm{co}}\circ\tau_{\mathrm{sm}}"'] & \mathcal{G}_{\mathrm{co}}(M) \arrow[r, hook, "\tau_{\mathrm{co}}"] & \mathcal{G}(M) \end{tikzcd}$$

As above, we obtain the following.

Lemma 3.2. *Let M be a manifold. The maps $\tau_{\mathrm{co}} \colon \mathcal{G}_{\mathrm{co}}(M) \to \mathcal{G}(M)$, $\tau_{\mathrm{sm}} \colon \mathcal{G}_{\mathrm{sm}}(M) \to \mathcal{G}_{\mathrm{co}}(M)$ and $\tau_{\mathrm{co}} \circ \tau_{\mathrm{sm}} \colon \mathcal{G}_{\mathrm{sm}}(M) \to \mathcal{G}(M)$, defined by $[(u_\varepsilon)_\varepsilon] \mapsto [(u_\varepsilon)_\varepsilon]$, are injective and unital algebra homomorphisms.*

Whenever convenient, we may therefore omit the natural embeddings and simply write $\tilde{\mathbb{K}}_{\mathrm{sm}} \subseteq \tilde{\mathbb{K}}_{\mathrm{co}} \subseteq \tilde{\mathbb{K}}$ and $\mathcal{G}_{\mathrm{sm}}(M) \subseteq \mathcal{G}_{\mathrm{co}}(M) \subseteq \mathcal{G}(M)$.

Remarkably, τ_{co} (and therefore also $\tau_{\mathrm{co}} \circ \tau_{\mathrm{sm}}$) is not surjective, but τ_{sm} is. Both of these results will be proved below. We start by examining the relation between arbitrary and continuous dependence on ε. To this end, we first determine the idempotents in the algebra of generalized functions and the ring of generalized numbers, respectively, in the case of continuous and smooth parameter dependence. We first note that the situation for arbitrary ε-dependence is completely characterized by the following two results. By [**2**, Theorem 4.1] the non-trivial idempotents in $\tilde{\mathbb{K}}$ are precisely the equivalence classes in $\tilde{\mathbb{K}}$ of characteristic functions e_S of some $S \subseteq I$ with $0 \in \bar{S} \cap \overline{S^{\mathrm{c}}}$. Furthermore, by [**30**, Proposition 5.3] any idempotent of $\mathcal{G}(M)$ for M connected is a generalized constant.

Contrary to the case of $\mathcal{G}(M)$ and $\tilde{\mathbb{K}}$, the following result shows that there are no non-trivial idempotents in the case of smooth or continuous parameter dependence.

Proposition 3.3. *Let M be a connected smooth manifold. Then there are no non-trivial idempotents in $\mathcal{G}_{\mathrm{co}}(M)$.*

Proof. Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathrm{co}}(M)$ such that $u_\varepsilon \cdot u_\varepsilon = u_\varepsilon + n_\varepsilon$ for some $(n_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{co}}(M)$.

We first consider an open, relatively compact and connected open set U. There are two possible solutions for the quadratic equation $u_\varepsilon(x) \cdot u_\varepsilon(x) = u_\varepsilon(x) + n_\varepsilon(x)$ on U:

$$u_{\varepsilon,1}(x) = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + n_\varepsilon(x)} \quad \text{and} \quad u_{\varepsilon,2}(x) = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + n_\varepsilon(x)}. \tag{3.1}$$

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As $(n_\varepsilon)_\varepsilon$ is negligible, there exists $\varepsilon_0 > 0$ such that $|n_\varepsilon(x)| < \frac{1}{8}$ for all $\varepsilon < \varepsilon_0$ and all $x \in U$. By continuity of u in ε and x, both of the sets

$$U_1 := \{(\varepsilon, x) \in (0, \varepsilon_0] \times U \mid u_\varepsilon(x) = u_{\varepsilon,1}(x)\},$$
$$U_2 := \{(\varepsilon, x) \in (0, \varepsilon_0] \times U \mid u_\varepsilon(x) = u_{\varepsilon,2}(x)\}$$

are closed and, as they form a partition of $(0, \varepsilon_0] \times U$, also open in $(0, \varepsilon_0] \times U$. Since the latter is connected, we have that either $U_1 = (0, \varepsilon_0] \times U$ or $U_2 = (0, \varepsilon_0] \times U$. Let us assume that it is U_1. Thus, for any $x \in U$, any $m \in \mathbb{N}$ and sufficiently small ε we obtain that

$$|u_\varepsilon(x) - 1| = \left|\sqrt{\tfrac{1}{4} + n_\varepsilon(x)} - \tfrac{1}{2}\right| < \varepsilon^m.$$

Therefore, $u|_U = 1$ in $\mathcal{G}_{\mathrm{co}}(U)$. In the case $U_2 = (0, \varepsilon_0] \times U$ we have that $u|_U = 0$.

Now consider

$$M_1 := \{x \in M \mid \exists \text{ a neighbourhood } V \text{ of } x \text{ such that } u|_V = 1\},$$
$$M_2 := \{x \in M \mid \exists \text{ a neighbourhood } V \text{ of } x \text{ such that } u|_V = 0\}.$$

Both sets are obviously open. Moreover, by the above, M is the disjoint union of M_1 and M_2. Connectedness of M implies that u is either 1 or 0. □

Consequently, there are no non-trivial idempotents in $\mathcal{G}_{\mathrm{sm}}(M)$, $\tilde{\mathbb{K}}_{\mathrm{co}}$ and $\tilde{\mathbb{K}}_{\mathrm{sm}}$.

Next we demonstrate that τ_{co} is not an isomorphism. Hence, $\tilde{\mathbb{K}}$ is strictly larger than $\tilde{\mathbb{K}}_{\mathrm{co}}$, and *a fortiori* $\mathcal{G}(M)$ is strictly larger than $\mathcal{G}_{\mathrm{co}}(M)$.

Lemma 3.4. $\tau_{\mathrm{co}} \colon \tilde{\mathbb{K}}_{\mathrm{co}} \to \tilde{\mathbb{K}}$ *is not surjective, i.e.* $\tilde{\mathbb{K}}_{\mathrm{co}} \subsetneq \tilde{\mathbb{K}}$.

First proof. Let $r = [(r_\varepsilon)_\varepsilon] \in \tilde{\mathbb{K}}$ be defined by

$$r_\varepsilon := \begin{cases} 1 & \text{if } \varepsilon = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose there exists a continuous representative $(s_\varepsilon)_\varepsilon$ of r. Then $r_\varepsilon = s_\varepsilon + n_\varepsilon$ for some $(n_\varepsilon)_\varepsilon \in \mathcal{N}$. For ε sufficiently small (say smaller than some $\varepsilon_0 > 0$) we have that $|n_\varepsilon| < \frac{1}{4}$ and therefore

$$\text{either } |s_\varepsilon| < \tfrac{1}{4} \text{ or } |s_\varepsilon| > \tfrac{3}{4}. \tag{3.2}$$

For $\mathbb{N} \ni n > 1/\varepsilon_0$ we have in particular that $|s_{1/n}| \geqslant |r_{1/n}| - |n_{1/n}| > \frac{3}{4}$ but (as $(2n+1)/2n(n+1) = \frac{1}{2}(1/n + 1/(n+1))$)

$$|s_{(2n+1)/2n(n+1)}| \leqslant |r_{(2n+1)/2n(n+1)}| + |n_{(2n+1)/2n(n+1)}| < 0 + \tfrac{1}{4} = \tfrac{1}{4}.$$

By the Intermediate Value Theorem, there must be an $\varepsilon \in ((2n+1)/2n(n+1), 1/n)$ such that $|s_\varepsilon| = \frac{1}{2}$. This contradicts (3.2). □

Second proof. If τ_{co} were surjective, it would be an isomorphism. Since by [**2**, Theorem 4.1] there exist non-trivial idempotents in $\tilde{\mathbb{K}}$, the same would be true of $\tilde{\mathbb{K}}_{\mathrm{co}}$, contradicting Proposition 3.3. □

This immediately implies the following.

Corollary 3.5. $\mathcal{G}_{\mathrm{co}}(M) \subsetneq \mathcal{G}(M)$.

Our next aim is to establish the surjectivity of the natural embeddings τ_{sm}, both in the case of the rings of generalized numbers $\tilde{\mathbb{K}}_{\mathrm{co}}$ and $\tilde{\mathbb{K}}_{\mathrm{sm}}$ and for the algebras of generalized functions $\mathcal{G}_{\mathrm{co}}$ and $\mathcal{G}_{\mathrm{sm}}$.

Theorem 3.6. *$\tilde{\mathbb{K}}_{\mathrm{sm}}$ is isomorphic to $\tilde{\mathbb{K}}_{\mathrm{co}}$ (via τ_{sm}).*

Proof. Let $(r_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{co}}$. By [**21**, Lemma A.9] (or its strengthening, Lemma 3.7, below) there exists $(s_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(I, \mathbb{K})$ such that

$$|s_\varepsilon - r_\varepsilon| \leqslant \mathrm{e}^{-1/\varepsilon} \quad \forall \varepsilon \in I,$$

so $|s_\varepsilon - r_\varepsilon| < \varepsilon^m$ for all $m \in \mathbb{N}$ and ε sufficiently small. This implies $(s_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{sm}}$ and $[(s_\varepsilon)_\varepsilon] = [(r_\varepsilon)_\varepsilon]$ in $\tilde{\mathbb{K}}_{\mathrm{co}}$. □

Alternatively, one could also apply the Weierstrass Approximation Theorem on compact intervals covering $(0,1]$ to prove Theorem 3.6.

The proof of surjectivity of $\tau_{\mathrm{sm}} \colon \mathcal{G}_{\mathrm{sm}}(M) \to \mathcal{G}_{\mathrm{co}}(M)$ will rely on the following extension of [**21**, Lemma A.9].

Lemma 3.7. *Let $U \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^m$ be open, and suppose that $h \colon I \times U \to W$, $(\varepsilon, x) \mapsto h(\varepsilon, x)$ is continuous with respect to ε and smooth with respect to x. Then for any continuous map $g \colon I \times U \to \mathbb{R}^+$, any $k \in \mathbb{N}_0$ and any open subset U_1 of U with $\overline{U_1} \subset\subset U$ there exists a smooth map $f \colon I \times U \to W$ such that, for all $|\alpha| \leqslant k$ and all $\varepsilon \in I$,*

$$\sup_{x \in U_1} \|\partial_x^\alpha h(\varepsilon, x) - \partial_x^\alpha f(\varepsilon, x)\| \leqslant \inf_{x \in U_1} g(\varepsilon, x).$$

Proof. Replacing, if necessary, g by $(\varepsilon, x) \mapsto \min(g(\varepsilon, x), \frac{1}{2} d(h(\varepsilon, x), \mathbb{R}^m \setminus W))$, we may without loss of generality suppose that $W = \mathbb{R}^m$.

By continuity, for each $\eta \in I$ there exists an open neighbourhood I_η of η in I such that

$$\sup_{x \in U_1} \|\partial_x^\alpha h(\varepsilon, x) - \partial_x^\alpha h(\eta, x)\| \leqslant \inf_{x \in U_1} g(\varepsilon, x), \quad |\alpha| \leqslant k, \ \varepsilon \in I_\eta.$$

Choose a smooth partition of unity $(\phi_\eta)_{\eta \in I}$ on I with $\operatorname{supp} \phi_\eta \subseteq I_\eta$ for each η and set $f(\varepsilon, x) := \sum_{\eta \in I} \phi_\eta(\varepsilon) h(\eta, x)$. Then $f \in \mathcal{C}^\infty(I \times U)$ and for any $\varepsilon \in I$, any $x, y \in U_1$ and any $|\alpha| \leqslant k$ we obtain

$$\begin{aligned}
\|\partial_x^\alpha h(\varepsilon, x) - \partial_x^\alpha f(\varepsilon, x)\| &\leqslant \sum_{\eta \in I} \phi_\eta(\varepsilon) \|\partial_x^\alpha h(\varepsilon, x) - \partial_x^\alpha h(\eta, x)\| \\
&\leqslant \sum_{\eta \in I} \phi_\eta(\varepsilon) g(\varepsilon, y) = g(\varepsilon, y),
\end{aligned}$$

so the claim follows. □

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Lemma 3.8. *Let U, U_1 be open subsets of $\mathbb{R}^n$ with $\overline{U_1} \subset\subset U$. Then, given any $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{co}}(U)$, there exists $(v_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{sm}}(U_1)$ such that $(u_\varepsilon|_{U_1} - v_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{co}}(U_1)$.*

Proof. By Lemma 3.7, for each $n \in \mathbb{N}_0$ there exists $v_n \in \mathcal{C}^\infty(I \times U)$ such that, for all $|\alpha| \leqslant n$ and all $\varepsilon \in I$,

$$\sup_{x \in U_1} \|\partial^\alpha u_\varepsilon(x) - \partial^\alpha v_{n,\varepsilon}(x)\| \leqslant \mathrm{e}^{-1/\varepsilon}. \tag{3.3}$$

Let $\mathcal{I} = (I_n)_{n \in \mathbb{N}}$ be the open cover of I defined by $I_n := (1/(n+2), 1/n)$ for $n \geqslant 2$ and $I_1 := (\frac{1}{3}, 1]$. Choose a smooth partition of unity, $(\chi_n)_{n\in\mathbb{N}}$, with $\operatorname{supp}\chi_n \subseteq I_n$ for all n. For $x \in U_1$ and $\varepsilon \in I$ let

$$v_\varepsilon(x) := \sum_{n=1}^{\infty} \chi_n(\varepsilon) v_{n,\varepsilon}(x). \tag{3.4}$$

Obviously, v is smooth in x and ε. It remains to be shown that $(v_\varepsilon - u_\varepsilon)_\varepsilon$ is negligible on U_1. Fix $K \subset\subset U_1$ and $k \in \mathbb{N}_0$. Then for $\varepsilon \leqslant 1/(k+2)$ and any α with $|\alpha| \leqslant k$ we have that

$$\sup_{x \in K} \|\partial^\alpha v_\varepsilon(x) - \partial^\alpha u_\varepsilon(x)\| \overset{(3.4)}{\leqslant} \sum_{n=k+1}^{\infty} \chi_n(\varepsilon) \sup_{x \in U_1} \|\partial^\alpha v_{n,\varepsilon}(x) - \partial^\alpha u_\varepsilon(x)\| \overset{(3.3)}{\leqslant} \mathrm{e}^{-1/\varepsilon}.$$

Thus, $(v_\varepsilon - u_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{co}}(U_1)$, and therefore also $(v_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{sm}}(U_1)$. $\square$

From these preparations we conclude the following.

Theorem 3.9. *The map $\tau_{\mathrm{sm}} \colon \mathcal{G}_{\mathrm{sm}}(M) \to \mathcal{G}_{\mathrm{co}}(M)$ is an isomorphism.*

Proof. Since both $\mathcal{G}_{\mathrm{co}}(\cdot)$ and $\mathcal{G}_{\mathrm{sm}}(\cdot)$ are sheaves of differential algebras we may without loss of generality suppose that M is an open subset of $\mathbb{R}^n$. Furthermore, by Lemma 3.2 it remains to be shown that τ_{sm} is surjective. To this end let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathrm{co}}(M)$. Choose a locally finite open cover $(U_\alpha)_{\alpha \in A}$ of M such that $\overline{U_\alpha} \subset\subset M$ for all α. Let $(\chi_\alpha)_{\alpha\in A}$ be a partition of unity on M with $\operatorname{supp}\chi_\alpha \subseteq U_\alpha$ for all α. By Lemma 3.8, for each $\alpha \in A$ there exists some $(v_{\alpha,\varepsilon})_\varepsilon \in \mathcal{E}_{M,\mathrm{sm}}(U_\alpha)$ such that $(u_\varepsilon|_{U_\alpha} - v_{\alpha,\varepsilon}) \in \mathcal{N}_{\mathrm{co}}(U_\alpha)$. Then $v_\varepsilon := \sum_\alpha \chi_\alpha v_{\alpha,\varepsilon}$ defines an element $(v_\varepsilon)_\varepsilon$ of $\mathcal{E}_{M,\mathrm{sm}}(M)$ and, by construction, $\tau_{\mathrm{sm}}([(v_\varepsilon)_\varepsilon]) = u$. $\square$

The set of generalized numbers $\tilde{\mathbb{K}}_{\mathrm{sm}}$ can be identified with the set of constant generalized functions in $\mathcal{G}_{\mathrm{sm}}(M)$ via $[(r_\varepsilon)_\varepsilon] \mapsto [(u_\varepsilon)_\varepsilon]$, $u_\varepsilon(x) := r_\varepsilon$ for all $\varepsilon \in I$, $x \in M$. The same is true for $\tilde{\mathbb{K}}_{\mathrm{co}}$ and the set of constant functions in $\mathcal{G}_{\mathrm{co}}(M)$. Thus, Theorem 3.6 can also be viewed as an immediate corollary of Theorem 3.9.

A result analogous to Theorem 3.9 also holds for manifold-valued generalized functions from M to N. Also in this case we define τ_{co} and τ_{sm} to be the natural embeddings, i.e. $[(u_\varepsilon)_\varepsilon] \to [(u_\varepsilon)_\varepsilon]$:

$$\begin{tikzcd}
\mathcal{G}_{\mathrm{sm}}[M,N] \arrow[r, hook, "\tau_{\mathrm{sm}}"] \arrow[rr, hook, bend right, "\tau_{\mathrm{co}} \circ \tau_{\mathrm{sm}}"'] & \mathcal{G}_{\mathrm{co}}[M,N] \arrow[r, hook, "\tau_{\mathrm{co}}"] & \mathcal{G}[M,N]
\end{tikzcd}$$

Similarly to Lemma 3.2, we have that these maps are well defined and injective, using [**16**, Definitions 2.2 and 2.4]. Building on Theorem 3.9 we can now show the following result.

Theorem 3.10. *The map* $\tau_{\mathrm{sm}}\colon \mathcal{G}_{\mathrm{sm}}[M,N] \to \mathcal{G}_{\mathrm{co}}[M,N]$ *is bijective.*

Proof. By [**19**, Proposition 2.2], given any Whitney embedding $i\colon N \hookrightarrow \mathbb{R}^s$, we may identify $\mathcal{G}_{\mathrm{sm}}[M,N]$ with the subspace $\tilde{\mathcal{G}}_{\mathrm{sm}}[M,i(N)]$ of $\mathcal{G}_{\mathrm{sm}}(M)^s$. The proof of that result carries over verbatim to the $\mathcal{G}_{\mathrm{co}}$ setting. Therefore, we may without loss of generality suppose that N is a submanifold of some $\mathbb{R}^s$. Let T be a tubular neighbourhood of N in $\mathbb{R}^s$ with retraction map $r\colon T \to N$ (see, for example, [**15**, **20**]).

Let $u \in \mathcal{G}_{\mathrm{co}}[M,N]$. By Theorem 3.9 there exists $v' \in \mathcal{G}_{\mathrm{sm}}(M)^s$ such that $(u_\varepsilon - v'_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{co}}(M)^s$ (hence, in particular, v' is c-bounded, i.e. $v' \in \mathcal{G}_{\mathrm{sm}}[M,\mathbb{R}^s]$). We will now suitably modify v' such that the resulting element of $\mathcal{G}_{\mathrm{sm}}[M,N]$ equals u in $\mathcal{G}_{\mathrm{co}}[M,N]$. To this end we follow a similar path to that in the proof of [**19**, Theorem 2.3]. Let $T' \subseteq T$ be a closed tubular neighbourhood of N. Using a partition of unity subordinate to $\{T, \mathbb{R}^s \setminus T'\}$ we obtain a map $\tilde{r}\colon \mathbb{R}^s \to \mathbb{R}^s$ which coincides with r on T'. Let $(K_l)_l$ be a compact exhaustion of M with $K_l \subseteq K^\circ_{l+1}$ for all l. Since each u_ε is c-bounded with values in N and $(u_\varepsilon - v'_\varepsilon)_\varepsilon$ is negligible, for each l there exists a compact set $K'_l \subseteq T'$ and an $\varepsilon_l > 0$ (without loss of generality $\varepsilon_l < \varepsilon_{l-1}$) such that $u_\varepsilon(K_l) \cup v'_\varepsilon(K_l) \subseteq K'_l$ for all $\varepsilon \leqslant \varepsilon_l$.

For $\varepsilon \in I$ we set $v''_\varepsilon := \tilde{r} \circ v'_\varepsilon$. Then $v'' \in \mathcal{G}_{\mathrm{sm}}[M,\mathbb{R}^s]$, and for $x \in K_l$ and $\varepsilon < \varepsilon_l$ we have (denoting by $\mathrm{ch}(K'_l)$ the convex hull of K'_l)

$$\begin{aligned}\|u_\varepsilon(x) - v''_\varepsilon(x)\| &= \|\tilde{r} \circ u_\varepsilon(x) - \tilde{r} \circ v'_\varepsilon(x)\| \\ &\leqslant \|D\tilde{r}\|_{L^\infty(\mathrm{ch}(K'_l))} \cdot \|u_\varepsilon(x) - v'_\varepsilon(x)\| \\ &= O(\varepsilon^m)\end{aligned}$$

for each m by construction of v'.

Now let $\eta\colon M \to \mathbb{R}$ be smooth such that $0 < \eta(x) \leqslant \varepsilon_l$ for all $x \in K_l \setminus K^\circ_{l-1}$ ($K_0 := \emptyset$) [**14**, Lemma 2.7.3]. Moreover, let $\nu\colon \mathbb{R}^+_0 \to [0,1]$ be a smooth function satisfying $\nu(t) \leqslant t$ for all t and

$$\nu(t) = \begin{cases} t, & 0 \leqslant t \leqslant \frac{1}{2}, \\ 1, & t \geqslant \frac{3}{2}. \end{cases}$$

For $(\varepsilon, x) \in I \times M$ let $\mu(\varepsilon, x) := \eta(x)\nu(\varepsilon/\eta(x))$. Then we may define $v_\varepsilon(x) := v''_{\mu(\varepsilon,x)}(x)$ for $(\varepsilon, x) \in I \times M$. It follows that $v \in \mathcal{G}_{\mathrm{sm}}[M,N]$. Since $v_\varepsilon|_{K^\circ_l} = v''_\varepsilon|_{K^\circ_l}$ for $\varepsilon \leqslant \frac{1}{2}\min_{x \in K_l}\eta(x)$ and any $l \in \mathbb{N}$, $(u_\varepsilon - v_\varepsilon)_\varepsilon$ satisfies the negligibility estimate of order 0 on any compact subset of M. Thus, by [**18**, Theorem 3.3], we conclude that $u = v$ in $\mathcal{G}_{\mathrm{co}}[M,N]$. □

Remark 3.11. Similar techniques can be used to show that the smooth and continuous variants of the spaces of generalized vector bundle homomorphisms and of hybrid generalized functions (see [**16**–**18**] for definitions and characterizations of these spaces) can be identified.

4. Algebraic properties of $\tilde{\mathbb{K}}_{\mathrm{sm}} = \tilde{\mathbb{K}}_{\mathrm{co}}$

Above we have seen that $\tilde{\mathbb{K}}_{\mathrm{co}}$ and $\tilde{\mathbb{K}}_{\mathrm{sm}}$ are algebraically isomorphic, and are proper subrings of $\tilde{\mathbb{K}}$. The aim of this section is to initiate the investigation of algebraic properties of $\tilde{\mathbb{K}}_{\mathrm{sm}}$ along the lines of [**1**, **2**, **29**]. In particular, we point out similarities and differences between the spaces $\tilde{\mathbb{K}}_{\mathrm{sm}}$ and $\tilde{\mathbb{K}}$.

4.1. Non-invertible elements are zero divisors

By [**29**, § 2.1], $\tilde{\mathbb{K}}$ is a reduced ring, i.e. a commutative ring without non-zero nilpotent elements. As $\tilde{\mathbb{K}}_{\mathrm{sm}}$ is a subring of $\tilde{\mathbb{K}}$, it inherits this property.

A fundamental property of $\tilde{\mathbb{K}}$ is that the non-invertible elements and the zero divisors in $\tilde{\mathbb{K}}$ coincide (see [**1**, Theorem 2.18] and [**14**, Theorem 1.2.39]). The same holds true for $\tilde{\mathbb{K}}_{\mathrm{sm}}$.

Proposition 4.1. *An element $r \in \tilde{\mathbb{K}}_{\mathrm{sm}}$ is non-invertible if and only if it is a zero divisor.*

Proof. Let r be non-invertible. By [**14**, Theorem 1.2.38] we have that r is not strictly non-zero (the proof carries over unchanged to the $\tilde{\mathbb{K}}_{\mathrm{sm}}$ setting; see also [**6**, Proposition 6.2.5] for a generalization), i.e. for all representatives $(r_\varepsilon)_\varepsilon$ of r and all $m \in \mathbb{N}$ there exists a strictly decreasing sequence $\varepsilon_k \searrow 0$ such that $|r_{\varepsilon_k}| < \varepsilon_k^m$. By varying m, we obtain a sequence $(\varepsilon_j)_j$, $\varepsilon_j \searrow 0$, such that

$$|r_{\varepsilon_j}| < \varepsilon_j^j \quad \forall j \in \mathbb{N}.$$

Since $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is discrete in $(0,1]$ we may find disjoint neighbourhoods $(a_j, b_j) \ni \varepsilon_j$ such that

$$|r_\varepsilon| < \varepsilon^j \quad \forall \varepsilon \in (a_j, b_j).$$

On each such interval (a_j, b_j) there exists a smooth bump function $\chi_j \in \mathcal{D}(a_j, b_j)$ such that $\chi_j(\varepsilon_j) = 1$, $0 \leqslant \chi_j \leqslant 1$. Let $(s_\varepsilon)_\varepsilon$ be defined by

$$s_\varepsilon := \begin{cases} \chi_j(\varepsilon) & \text{if } \varepsilon \in (a_j, b_j), \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $(s_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{sm}} \setminus \mathcal{N}_{\mathrm{sm}}$. Moreover, $(r_\varepsilon s_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{sm}}$; hence, r is a zero divisor of $\tilde{\mathbb{K}}_{\mathrm{sm}}$. $\square$

4.2. Exchange rings

There are various equivalent definitions for exchange rings (see, for example, [**29**, § 2.2]). The most convenient one for our purposes is the following.

Definition 4.2. A commutative ring R with 1 is an *exchange ring* if, for each $r \in R$, there exists an idempotent $e \in R$ such that $r + e$ is invertible.

By [**29**, Proposition 2.1], $\tilde{\mathbb{K}}$ is an exchange ring. The situation is different for $\tilde{\mathbb{K}}_{\mathrm{sm}}$.

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Lemma 4.3. *$\tilde{\mathbb{K}}_{\mathrm{sm}}$ is not an exchange ring.*

Proof. By Proposition 3.3, there are no non-trivial idempotents in $\tilde{\mathbb{K}}_{\mathrm{sm}}$. Moreover, $r \in \tilde{\mathbb{K}}_{\mathrm{sm}}$, defined by $r_\varepsilon := \sin(1/\varepsilon)$, is both non-zero and non-invertible, and also $r \pm 1$ is non-invertible. □

4.3. Gelfand rings

Definition 4.4. A ring R is called a *Gelfand ring* if for $a, b \in R$ with $a + b = 1$ there exist $r, s \in R$ such that $(1 + ar)(1 + bs) = 0$.

That $\tilde{\mathbb{K}}$ is a Gelfand ring is a direct consequence of the fact that it is an exchange ring. For $\tilde{\mathbb{K}}_{\mathrm{sm}}$ we need a different approach.

Lemma 4.5. *$\tilde{\mathbb{K}}_{\mathrm{sm}}$ is a Gelfand ring.*

Proof. Assume that $(a_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ are representatives of a and b such that $a_\varepsilon + b_\varepsilon = 1$ for all ε. If $a = 0$, then $b = 1$, and $r = 0$ and $s = -1$ satisfy $(1 + ar)(1 + bs) = 1 \cdot 0 = 0$ (similarly for $b = 0$).

Let $a \neq 0$ and $b \neq 0$. Let $S := \{\varepsilon \in I \colon |a_\varepsilon| \geqslant \frac{1}{2}\}$ and let $\chi \in \mathcal{C}^\infty(\mathbb{R}, I)$ such that $\chi|_{(-\infty,1/2]} = 0$ and $\chi|_{[1,\infty)} = 1$. Then $(r_\varepsilon)_\varepsilon$, defined by

$$r_\varepsilon := \begin{cases} -\dfrac{\chi(2|a_\varepsilon|)}{a_\varepsilon} & \text{if } \varepsilon \in S, \\ 0 & \text{otherwise,} \end{cases}$$

is well defined and smooth. It is moderate, since for ε such that $|a_\varepsilon| \geqslant \frac{1}{4}$ we have that

$$|r_\varepsilon| = \frac{|\chi(2|a_\varepsilon|)|}{|a_\varepsilon|} \leqslant \frac{1}{|a_\varepsilon|} \leqslant 4,$$

and for ε such that $|a_\varepsilon| < \frac{1}{4}$ we even have that $|r_\varepsilon| = 0$. Furthermore, $a_\varepsilon r_\varepsilon = -1$ on S.

Similarly, there exists $(s_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{sm}}$ such that $b_\varepsilon s_\varepsilon = -1$ on $\{\varepsilon \in I \colon |b_\varepsilon| \geqslant \frac{1}{2}\}$. Altogether, $(1 + a_\varepsilon r_\varepsilon)(1 + b_\varepsilon s_\varepsilon) = 0$ for all $\varepsilon \in (0, 1]$. □

4.4. Partial order and absolute value

The order on $\tilde{\mathbb{R}}_{\mathrm{sm}}$ (and similarly on $\tilde{\mathbb{R}}_{\mathrm{co}}$) is inherited by the order on $\tilde{\mathbb{R}}$ [**14**, § 1.2.4].

Definition 4.6. Let $r, s \in \tilde{\mathbb{R}}_{\mathrm{sm}}$. We write $r \leqslant s$ if there are representatives $(r_\varepsilon)_\varepsilon$, $(s_\varepsilon)_\varepsilon$ with $r_\varepsilon \leqslant s_\varepsilon$ for all ε.

Remark 4.7. Note that this is equivalent to the fact that for any representatives $\bar{r}$, $\bar{s}$ of r and s there exists some $(n_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathrm{sm}}$ with $\bar{r}_\varepsilon \leqslant \bar{s}_\varepsilon + n_\varepsilon$.

Moreover, by [**26**], $r \leqslant s$ if and only if for all representatives (r_ε), $(s_\varepsilon)_\varepsilon$ and any $a > 0$ there exists some $\varepsilon_0 > 0$ such that $r_\varepsilon \leqslant s_\varepsilon + \varepsilon^a$ for all $\varepsilon < \varepsilon_0$. Further properties of the order structure in $\tilde{\mathbb{R}}$ and $\mathcal{G}$ can be found in [**23**, **26**].

The same argument as in the case of $\tilde{\mathbb{R}}$ yields the following.

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Proposition 4.8. $(\tilde{\mathbb{R}}_{\mathrm{sm}}, \leqslant)$ *is a partially ordered ring.*

By the identification of $\tilde{\mathbb{K}}_{\mathrm{sm}}$ with $\tilde{\mathbb{K}}_{\mathrm{co}}$ in Theorem 3.6 we can even define the absolute value of generalized numbers in $\tilde{\mathbb{K}}_{\mathrm{sm}}$ (note that generally $(|r_\varepsilon|)_\varepsilon \in \mathcal{E}_{M,\mathrm{co}}$ but $(|r_\varepsilon|)_\varepsilon \notin \mathcal{E}_{M,\mathrm{sm}}$).

Definition 4.9. Let $r = [(r_\varepsilon)_\varepsilon] \in \tilde{\mathbb{K}}_{\mathrm{sm}}$. The *absolute value* of r, denoted by $|r|$, is defined as the generalized number

$$|r| := \tau_{\mathrm{sm}}^{-1}([(|r_\varepsilon|)_\varepsilon]),$$

where $|\cdot|$ denotes the absolute value in $\mathbb{C}$ and $\tau_{\mathrm{sm}} \colon \tilde{\mathbb{K}}_{\mathrm{sm}} \to \tilde{\mathbb{K}}_{\mathrm{co}}$ is the canonical isomorphism (see Theorem 3.6).

By the identification of $\tilde{\mathbb{R}}_{\mathrm{sm}}$ with $\tilde{\mathbb{R}}_{\mathrm{co}}$ we can show that it is a lattice.

Definition 4.10. A *lattice* is a partially ordered set R such that any two elements $r, s \in R$ have a *join* (or *supremum*) $r \vee s$ and a *meet* (or *infimum*) $r \wedge s$.

A partially ordered ring that is a lattice for this order is called an *l-ring* (or *lattice-ordered ring*).

Definition 4.11. The *minimum* $\min(r, s)$ and the *maximum* $\max(r, s)$ for $r = [(r_\varepsilon)_\varepsilon]$, $s = [(s_\varepsilon)_\varepsilon] \in \tilde{\mathbb{R}}_{\mathrm{sm}}$ are defined as follows:

$$\min(r, s) := \tau_{\mathrm{sm}}^{-1}([(\min(r_\varepsilon, s_\varepsilon))_\varepsilon]),$$
$$\max(r, s) := \tau_{\mathrm{sm}}^{-1}([(\max(r_\varepsilon, s_\varepsilon))_\varepsilon]).$$

These notions are well defined for $\tilde{\mathbb{R}}_{\mathrm{sm}}$ since the min and max of real-valued continuous functions are continuous themselves. Clearly,

$$\min((r_\varepsilon)_\varepsilon + \mathcal{N}_{\mathrm{co}}, (s_\varepsilon)_\varepsilon + \mathcal{N}_{\mathrm{co}}) = (\min(r_\varepsilon, s_\varepsilon))_\varepsilon + \mathcal{N}_{\mathrm{co}},$$

etc. Thus, by Remark 4.7 we have the following.

Lemma 4.12. *The minimum and maximum as defined above are well defined and compatible with the partial order structure of* $(\tilde{\mathbb{R}}_{\mathrm{sm}}, \leqslant)$.

This result is remarkable since the underlying ring in the definition of $\tilde{\mathbb{R}}_{\mathrm{sm}}$ (namely, $\mathcal{C}^\infty(I, \mathbb{R})$) does not satisfy these properties. Setting $r \vee s = \max(r, s)$ and $r \wedge s = \min(r, s)$ we obtain [**29**, § 2.3] the following.

Proposition 4.13. $\tilde{\mathbb{R}}_{\mathrm{sm}}$ *is an l-ring.*

Clearly, the absolute value as introduced in Definition 4.9 is compatible with the order structure on $\tilde{\mathbb{R}}_{\mathrm{sm}}$, i.e. $|r| = \max(r, -r)$ for any $r \in \tilde{\mathbb{R}}_{\mathrm{sm}}$.

Definition 4.14. A commutative ring R with 1 is called an *f-ring* if it is an l-ring and, for all $r, s, t \in R$ with $t \geqslant 0$, $(r \wedge s)t = rt \wedge st$.

By [**29**, Proposition 2.2], $\tilde{\mathbb{R}}$ is an f-ring. The same holds true for $\tilde{\mathbb{R}}_{\mathrm{sm}}$.

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Proposition 4.15. *$\tilde{\mathbb{R}}_{\mathrm{sm}}$ is an f-ring.*

Proof. Let $r, s, t \in \tilde{\mathbb{R}}_{\mathrm{co}}$ with representatives $(r_\varepsilon)_\varepsilon$ of r, $(s_\varepsilon)_\varepsilon$ of s and $(t_\varepsilon)_\varepsilon$ of t such that $t_\varepsilon \geqslant 0$ for all ε. Then $\min(r_\varepsilon, s_\varepsilon)t_\varepsilon = \min(r_\varepsilon t_\varepsilon, s_\varepsilon t_\varepsilon)$ for all $\varepsilon \in I$, and therefore $(r \wedge s)t = rt \wedge st$. By Theorem 3.6 and Definition 4.11, the claim follows. $\square$

For some properties of l- and f-rings see [**3**].

A main technical tool in the algebraic investigation of $\tilde{\mathbb{K}}$ is the use of characteristic functions e_S of subsets $S \subseteq I$ (see [**1**, **2**, **29**, **30**]). Obviously, such functions on I are not continuous unless in trivial cases and therefore cannot be used in the $\tilde{\mathbb{K}}_{\mathrm{sm}}$ setting. This forecloses a direct adaptation of many proof techniques from $\tilde{\mathbb{K}}$ to $\tilde{\mathbb{K}}_{\mathrm{sm}}$. In certain situations, however, a substitute for these techniques can be based on the notion of characteristic set.

Definition 4.16. A subset S of I is called *characteristic set* if $0 \in \bar{S}$.

If $r \in \tilde{\mathbb{K}}_{\mathrm{sm}}$ and S is a characteristic set, then by $r|_S = 0$ we mean that for any $m \in \mathbb{N}$ there exists some ε_0 such that $|r_\varepsilon| < \varepsilon^m$ for all $\varepsilon \in S$ with $\varepsilon < \varepsilon_0$ (which clearly is independent of the representative of r).

Lemma 4.17. *Let $r, s \in \tilde{\mathbb{K}}_{\mathrm{sm}}$, $r, s \neq 0$ and $rs = 0$. Then there exists a characteristic set S such that $r|_S = s|_S = 0$.*

Proof. Since r and s are non-zero there exist characteristic sets S_r and S_s and $K \in \mathbb{N}$ such that

$$|r_\varepsilon| > \varepsilon^K \quad \forall \varepsilon \in S_r \qquad \text{and} \qquad |s_\varepsilon| > \varepsilon^K \quad \forall \varepsilon \in S_s. \tag{4.1}$$

Let $m = 2K$. Since $rs = 0$ there exists $\varepsilon_m > 0$ such that

$$|r_\varepsilon s_\varepsilon| < \varepsilon^m \quad \forall \varepsilon < \varepsilon_m. \tag{4.2}$$

Moreover, S_r and S_s are disjoint on $(0, \varepsilon_m)$, i.e. $S_r \cap S_s \cap (0, \varepsilon_m) = \emptyset$: for $\varepsilon \in S_r$, $\varepsilon < \varepsilon_m$ we have that $\varepsilon^K |s_\varepsilon| < |r_\varepsilon s_\varepsilon| < \varepsilon^m$. Therefore, $|s_\varepsilon| < \varepsilon^{m-K} = \varepsilon^K$, i.e. $\varepsilon \notin S_s$ by (4.1).

For all $\varepsilon \in S_r \cap (0, \varepsilon_m)$ we have by (4.1) and (4.2) that $|r_\varepsilon| - |s_\varepsilon| > \varepsilon^K - \varepsilon^K = 0$. In particular, since $S_r \cap (0, \varepsilon_m) \neq \emptyset$ (S_r being a characteristic set), there exists $\varepsilon_r < \varepsilon_m$ such that $|r_{\varepsilon_r}| - |s_{\varepsilon_r}| > 0$. Similarly, there exists $\varepsilon_s < \varepsilon_m$ such that $|r_{\varepsilon_s}| - |s_{\varepsilon_s}| < 0$. Hence, by continuity in ε there exists $\delta_m \in (0, \varepsilon_m)$ such that $|r_{\delta_m}| = |s_{\delta_m}|$. We even know that $\delta_m \notin S_r \cup S_s$. In fact, as $\delta_m < \varepsilon_m$, (4.2) implies that $|r_{\delta_m} s_{\delta_m}| < \delta_m^m$, and therefore $|r_{\delta_m}| = |s_{\delta_m}| < \delta_m^{m/2}$.

To construct the characteristic set S we proceed by induction. Let $\bar{\varepsilon}_1 := \delta_m = \delta_{2K}$. Suppose we have already constructed $\bar{\varepsilon}_i < \min(\bar{\varepsilon}_{i-1}, 1/i)$ such that

$$|r_{\bar{\varepsilon}_i}| = |s_{\bar{\varepsilon}_i}| < \bar{\varepsilon}_i^{(m+2(i-1))/2}. \tag{4.3}$$

As above we find $\varepsilon_{m+2i} < \min(\bar{\varepsilon}_i, 1/(i+1))$ such that $|r_\varepsilon s_\varepsilon| < \varepsilon^{m+2i}$ for all $\varepsilon < \varepsilon_{m+2i}$. Since $m + 2i > 2K$ all other arguments hold as well and we finally obtain $\bar{\varepsilon}_{i+1} < \min(\bar{\varepsilon}_i, 1/(i+1))$ such that (4.3) holds for $i+1$ instead of i.

Since $\bar{\varepsilon}_j \searrow 0$ we have that $S := \{\bar{\varepsilon}_j | j \in \mathbb{N}\}$ is a characteristic set and by (4.3) it follows that $r|_S = s|_S = 0$. $\square$

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Let $S \subseteq I$ be a characteristic set and let $\mathcal{A}$ denote the algebra $\tilde{\mathbb{K}}_{\mathrm{sm}}$ or $\mathcal{G}_{\mathrm{sm}}(M)$. An element $u \in \mathcal{A}$ is called invertible with respect to S if there exist $v \in \mathcal{A}$ and $r \in \tilde{\mathbb{K}}_{\mathrm{sm}}$ such that

$$uv = r1 \text{ in } \mathcal{A} \quad \text{and} \quad (r-1)|_S = 0 \text{ in } \tilde{\mathbb{K}}_{\mathrm{sm}}.$$

By [**7**, Proposition 4.2], an element r of $\tilde{\mathbb{K}}_{\mathrm{sm}}$ is non-zero if and only if there exists a characteristic set S such that r is invertible with respect to S. The proof of this result also shows that r is invertible with respect to S if and only if it is strictly non-zero with respect to S, which gives a generalization of [**14**, Proposition 1.2.38]. Analogous results hold for generalized functions with smooth parameter dependence [**6**, § 6.2].

Definition 4.18. Let R be a ring and $r \in R$. The *annihilator* of r is defined as the set $\mathrm{Ann}(r) := \{s \in R \colon rs = 0\}$.

Theorem 4.19. *Let $r, s \in \tilde{\mathbb{K}}_{\mathrm{sm}}$. The following are equivalent:*

(i) $rs = 0$;

(ii) *there exists $x \in \tilde{\mathbb{K}}_{\mathrm{sm}}$ such that $rx = 0$ and $s(1-x) = 0$;*

(iii) $\mathrm{Ann}(r) + \mathrm{Ann}(s) = \tilde{\mathbb{K}}_{\mathrm{sm}}$;

(iv) $|r| \wedge |s| = 0$.

We show this along the lines of the proof of [**29**, Lemma 2.3], where the result was verified for $\tilde{\mathbb{K}}$.

Proof. (i) $\implies$ (ii). Let $rs = 0$. The cases $r = 0$ or $s = 0$ being obvious, we may assume that both r and s are zero divisors. For all $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ such that $|r_\varepsilon s_\varepsilon| < \varepsilon^m$ for all $\varepsilon < \varepsilon_m$ by (i). Without loss of generality we can assume that $(\varepsilon_m)_m$ is a decreasing sequence. Moreover, by moderateness of $(r_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ we have an $N \in \mathbb{N}$ such that $|r_\varepsilon| < \varepsilon^{-N}$ and $|s_\varepsilon| < \varepsilon^{-N}$ for ε sufficiently small. Using a partition-of-unity argument (see, for example, [**14**, Lemma 2.7.3]) we obtain a function $\eta \in \mathcal{C}^\infty(I, \mathbb{R})$ such that

$$0 < \eta(\varepsilon) \leqslant \varepsilon^{m+N} \quad \text{for } \varepsilon \in [\varepsilon_{m+1}, \varepsilon_m].$$

Let

$$U := \{\varepsilon \in I \colon |r_\varepsilon| < |s_\varepsilon| + \eta(\varepsilon)\},$$
$$V := \{\varepsilon \in I \colon |r_\varepsilon| \leqslant |s_\varepsilon| - \eta(\varepsilon)\}.$$

By continuous dependence of r, s and η on ε, U is open and V is closed in I. Using a partition of unity subordinate to $\{I \setminus V, U\}$ we obtain a smooth bump function $I \to [0,1]$, $\varepsilon \mapsto x_\varepsilon$ with $x|_V = 1$, $x|_U \leqslant 1$ and $x|_{I \setminus U} = 0$. In particular, $(x_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{sm}}$.

Therefore, we have that (using $\tilde{\mathbb{K}}_{\mathrm{sm}} \cong \tilde{\mathbb{K}}_{\mathrm{co}}$ by Theorem 3.6 and calculating in $\mathcal{E}_{M,\mathrm{co}}$)

$$\begin{aligned}
0 &\leqslant (|r|x)^2 \\
&= \begin{cases} (|r_\varepsilon| x_\varepsilon)^2 & \text{if } \varepsilon \in U, \\ 0 & \text{otherwise,} \end{cases} \\
&\overset{(x_\varepsilon^2 \leqslant 1)}{\leqslant} \begin{cases} |r_\varepsilon|(|s_\varepsilon| + \varepsilon^{m+N}) & \text{if } \varepsilon \in U, \\ 0 & \text{otherwise,} \end{cases} \\
&< 2\varepsilon^m \quad \text{for } \varepsilon \text{ sufficiently small,}
\end{aligned}$$

since $|r_\varepsilon s_\varepsilon| < \varepsilon^m$ and $|r_\varepsilon|\varepsilon^{m+N} < \varepsilon^{-N}\varepsilon^{m+N} = \varepsilon^m$ for such ε. Hence,

$$rx = 0.$$

Similarly,

$$\begin{aligned}
0 &\leqslant (|s|(1-x))^2 \\
&= \begin{cases} (|s_\varepsilon|(1 - x_\varepsilon))^2 & \text{if } \varepsilon \notin V, \\ 0 & \text{otherwise,} \end{cases} \\
&\overset{((1-x_\varepsilon)^2 \leqslant 1)}{\leqslant} \begin{cases} |s_\varepsilon|(|r_\varepsilon| + \varepsilon^{m+N}) & \text{if } \varepsilon \notin V, \\ 0 & \text{otherwise,} \end{cases} \\
&< 2\varepsilon^m \quad \text{for } \varepsilon \text{ sufficiently small.}
\end{aligned}$$

Thus, also

$$s(1-x) = 0.$$

(ii) $\implies$ (iii). By (ii) there exists $x \in \tilde{\mathbb{K}}_{\mathrm{sm}}$ such that $x \in \mathrm{Ann}(r)$ and $1 - x \in \mathrm{Ann}(s)$. For any $t \in \tilde{\mathbb{K}}_{\mathrm{sm}}$, $t = xt + (1-x)t$. Since annihilators are ideals in the ring, $xt \in \mathrm{Ann}(r)$ and $(1-x)t \in \mathrm{Ann}(s)$.

(iii) $\implies$ (i). By (iii) we may write $1 = x + y$ for $x \in \mathrm{Ann}(r)$ and $y \in \mathrm{Ann}(s)$. Therefore,

$$rs = rs1 = rs(x+y) = (rx)s + r(sy) = 0.$$

(i) $\iff$ (iv). As $rs = 0$ is equivalent to $|r|\,|s| = 0$, we may assume that $r, s \in \tilde{\mathbb{R}}_{\mathrm{sm}}$. By § 4.1, $\tilde{\mathbb{K}}_{\mathrm{sm}}$ is a reduced ring and by Proposition 4.15 it is an f-ring. Since the equivalence holds in any reduced f-ring (see [**5**, Theorem 9.3.1]), the proof is complete. $\square$

From the equivalence of (i) and (iii) we can deduce another property of rings of generalized numbers, namely normality. Since we are dealing with reduced rings, we may use the following definition (see [**29**, § 2.3] for different equivalent conditions).

Definition 4.20. A reduced commutative f-ring R with 1 is called *normal* if for all $r, s \in R$ with $rs = 0$ we can write $R = \mathrm{Ann}(r) + \mathrm{Ann}(s)$.

Corollary 4.21. *$\tilde{\mathbb{R}}$ and $\tilde{\mathbb{R}}_{\mathrm{sm}}$ are (reduced) normal f-rings.*

Proof. The property of being a reduced ring was noted at the beginning of § 4.1; the other claims follow from Proposition 4.15 and Theorem 4.19. □

4.5. Ideals

In recent years, various properties of ideals in the ring $\tilde{\mathbb{K}}$ of generalized numbers have been studied. Previous investigations have led to, among other things, a complete description of the maximal ideals (see [**1**, Theorem 4.20]), minimal prime ideals (see [**2**, Corollary 4.7]) and prime ideals (see [**29**, Theorem 3.6]) in $\tilde{\mathbb{K}}$. In this section we initiate a similar study for the ring $\tilde{\mathbb{K}}_{\mathrm{sm}}$ of generalized numbers with smooth parameter dependence and provide some basic properties of its ideals.

Let R be a commutative ring with 1. An ideal J in R is denoted by $J \trianglelefteq R$; a proper ideal is denoted by $J \triangleleft R$. Moreover, we call $J \triangleleft R$ *prime* if, for all $r, s \in R$ with $rs \in J$, we have that $r \in J$ or $s \in J$. A proper ideal J is called *maximal* if the only ideal properly containing it is R itself. $J \trianglelefteq R$ is called *idempotent* if $J^2 = J$.

The radical of an ideal $J \triangleleft R$ is denoted by

$$\sqrt{J} = \{r \in R \mid \exists n \in N \colon x^n \in J\} = \bigcap_{\substack{J \subseteq P \\ P \text{ prime}}} P$$

(see, for example, [**13**, Corollary 0.18]). An ideal $J \trianglelefteq R$ is called *radical* if $J = \sqrt{J}$.

To begin with, we investigate convexity of ideals in $\tilde{\mathbb{K}}_{\mathrm{sm}}$.

Definition 4.22. Let R be a partially ordered ring and $J \trianglelefteq R$ be an ideal. J is said to be *convex* if $0 \leqslant y \leqslant x$ and $x \in J$ imply that $y \in J$.

An ideal J in an l-ring R is called *absolutely convex* (or *l-ideal*) if $|y| \leqslant |x|$ and $x \in J$ imply $y \in J$.

In [**2**, Proposition 3.7] it was shown that every ideal in $\tilde{\mathbb{K}}$ is absolutely convex. For $\tilde{\mathbb{R}}_{\mathrm{sm}}$ we firstly have the following.

Proposition 4.23. *All ideals in $\tilde{\mathbb{R}}_{\mathrm{sm}}$ are convex.*

Proof. Let $J \trianglelefteq \tilde{\mathbb{R}}_{\mathrm{sm}}$, $x \in J$ and $0 \leqslant y \leqslant x$. Without loss of generality we may consider representatives $(x_\varepsilon)_\varepsilon$, $(y_\varepsilon)_\varepsilon$ such that $0 < y_\varepsilon \leqslant x_\varepsilon$ for all $\varepsilon \in I$ (otherwise add $(\exp(-1/\varepsilon))_\varepsilon \in \mathcal{N}_{\mathrm{sm}}$ to non-negative representatives). Thus, $(a_\varepsilon)_\varepsilon$, defined by

$$a_\varepsilon := \frac{y_\varepsilon}{x_\varepsilon} \quad \forall \varepsilon \in I,$$

is well defined, smooth and bounded by 1, and hence moderate. Since $x \in J$ and $y = ax$, we also have that $y \in J$. □

In order to prove that ideals are in fact absolutely convex, we show the following lemma on $\tilde{\mathbb{R}}_{\mathrm{sm}}$ and $\tilde{\mathbb{C}}_{\mathrm{sm}}$.

Lemma 4.24. *Let $J \trianglelefteq \tilde{\mathbb{K}}_{\mathrm{sm}}$ and $x \in J$. Then $|x| \in J$.*

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Proof. According to Theorem 3.6 we can work in $\tilde{\mathbb{C}}_{\mathrm{co}}$. The proof for $\tilde{\mathbb{R}}_{\mathrm{co}}$ proceeds along the same lines. Let $(x_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{co}}$ be a representative of x. We construct $(a_\varepsilon)_\varepsilon \in \mathcal{E}_{M,\mathrm{co}}$ such that $ax = |x|$.

Fix $m \in \mathbb{N}$. Let $b_m \colon (0,1] \to (0,1]$ be defined by

$$b_m(\varepsilon) := \begin{cases} \dfrac{\varepsilon^m}{|x_\varepsilon|} & \text{if } |x_\varepsilon| \geqslant \varepsilon^m, \\ 1 & \text{otherwise.} \end{cases} \tag{4.4}$$

Then b_m is continuous. In order to obtain the necessary asymptotic behaviour, we patch the b_m together. Thus, we consider the open cover

$$\mathcal{I} := \left\{ \left(\frac{1}{m+1}, \frac{1}{m-1} \right) \right\}_{m>1} \cup \{(\tfrac{1}{3}, 1]\}$$

of the interval $(0,1]$, and a corresponding (continuous) partition of unity $(\chi_m)_{m\in\mathbb{N}}$. By $\arg(z)$ we denote the argument of the complex number z. Let

$$a_\varepsilon := \begin{cases} \mathrm{e}^{-\mathrm{i}\arg(x_\varepsilon)} \left(1 - \displaystyle\sum_{m=1}^{\infty} b_m(\varepsilon)\chi_m(\varepsilon) \right) & \text{if } x_\varepsilon \neq 0, \\ 0 & \text{if } x_\varepsilon = 0, \end{cases}$$

for all $\varepsilon \in (0,1]$. Suppose that $x_{\bar\varepsilon} = 0$, $x_{\varepsilon_k} \neq 0$ and $\varepsilon_k \to \bar\varepsilon$. Then

$$\lim_{k\to\infty} a_{\varepsilon_k} = \lim_{k\to\infty} \left(\underbrace{\mathrm{e}^{-\mathrm{i}\arg(x_{\varepsilon_k})}}_{|\cdot|\leqslant 1} \left(1 - \sum_{m\in\mathbb{N}} b_m(\varepsilon_k)\chi_m(\varepsilon_k) \right) \right) = 0$$

due to (4.4). Thus, $(a_\varepsilon)_\varepsilon \in \mathcal{C}(I, \mathbb{C})$. Furthermore, $(a_\varepsilon)_\varepsilon$ is moderate:

$$|a_\varepsilon| \leqslant |\mathrm{e}^{-\mathrm{i}\arg(x_\varepsilon)}| \cdot \left| 1 - \sum_{m=0}^{\infty} b_m(\varepsilon)\chi_m(\varepsilon) \right| \leqslant 2. \tag{4.5}$$

It remains to show that $(a_\varepsilon x_\varepsilon - |x_\varepsilon|)_\varepsilon \in \mathcal{N}_{\mathrm{co}}$. Since all terms are continuous in ε and $x_\varepsilon = \mathrm{e}^{\mathrm{i}\arg(x_\varepsilon)}|x_\varepsilon|$, it is sufficient to consider

$$|a_\varepsilon - \mathrm{e}^{-\mathrm{i}\arg(x_\varepsilon)}|\,|x_\varepsilon| \tag{4.6}$$

in the following cases (we assume that $\varepsilon \in (1/(m+1), 1/m]$ throughout).

(i) $|x_\varepsilon| < \varepsilon^{m+1}$: by (4.4) and (4.5), $a_\varepsilon = 0$, so (4.6) is equal to $1 \cdot |x_\varepsilon| < \varepsilon^{m+1}$.

(ii) $\varepsilon^{m+1} \leqslant |x_\varepsilon| < \varepsilon^m$: in this case

$$a_\varepsilon = \mathrm{e}^{-\mathrm{i}\arg(x_\varepsilon)} \left(1 - \frac{\varepsilon^{m+1}}{|x_\varepsilon|}\chi_{m+1}(\varepsilon) - \chi_m(\varepsilon) \right),$$

so (4.6) is less than or equal to

$$\left(\frac{\varepsilon^{m+1}}{|x_\varepsilon|} + 1 \right)|x_\varepsilon| < \varepsilon^{m+1} + \varepsilon^m.$$

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(iii) $|x_\varepsilon| \geqslant \varepsilon^m$: here,

$$a_\varepsilon = \mathrm{e}^{-\mathrm{i}\arg(x_\varepsilon)}\left(1 - \frac{\varepsilon^{m+1}}{|x_\varepsilon|}\chi_{m+1}(\varepsilon) - \frac{\varepsilon^m}{|x_\varepsilon|}\chi_m(\varepsilon)\right),$$

so as above (4.6) is less than or equal to $\varepsilon^{m+1} + \varepsilon^m$.

Summing up, we obtain for all $m \in \mathbb{N}$ that

$$|a_\varepsilon - \mathrm{e}^{-\mathrm{i}\arg(x_\varepsilon)}|\,|x_\varepsilon| < 2\varepsilon^m \quad \text{for } \varepsilon \leqslant \frac{1}{m}. \tag{4.7}$$

Thus, $(a_\varepsilon x_\varepsilon - |x_\varepsilon|)_\varepsilon \in \mathcal{N}_{\mathrm{co}}$, and hence $|x| = [(|x_\varepsilon|)_\varepsilon] \in J$. □

Proposition 4.25. *All ideals in $\tilde{\mathbb{R}}_{\mathrm{sm}}$ are absolutely convex.*

Proof. By Proposition 4.23, all ideals in $\tilde{\mathbb{R}}_{\mathrm{sm}}$ are convex. According to [**13**, Theorem 5.3], a convex ideal $J \trianglelefteq \tilde{\mathbb{R}}_{\mathrm{sm}}$ is absolutely convex if and only if $x \in J$ implies that $|x| \in J$. This is Lemma 4.24. □

Moreover, we can deduce from Lemma 4.24 that all finitely generated ideals in $\tilde{\mathbb{R}}_{\mathrm{sm}}$ and $\tilde{\mathbb{C}}_{\mathrm{sm}}$ are in fact principal ideals.

Proposition 4.26. *Let $r, s \in \tilde{\mathbb{K}}_{\mathrm{sm}}$. Then*

(i) $r\tilde{\mathbb{K}}_{\mathrm{sm}} + s\tilde{\mathbb{K}}_{\mathrm{sm}} = (|r| + |s|)\tilde{\mathbb{K}}_{\mathrm{sm}} = (|r| \vee |s|)\tilde{\mathbb{K}}_{\mathrm{sm}}$,

(ii) $r\tilde{\mathbb{K}}_{\mathrm{sm}} \cap s\tilde{\mathbb{K}}_{\mathrm{sm}} = (|r| \wedge |s|)\tilde{\mathbb{K}}_{\mathrm{sm}}$.

Proof. Both statements can be proved along the same lines as the corresponding ones for $\tilde{\mathbb{K}}$ in [**29**, Lemma 3.1]: $\tilde{\mathbb{R}}_{\mathrm{sm}}$ is an f-ring (by Proposition 4.15), and all ideals are absolutely convex (by Proposition 4.25). Thus, (i) and (ii) follow from [**5**, Proposition 8.2.8] and [**5**, Proposition 9.1.8], respectively. The results can be transferred to ideals in $\tilde{\mathbb{C}}_{\mathrm{sm}}$ by using the bijective correspondence between ideals in $\tilde{\mathbb{C}}_{\mathrm{sm}}$ and $\tilde{\mathbb{R}}_{\mathrm{sm}}$ (analogous to [**29**, § 2.4]). □

Furthermore, we can characterize powers and radicals of ideals in $\tilde{\mathbb{K}}_{\mathrm{sm}}$. In what follows, $\langle A\rangle$ denotes the ideal generated by A.

Lemma 4.27. *Let $J \trianglelefteq \tilde{\mathbb{K}}_{\mathrm{sm}}$ and $m \in \mathbb{N}$. Then the following properties hold.*

(i) $J^m = \{r \in \tilde{\mathbb{K}}_{\mathrm{sm}} \colon \sqrt[m]{|r|} \in J\}$.

(ii) *Let $L \trianglelefteq \tilde{\mathbb{K}}_{\mathrm{sm}}$ and $L^m \subseteq J^m$. Then $L \subseteq J$. In particular, if $r \in \tilde{\mathbb{K}}_{\mathrm{sm}}$ and $r^m \in J^m$, then $r \in J$.*

(iii) $\sqrt{J} = \langle \sqrt[n]{|r|} \colon n \in \mathbb{N}, r \in J\rangle$, *and in particular, for $s \in \tilde{\mathbb{K}}_{\mathrm{sm}}$,*

$$\sqrt{s\tilde{\mathbb{K}}_{\mathrm{sm}}} = \langle \sqrt[n]{|s|} \colon n \in \mathbb{N}\rangle.$$

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Proof. The extraction of roots is a continuous function, and hence is an inner operation in $\tilde{\mathbb{K}}_{\mathrm{co}}$ and therefore $\tilde{\mathbb{K}}_{\mathrm{sm}}$. Thus, the proof is identical to the case of arbitrary parametrization by making use of [**5**, Proposition 8.2.11]. See [**29**, Lemma 3.2] for details. □

The idempotent ideals are exactly the radical ideals.

Proposition 4.28. *Let $J \trianglelefteq \tilde{\mathbb{K}}_{\mathrm{sm}}$. The following are equivalent:*

(i) *J is idempotent;*

(ii) *J is radical;*

(iii) *for all $r \in J$, $\sqrt{|r|} \in J$;*

(iv) *J is an intersection of prime ideals.*

Proof. This is identical to that of [**29**, Proposition 3.3]. □

The next result shows, in particular, that the sum and the intersection of a family of radical ideals is again radical (see (i) and (iv)).

Proposition 4.29. *Let $J_\lambda \trianglelefteq \tilde{\mathbb{K}}_{\mathrm{sm}}$ for all $\lambda \in \Lambda$. Then we have the following.*

(i) $\sqrt{\sum_{\lambda\in\Lambda} J_\lambda} = \sum_{\lambda\in\Lambda} \sqrt{J_\lambda}$.

(ii) *Let $I, J \trianglelefteq \tilde{\mathbb{K}}_{\mathrm{sm}}$. Then $\sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J}$.*

(iii) *Let $J \trianglelefteq \tilde{\mathbb{K}}_{\mathrm{sm}}$. Then*

$$J^{\surd} := \bigcap_{n\in\mathbb{N}} J^n = \{r \in \tilde{\mathbb{K}}_{\mathrm{sm}} \mid \forall n \in \mathbb{N}\colon\ \sqrt[n]{|r|} \in J\} = \{r \in \tilde{\mathbb{K}}_{\mathrm{sm}} \mid \sqrt{r\tilde{\mathbb{K}}_{\mathrm{sm}}} \subseteq J\}$$

is the largest radical ideal that is contained in J. J is radical if and only if $J = J^{\surd}$.

(iv) $\bigcap_{\lambda\in\Lambda} J_\lambda^{\surd} = \Big(\bigcap_{\lambda\in\Lambda} J_\lambda\Big)^{\surd}$.

Proof. Based on the above results, this is analogous to [**29**, Proposition 3.4]. □

Remark 4.30. We have seen that many characterizations of ideals in $\tilde{\mathbb{K}}_{\mathrm{sm}}$ can be carried over from $\tilde{\mathbb{K}}$. The characterization of prime ideals, however, relies heavily on the structure of $\tilde{\mathbb{K}}$ and makes use of the idempotents therein [**29**, Theorems 3.5, 3.6]. Thus, a characterization of prime ideals in $\tilde{\mathbb{K}}_{\mathrm{sm}}$ will have to proceed along different lines.

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