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On a problem of Kurt Mahler concerning binomial coefficents

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Recently Kurt Mahler asked: for which natural numbers N is the least common multiple of all the binomial coefficients $\binom{N}{k}$ the product of the primes less than or equal to N ?

We obtain a formula for the least common multiple of all the binomial coefficients of any natural number N and hence show that $2, 11$, and 23 are the only solutions to Mahler's problem.

Write LCM_N for the least common multiple of the binomial coefficients $\binom{N}{k}$ of N, $k = 0, \ldots, N$.

LEMMA. *Let* N *be any natural number and* $p_i^{r_i}$ *be a prime power such that* $p_i^{r_i} \le N+1 < p_i^{r_i+1}$. *Then*

$$\text{LCM}_N = \left(\prod_i p_i^{r_i}\right)/(N+1)$$

where the product is over all primes $p_i \le N+1$.

Proof. Let q be any prime power less than or equal to $N + 1$.

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Then q divides one and only one of

$$N+1,\ N,\ N-1,\ \ldots,\ N+2-q\ .$$

Hence q divides the numerator and not the denominator of

$$\text{(1)}\qquad \frac{(N+1)N(N-1)\ldots(N+2-q)}{1.2.3\ldots(q-1)} = \binom{N}{q-1}(N+1)\ .$$

So if $q = p_i^{r_i}$, where $p_i^{r_i} \le N+1 < p_i^{r_i+1}$, then $p_i^{r_i} \mid (N+1)\binom{N}{q-1}$. But this is true for any prime power $p_i^{r_i}$. Hence

$$\prod_i p_i^{r_i} \mid (N+1)\mathrm{LCM}_N\ .$$

It remains to show that if $p \mid q = p^a$, then no higher power of p than p^a divides the left-hand side of (1). This could only happen if some power of p, say $p^b < p^a$ occurred more times in the numerator than in the denominator; this clearly cannot happen, as p^b divides exactly $p^{a-b} - 1$ other terms in both the numerator and denominator.

Hence we have the result.

THEOREM. *Let* N *be a natural number. Suppose the least common multiple of the binomial coefficients of* N *is the product of the primes less than or equal to* N . *Then* N *is* 2, 11 , *or* 23 .

Proof. Write $\mathrm{LCM}_N = \prod_i p_i$ for the least common multiple of the primes less than or equal to N . Now using the previous lemma,

$$\mathrm{LCM}_N = \left(\prod_j p_j^{r_j}\right)/(N+1)$$

where p_j runs over all primes less than or equal to $N+1$, and where $p_j^{r_j} \le N+1 < p_j^{r_j+1}$. Hence, unless $N+1$ is a prime or prime power,

$$\text{(2)}\qquad N = \prod_j p_j^{r_j-1} - 1\ .$$

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Clearly $r_j = 1$ for any prime greater than $\sqrt{N+1}$, and so such a prime does not appear in (2). Similarly for any prime less than or equal to $\sqrt{N+1}$ the condition $p_j^{r_j} \le N+1 < p_j^{r_j+1}$ ensures that $r_j \ge 2$, and so it does appear in (2). Thus every prime less than $\sqrt{N+1}$ is a factor of $N + 1$ and not of N. Hence N is a prime.

Therefore we consider the primes N of the form

$$N = \prod p_j^{r_j-1} - 1$$

in which every prime $p_j \le \sqrt{N+1}$ occurs and for which $p_j^{r_j} \le N+1 < p_j^{r_j+1}$

First we show $N + 1$ cannot be divisible by 3 to a power greater than or equal to 2. Suppose the contrary; then

$$N = 2^a 3^{2+b} k - 1 , \quad a > 0 , \quad b \ge 0 ,$$

satisfies all the assumptions. Thus $3^{3+b} \le N+1 < 3^{4+b}$ and $a \ge 3$ (since $2^{a+1} \le N+1 < 2^{a+2}$). But this means

$$3^{3+b} \le 2^a 3^{2+b} k < 3^{4+b} ,$$

$$3 \le 2^a k < 3^2$$

which can only be satisfied for $a = 3$, $k = 1$. In this case $N = 2^3 3^{2+b} - 1$, $2^{a+1} = 16 \le N+1 < 2^{a+2} = 32$ and so $b < 0$, which is a contradiction.

Hence the power of 3 is 0 or 1. Now consider $b = 0$; then

$$N = 2^a - 1$$

and $3 > \sqrt{N+1}$; so $N = 7$ or 3 which clearly do not satisfy the assumption that $2^{a+1} \le N+1 < 2^{a+2}$.

It remains to consider natural numbers $N = 2^a 3 - 1$. We must have

$$3^2 \le N+1 < 3^3 , \quad 2^{a+1} \le N+1 < 2^{a+2} , \text{ and } 5 > \sqrt{N+1} .$$

Clearly $N = 11 = 2^2.3 - 1$ and $N = 23 = 2^3.3 - 1$ are the only natural numbers which satisfy all the assumptions, when $N + 1$ is not a prime or prime power.

If $N + 1$ is a prime power, $p_i^{r_i}$, then from the lemma, p_i will not divide LCM_N and so LCM_N is not a product of the primes less than or equal to N.

If $N + 1$ is prime, then N can only equal 2, as N must be prime using the same argument as above. Clearly $\mathrm{LCM}_2 = \frac{2.3}{2} = 2$ satisfies the assumptions.

Hence we have the result.

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