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The Hodge diamond of O'Grady's six-dimensional example

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Abstract

We realize O'Grady's six-dimensional example of an irreducible holomorphic symplectic (IHS) manifold as a quotient of an IHS manifold of $K3^{[3]}$ type by a birational involution, thereby computing its Hodge numbers.

1. Introduction

In this paper we present a new way of obtaining O'Grady's six-dimensional example of an irreducible holomorphic symplectic manifold and use this to compute its Hodge numbers. Further applications, such as the description of the movable cone or the answer to Torelli-type questions for this deformation class of irreducible holomorphic symplectic manifolds, will be the topic of a subsequent paper.

Recall that irreducible holomorphic symplectic (IHS) manifolds are simply connected compact Kähler manifolds that have a holomorphic symplectic form, unique up to scalars. They arise naturally as one of the three building blocks of manifolds with trivial first Chern class according to the Beauville–Bogomolov decomposition [Bog78, Bea83], the other two blocks being abelian varieties and Calabi–Yau manifolds. By definition, IHS manifolds are higher-dimensional generalizations of K3 surfaces; moreover, they have a canonically defined quadratic form on their integral second cohomology group, which allows one to speak of their periods and to develop their theory in a way which is analogous to the theory of K3 surfaces. The interested reader can see [Huy99] and [O'Gr12] for a general introduction on the topic.

Two deformation classes of IHS manifolds in every even dimension greater than two were introduced by Beauville in [Bea83]. They are the Hilbert scheme of n points on a K3 and the generalized Kummer variety of dimension $2n$ of an abelian surface (i.e. the Albanese fiber of the Hilbert scheme of $n+1$ points of the abelian surface). Elements of these two deformation classes have second Betti number equal to 23 and 7, respectively, and are referred to as IHS manifolds of $K3^{[n]}$ type and of generalized Kummer type, respectively. There are two more examples, found by O'Grady in [O'Gr99] and [O'Gr03], of dimensions ten and six, respectively, which are obtained from a symplectic resolution of certain singular moduli spaces of sheaves on a K3 surface and on an abelian surface, respectively. They are referred to as the exceptional examples of IHS, and their deformation classes are denoted by OG10, respectively OG6.

These exceptional examples have not been studied as much as IHSs of $K3^{[n]}$ type and of generalized Kummer type and their geometry is less well understood. Though their topological Euler characteristic is known (see [Rap04] and [Moz06]), even other basic invariants such as their Hodge numbers have not been computed yet. In the case of manifolds of $K3^{[n]}$ type or

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of generalized Kummer type, the Hodge numbers were computed by Göttsche in [Got90] and Göttsche and Sörgel in [GS93].

One of the main results of this paper is to realize O'Grady's six-dimensional example as a quotient of an IHS manifold of $K3^{[3]}$ type by a birational symplectic involution: we therefore relate this deformation class to the most studied deformation class of IHS manifolds and this allows us, by resolving the indeterminacy locus of the involution and by describing explicitly its fixed locus (which has codimension 2), to compute the Hodge numbers. The involution we use was first introduced in [Rap04] in order to compute the Beauville–Bogomolov form for IHS of type OG6 and then used in [MW17] to determine a special subgroup of the automorphisms group of such manifolds. The main result of the present paper is the following.

THEOREM 1.1. *Let $\widetilde{K}$ be an irreducible holomorphic symplectic of type OG6. The odd Betti numbers of $\widetilde{K}$ are zero, and its non-zero Hodge numbers are collected in the following table:*

$$\begin{array}{ccccccc} & & & H^{0,0}=1 & & & \\ & & H^{2,0}=1 & H^{1,1}=6 & H^{0,2}=1 & & \\ & H^{4,0}=1 & H^{3,1}=12 & H^{2,2}=173 & H^{1,3}=12 & H^{0,4}=1 & \\ H^{6,0}=1 & H^{5,1}=6 & H^{4,2}=173 & H^{3,3}=1144 & H^{2,4}=173 & H^{1,5}=6 & H^{0,6}=1 \\ & H^{6,2}=1 & H^{5,3}=12 & H^{4,4}=173 & H^{3,5}=12 & H^{2,6}=1 & \\ & & H^{6,4}=1 & H^{5,5}=6 & H^{4,6}=1 & & \\ & & & H^{6,6}=1. & & & \end{array}$$

As a corollary, we also get the Chern numbers of this sixfold, see Proposition 6.8 for details.

Let us outline the main ideas that go into the proof of our main result.

Recall that O'Grady's six-dimensional example is obtained as a symplectic resolution of a certain natural subvariety of a moduli space of sheaves on an abelian surface A. In order to describe how to obtain it as a 'quotient' of another IHS by a birational symplectic automorphism, we first need to introduce some notation.

Let X be a K3 or an abelian surface. Fix an effective Mukai vector[1] $v \in H^*_{\mathrm{alg}}(X,\mathbb{Z})$, with $v^2 \geqslant -2$, and let H be a sufficiently general ample line bundle on X. It is well known [Muk88, Yos07] that, if v is primitive, the moduli space $M_v(X,H)$ of H-stable sheaves on X with Mukai vector v is a smooth projective manifold of dimension v^2+2 and that, if $v^2 \geqslant 0$, it admits a holomorphic symplectic form. If X is a K3 surface, then $M_v(X,H)$ is an IHS variety of $K3^{[n]}$ type, for $n = v^2/2+1$. Whereas, if $X = A$ is an abelian surface and if $v^2 \geqslant 4$, there is a nontrivial Albanese variety and, in order to get an *irreducible* holomorphic symplectic manifold, one needs to consider a fiber

$$K_v(A,H) := \mathrm{alb}^{-1}(0) \tag{1.1}$$

of the Albanese morphism (which is isotrivial)

$$\mathrm{alb} : M_v(A,H) \to A \times A^\vee.$$

Recall that if $v^2 \geqslant 6$, $K_v(A,H)$ is deformation equivalent to the generalized Kummer variety $K^{[n]}(A) := \sum^{-1}(0)$, where $\sum : A^{[n+1]} \to A$ is the summation morphism.

If we consider an H-stable sheaf F with a primitive Mukai vector v_0, then for $m \geqslant 2$, the sheaf $F^{\oplus m}$ is strictly H-semistable. If we set $v = mv_0$ and assume $v_0^2 > 0$, this sheaf determines a singular point of the moduli space $M_v(X,H)$, whose smooth locus still carries a holomorphic

[1] Vector v is effective if it is the Mukai vector of a coherent sheaf on X.

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symplectic form. In [O'Gr99] and [O'Gr03], O'Grady considered the case of $v_0 = (1, 0, -1)$ and $m = 2$, and showed that the singular symplectic variety $M_v(X, H)$ admits a symplectic resolution $\widetilde{M}_v(X, H)$. For X K3, this resolution gives a 10-dimensional IHS manifold of type OG10. For $X = A$, fix $F_0 \in M_v(A, H)$ and denote by $K_v(A, H)$ the fiber over 0 of the isotrivial fibration

$$\begin{array}{rccc} \mathbf{a}_v : & M_v(A,H) & \longrightarrow & A \times A^\vee, \\ & F & \longmapsto & (\mathrm{Alb}(c_2(F)), \det(F) \otimes \det(F_0)^{-1}), \end{array} \tag{1.2}$$

where $\mathrm{Alb} : CH_0(A) \to A$ is the Albanese homomorphism. The proper transform $\widetilde{K}_v(A, H)$ of $K_v(A, H)$ in $\widetilde{M}_v(X, H)$ is smooth and the induced map

$$f_v : \widetilde{K}_v(A, H) \to K_v(A, H) \tag{1.3}$$

is a symplectic resolution. The gives the six-dimensional IHS $\widetilde{K}_v(A, H)$, whose deformation type is called OG6, that is the object of this paper.

Lehn and Sorger proved in [LS06] that for *any* primitive v_0, with $v_0^2 = 2$, the moduli space $M_{2v_0}(X, H)$ admits a symplectic resolution. Finally, Perego and the second named author [PR13] showed that for any choice of v_0, with $v_0^2 = 2$, on a K3 or abelian surface, the IHS manifolds that one gets are deformation equivalent to OG10 and OG6, respectively.

When A is a general principally polarized abelian surface and $M_v(X, H)$ parametrizes pure one-dimensional sheaves, the IHS manifold $\widetilde{K}_v(A, H)$ is the image of a degree-2 rational map whose domain is an IHS manifold of $\mathrm{K3}^{[3]}$ type as we now briefly sketch: this is the starting point of our proof.

Let us consider a principal polarization $\Theta \subset A$. The Mukai vector $v_0 = (0, \Theta, 1)$ satisfies $v_0^2 = 2$, and hence, if we set $v = 2v_0$, there is symplectic resolution $\widetilde{K}_v(A, H) \to K_v(A, H)$ that is deformation equivalent to OG6. There is a natural support morphism $K_v(A, H) \to |2\Theta| = \mathbb{P}^3$, realizing $K_v(A, H)$ as a Lagrangian fibration. By definition of $K_v(A, H)$, the fiber over a smooth curve $C \in |2\Theta|$ is the kernel of the natural morphism $\mathrm{Pic}^6(C) \to A$ (which is also the restriction of $\mathbf{a}_v$ to $\mathrm{Pic}^6(C) \subset M_v(A, H)$).

It is well known that the morphism associated to the linear system $|2\Theta|$ is the quotient morphism $A \to A/\pm 1 \subset \mathbb{P}^3$ onto the singular Kummer surface of A. Let $S \to A/\pm 1$ be the minimal resolution of A. It is well known that S, the Kummer surface of A, is a K3 surface. Note that S comes naturally equipped with the degree-4 nef line bundle D obtained by pulling back the hyperplane section of $A/\pm 1 \subset \mathbb{P}^3$. Consider the diagram

$$\begin{array}{ccc} \widetilde{A} & \xrightarrow{a} & S \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[q]{} & A/\pm 1 \end{array} \tag{1.4}$$

where $\widetilde{A}$ is the blow up of A at its 16 2-torsion points or, equivalently, the ramified cover of S along the exceptional curves $E_1, \ldots, E_{16}$ of p. Consider the moduli space $M_w(S)$ of sheaves on S with Mukai vector $w = (0, D, 1)$ that are stable with respect to a chosen, sufficiently general, polarization. This is an IHS manifold birational to the Hilbert cube of S and it has a natural morphism $M_w(S) \to |D| = \mathbb{P}^3$ realizing it as the relative compactified Jacobian of the linear system $|D|$ (also a Lagrangian fibration).

The morphisms in diagram (1.4) induce a rational map

$$\Phi : M_w(S) \dashrightarrow K_v(A, H) \subset M_v(A, H),$$

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which maps the generic $[F] \in M_w(S)$ to $[q^*p_*F] \in M_v(A,H)$ and is generically 2:1 onto its image. Since $M_w(S)$ is simply connected, the image of this map lies in a fiber of $\mathbf{a}_v$, giving a generically 2:1 map $\Phi : M_w(S) \dashrightarrow K_v(A,H)$.

The map Φ commutes with the Lagrangian fibrations on $M_w(S)$ and $K_v(A,H)$. If C' is a smooth curve in $|D|$, $\mathrm{Pic}^3(C')$ embeds into $M_w(S)$ as a fiber of the Lagrangian fibration. If $C \in |2\Theta|$ is the étale cover of C', analogously $\ker[\mathrm{Pic}^6(C) \to A]$ embeds into $K_v(A,H)$ as a fiber of the Lagrangian fibration. On these fibers, the map Φ restricts to the natural 2:1 pull back morphism $\mathrm{Pic}^3(C') \to \mathrm{Pic}^6(C)$, whose image is precisely $\ker[\mathrm{Pic}^6(C) \to A]$. Recall that $\sum_i E_i$ is divisible by 2 in $H^2(S,\mathbb{Z})$ and that the line bundle $\eta := \mathcal{O}_S(\frac{1}{2}\sum E_i)$ determines the double cover q. It follows that the involution on $M_w(S)$ corresponding to Φ is given by tensoring by η and $\widetilde{K}_v(A,H)$ is a birational model of the 'quotient' of $M_w(S)$ by the birational involution induced by tensorization by η.

In this paper, for any abelian surface A and for an effective Mukai vector $v = 2v_0$ with $v_0^2 = 2$ on A, we show that $\widetilde{K}_v(A,H)$ admits a rational double cover from an IHS manifold $\underline{Y}_v(A,H)$ of $\mathrm{K3}^{[3]}$ type. Recall that the singular locus $\Sigma_v \subset K_v(A,H)$ has codimension 2 and can be identified with $(A \times A^\vee)/\pm 1$ (for more details, see § 2). Following [O'Gr03], the symplectic resolution (1.3) can be obtained by two subsequent blow ups followed by a contraction: first, one blows up the singular locus of Σ_v, then one blows up the proper transform of Σ_v itself (which is smooth). These two operations produce a manifold $\widehat{K}_v(A,H)$ that has a holomorphic two form degenerating along the strict transform of the exceptional divisor of the first blow up; contracting this exceptional divisor finally gives the manifold $\widetilde{K}_v(A,H)$ that has a symplectic two form and a regular morphism $\widetilde{K}_v(A,H) \to K_v(A,H)$, which is, therefore, a symplectic resolution. The inverse image $\widetilde{\Sigma}_v$ of Σ_v in $\widetilde{K}_v(A,H)$ is an irreducible divisor, which is divisible by two in the integral cohomology by results of the second named author [Rap04]. We show that the associated ramified double cover is a projective variety birational to an IHS manifold of $\mathrm{K3}^{[3]}$ type, which we denote by $\underline{Y}_v(A,H)$ and which is equipped with a birational symplectic involution.

This enables us to reconstruct $\widetilde{K}(A,H)$ starting from $\underline{Y}_v(A,H)$, and its symplectic birational involution

$$\underline{\tau}_v : \underline{Y}_v(A,H) \dashrightarrow \underline{Y}_v(A,H).$$

More specifically, $\underline{Y}_v(A,H)$ contains 256 $\mathbb{P}^3$, the birational involution $\underline{\tau}_v$ is regular on the complement of these $\mathbb{P}^3$, and, moreover, this involution lifts to a regular involution on the blow up $\overline{Y}_v(A,H)$ of $\underline{Y}_v(A,H)$ along the 256 $\mathbb{P}^3$. The fixed locus $\overline{\Delta}_v$ of the induced involution on $\overline{Y}_v(A,H)$ is smooth and four-dimensional, hence the blow up $\widehat{Y}_v(A,H)$ of $\overline{Y}_v(A,H)$ along this fixed locus carries an involution $\widehat{\tau}_v$ admitting a smooth quotient $\widehat{Y}_v(A,H)/\widehat{\tau}_v$. This quotient is $\widehat{K}_v(A,H)$, and $\widehat{Y}_v(A,H)$ is its double cover branched over $\widehat{\Sigma}_v$. Finally, $\widehat{K}_v(A,H)$ is the blow up of $\widetilde{K}_v(A,H)$ along 256 smooth three-dimensional quadrics.

This construction allows one to relate the Hodge numbers of $\widetilde{K}_v(A,H)$ to the invariant Hodge numbers of $\underline{Y}_v(A,H)$. Finally, the invariant Hodge numbers of $\underline{Y}_v(A,H)$ may be determined by using monodromy results of Markman [Mar02]. In particular, the computation of the Betti numbers can be carried out more easily by observing that $\overline{\Delta}_v \subset \overline{Y}_v(A,H)$ is a smooth submanifold with vanishing rational odd cohomology. As the odd Betti numbers of $\underline{Y}_v(A,H)$ are zero, the same holds for the odd Betti numbers of $\widehat{Y}_v(A,H)$. Since the rational cohomology of $\widetilde{K}_v(A,H)$ injects into the rational cohomology of $\widehat{Y}_v(A,H)$, the odd Betti numbers of $\widetilde{K}_v(A,H)$ are zero too. Since the Betti numbers b_0 and b_2 are known, it remains to determine b_4 and b_6: this can be done by using the knowledge of the Euler characteristic and Salomon's universal relation on Betti numbers of IHS manifolds (see Proposition 6.1).

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We should point out that this construction *cannot* be carried out for IHS manifolds of type OG10, since the exceptional divisor of the second blow up (the procedure to obtain the symplectic resolution is the same) is *not* divisible by 2 in the integral cohomology (see [Rap08] for the proof).

Remark 1.2. Recently, there has been considerable interest in exhibiting and classifying symplectic automorphisms of IHS manifolds [BS12, HM14, Men15, Mon16]. Note that quotients of IHS by symplectic automorphisms rarely admit a symplectic resolution since for this to happen the fixed locus has to be of codimension 2 (see [Kaw09] for one of the few cases where this happens). Our construction, however, indicates that 'quotients' by *birational* symplectic automorphisms can have a symplectic resolution, and thus they are potentially interesting. In upcoming work, we will study some of these birational morphisms for manifolds of $\mathrm{K3}^{[n]}$ type.

The structure of the paper is as follows. In § 2, we recall local and global properties of O'Grady's and Lehn–Sorger symplectic resolution. In § 3, we construct an affine double of the Lehn–Sorger local model of the deepest stratum of the singularity of $K_v(A,H)$, branched over the singular locus. In § 4, we globalize the previous results to construct global double covers Y_v of $K_v(A,H)$ branched over the singular locus. In § 5, we prove that Y_v is birational to an IHS manifold of $\mathrm{K3}^{[3]}$ type. Finally, in § 6, we use the previous results to compute the Hodge numbers.

Notation

For a closed embedding $X_1 \subset X_2$ of algebraic varieties we denote with $Bl_{X_1}X_2$ the blow up of X_2 along X_1.

For a vector bundle $X_3 \to Y$, we denote by $\mathbb{P}(X_3) \to Y$ its projectification.

For an affine cone X_4 we denote by $\mathbb{P}(X_4)$ its projectification.

Finally we denote by $H^k(X_1)$ the kth singular cohomology group of X_1 with rational coefficient and by $h^k(X_1)$ its dimension.

2. The resolution

Let us fix a primitive Mukai vector $v_0 \in H^*_{\mathrm{alg}}(A,\mathbb{Z})$ with $v_0^2 = 2$, set $v = 2v_0$, and consider a v-generic ample line bundle H on A (see [PR13, § 2.1]). By [LS06, Théorème 1.1] the projective variety $K_v := K_v(A,H)$ admits a symplectic resolution $\widetilde{K}_v$ which is deformation equivalent to O'Grady's six-dimensional example by [PR13, Theorem 1.6(2)]. In this section we recall the description of the singularities of K_v and of the symplectic resolution $f : \widetilde{K}_v \to K_v$ following both the papers of O'Grady [O'Gr99, O'Gr99] and Lehn and Sorger [LS06].

Since the singular locus Σ_v of K_v parametrizes polystable sheaves of the form $F_1 \oplus F_2$, with $F_i \in M_{v_0}(X,H)$, we have $\Sigma_v = K_v \cap \mathrm{Sym}^2 M_{v_0}(A,H)$. Since $v_0^2 = 2$ the smooth moduli space M_{v_0} is isomorphic to $A \times A^\vee$ and, as the Albanese map alb is an isotrivial fibration, the singular locus Σ_v is isomorphic to $(A \times A^\vee)/\pm 1$. This also implies that the singular locus Ω_v of Σ_v consists of 256 points representing sheaves of the form $F^{\oplus 2}$ with $F \in M_{v_0}(X,H)$.

The analytic type of the singularities appearing in K_v is completely understood. If $p \in \Sigma_v \backslash \Omega_v$, i.e. p represents a polystable sheaf of the form $F_1 \oplus F_2$ where $F_1 \neq F_2$, there exists a neighborhood $U \subset K_v$ of p, in the classical topology, biholomorphic to a neighborhood of the origin in the hypersurface defined in $\mathbb{A}^7$ by the equation $\sum_{i=1}^3 x_i^2 = 0$ (see for example [AS18, Proposition 4.4] or [O'Gr99, Proposition 1.4.1]), i.e. K_v has an A_1 singularity along $\Sigma_v \backslash \Omega_v$.

If $p \in \Omega_v$, the description of the analytic type of the singularity of K_v at p is due to Lehn and Sorger and it is contained in [LS06, Théorème 4.5]. To recall this description, let V be a four-dimensional vector space, let σ be a symplectic form on V, and let $\mathfrak{sp}(V)$ be the symplectic

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Lie algebra of (V,σ), i.e. the Lie algebra of the Lie group of the automorphisms of V preserving the symplectic form σ.

We let

$$Z := \{A \in \mathfrak{sp}(V) \,|\, A^2 = 0\}$$

be the subvariety of matrices in $\mathfrak{sp}(V)$ having square zero. It is known that Z is the closure of the nilpotent orbit of type $\mathfrak{o}(2,2)$, which parametrizes rank-2 square-zero matrices. Moreover, by Criterion 2 of [Hes79], Z is also a normal variety.

By [LS06, Théorèm 4.5], if $p \in \Omega_v$, there exists a Euclidean neighborhood of p in K_v, biholomorphic to a neighborhood of the origin in Z. Hence the local geometry of a symplectic desingularization of K_v is encoded in the local geometry of a symplectic desingularization of Z.

Let Σ be the singular locus of Z and let Ω be the singular locus of Σ. Let us recall that $\dim Z = 6$, $\dim \Sigma = 4$, $\dim \Omega = 0$ and, more precisely,

$$\Sigma = \{A \in Z \,|\, \operatorname{rk} A \leqslant 1\} \quad \text{and} \quad \Omega = \{0\}.$$

Let $G \subset \mathrm{Gr}(2,V) \subset \mathbb{P}(\wedge^2 V)$ be the Grassmannian of Lagrangian subspaces of V, note that G is a smooth three-dimensional quadric and set

$$\widetilde{Z} := \{(A,U) \,|\, A(U) = 0\} \subset Z \times G.$$

The restriction $\pi_G : \widetilde{Z} \to G$ of the second projection of $Z \times G$ makes $\widetilde{Z}$ the total space of a three-dimensional vector bundle, the cotangent bundle of G. In particular, $\widetilde{Z}$ is smooth and the restriction

$$f : \widetilde{Z} \to Z$$

of the first projection of $Z \times G$, which is an isomorphism when restricted to the locus of rank-2 matrices, is a resolution of the singularities. The fiber $f^{-1}(A)$, over a point $A \in \Sigma$, is the $\mathbb{P}^1$ parametrizing Lagrangian subspaces contained in the three-dimensional kernel of A and the central fiber $f^{-1}(0)$ is the whole G. As Z has an A_1 singularity along $\Sigma \backslash \Omega$ and G has dimension 3, it follows that $f : \widetilde{Z} \to Z$ is a symplectic resolution.

Remark 2.1. Let $\mathcal{U} \subset V \otimes \mathcal{O}_G$ be the rank-2 tautological bundle. The smooth symplectic variety $\widetilde{Z}$ is isomorphic to the total space $\mathbf{Sym}^2_G \mathcal{U}$ of the second symmetric power $\mathrm{Sym}^2_G \mathcal{U}$ of $\mathcal{U}$. In fact, an endomorphism $A \in \mathfrak{gl}(V)$ belongs to Z if and only if the following conditions hold:

(1) $A^2 = 0$;
(2) $\sigma(Av_1, v_2) = \sigma(Av_2, v_1)$ for any $v_1, v_2 \in V$.

By (2) the kernel $\ker A$ and the image $\operatorname{Im} A$ of A are orthogonal with respect to σ. Hence, for $(A,U) \in \widetilde{Z}$, we have $V \to \operatorname{Im} A \subset U \subset \ker A \subset V$. Since $U \subset V$ is Lagrangian we have $V/U \cong U^\vee$, so A has a factorization of the form $V \twoheadrightarrow U^\vee \to U \hookrightarrow V$. Moreover the induced linear map $\varphi_A \in \operatorname{Hom}(U^\vee, U) = U \otimes U$ defines a bilinear form on $U^\vee$ that is symmetric if and only if (2) holds.

Remark 2.2. Set

$$\widetilde{\Sigma} := \{(A,U) \in \widetilde{Z} \,|\, \operatorname{rank}(A) \leqslant 1\}.$$

The variety $\widetilde{\Sigma}$ is the exceptional locus of f. It is a locally trivial bundle over G with fiber the affine cone over a conic in $\mathbb{P}^2$. Using the isomorphism $\widetilde{Z} = \mathbf{Sym}^2_G \mathcal{U}$, the variety $\widetilde{\Sigma}$ is identified with the locus parametrizing singular symmetric bilinear forms on the fibers of the dual of the

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tautological rank-2 vector bundle $\mathcal{U}$. In particular, $\widetilde{\Sigma}$ is a fibration over G in cones over a smooth conic, i.e. $\widetilde{\Sigma}$ is singular only along the zero-section $\widetilde{\Omega} \simeq G$ of $\mathbf{Sym}^2_G\mathcal{U}$ and it has an A_1 singularity along it.

The following theorem due to Lehn and Sorger [LS06] gives an intrinsic reformulation of the symplectic desingularization $f : \widetilde{Z} \to Z$.

Theorem 2.3 [LS06]. *Let $p \in \Omega$ be a singular point of the singular locus $\Sigma \subset K_v$. Then,*

(a) *[LS06, Théorème 4.5] there is a local analytic isomorphism*

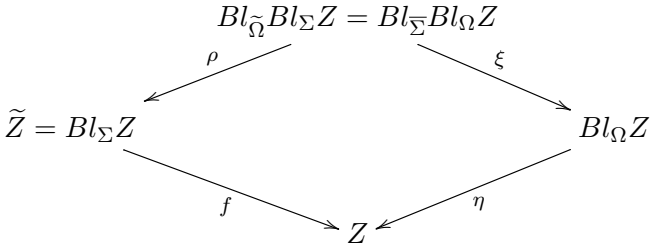
$$(Z, 0) \overset{\text{loc}}{\cong} (K_v, p);$$

(b) *[LS06, Théorème 3.1] the resolution $f : \widetilde{Z} \to Z$, defined above, coincides with the blow up of Z along its singular locus Σ.*

In order to discuss the topology of the symplectic desingularization of $f_v : \widetilde{K}_v \to K_v$, we are going to describe f in terms of blow ups along smooth subvarieties.

Proposition 2.4. *Let $\overline{\Sigma}$ be the strict transform of Σ in $Bl_\Omega Z$.*

(1) *$\overline{\Sigma}$ is the singular locus of Bl_Ω, $\overline{\Sigma}$ is smooth and $Bl_\Omega Z$ has an A_1 singularity along $\overline{\Sigma}$.*
(2) *The varieties $Bl_{\widetilde{\Omega}}Bl_\Sigma Z$ and $Bl_{\overline{\Sigma}}Bl_\Omega Z$ are smooth and isomorphic over Z. In particular, the diagram*



where the arrows are blow up maps, is commutative.

Proof. (1) Let $\mathbb{P}(Z) := Z/\mathbb{C}^*$ be the projectivization of the affine cone Z. As Z is a cone, its blow up $Bl_\Omega Z$ at the origin is the total space of the tautological line bundle over $\mathbb{P}(Z)$. The singular locus Σ of Z is a subcone, hence its strict transform $\overline{\Sigma} = Bl_\Omega\Sigma$ is the total space of the restriction to $\mathbb{P}(\Sigma) \subset \mathbb{P}(Z)$ of the tautological line bundle. As $\Sigma\backslash\{0\}$ is smooth, $\mathbb{P}(\Sigma)$ is smooth. Moreover, since Z has an A_1 singularity along $\Sigma\backslash\{0\}$, the singular locus of $\mathbb{P}(Z)$ is $\mathbb{P}(\Sigma)$, and $\mathbb{P}(Z)$ has an A_1 singularity along $\mathbb{P}(\Sigma)$. Passing to the total spaces of the tautological line bundles we get item (1).

(2) We only need to show that $Bl_{\widetilde{\Omega}}Bl_\Sigma Z$ and $Bl_{\overline{\Sigma}}Bl_0 Z$ are isomorphic. By Remark 2.1, $\widetilde{Z}$ is isomorphic to $\mathbf{Sym}^2_G\mathcal{U}$ and, by Remark 2.2, $Bl_{\widetilde{\Omega}}Bl_\Sigma Z$ is the blow up of $\mathbf{Sym}^2_G\mathcal{U}$ along its zero section. Letting $\mathbb{P}(\mathrm{Sym}^2_G\mathcal{U})$ be the projective bundle associated to $\mathrm{Sym}^2_G\mathcal{U}$, the blow up $Bl_{\widetilde{\Omega}}Bl_\Sigma Z$ is isomorphic to the total space $T \subset \mathbb{P}(\mathrm{Sym}^2_G\mathcal{U}) \times_G \mathbf{Sym}^2_G\mathcal{U}$ of the tautological line bundle of the projective bundle $\mathbb{P}(\mathrm{Sym}^2_G\mathcal{U})$. The isomorphism

$$\mathbf{Sym}^2_G\mathcal{U} = \widetilde{Z} := \{(A, U) \,|\, A(U) = 0\} \subset Z \times G$$

also implies

$$\mathbb{P}(\mathrm{Sym}^2_G\mathcal{U}) = \{([A], U) \,|\, A(U) = 0\} \subset \mathbb{P}(Z) \times G$$

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and, using this identification, we conclude that

$$Bl_{\widetilde{\Omega}}Bl_{\Sigma}Z = T = \{([A], B, U) \mid A(U) = 0,\ B \in [A]\} \subset \mathbb{P}(Z) \times Z \times G.$$

On the other side of the diagram, as Z is a cone, its blow up at the origin can explicitly be given as

$$Bl_{\Omega}Z = \{([A], B) \mid B \in [A]\} \subset \mathbb{P}(Z) \times Z.$$

It remains to show that the map $\xi : Bl_{\widetilde{\Omega}}Bl_{\Sigma}Z \to Bl_{\Omega}Z$ induced by the projection $\pi_{1,2} : \mathbb{P}(Z) \times Z \times G \to \mathbb{P}(Z) \times Z$ is the blow up of $\mathbb{P}(Z) \times Z$ along $\overline{\Sigma}$. Since for $q \in \overline{\Sigma}$ the schematic fiber $\xi^{-1}(q)$ is isomorphic to $\mathbb{P}^1$ and $\overline{\Sigma}$ is smooth, the schematic inverse image $\xi^{-1}(\overline{\Sigma})$ is a smooth, hence reduced and irreducible Cartier divisor. By the universal property of blow ups, ξ factors through a proper map $\iota : Bl_{\widetilde{\Omega}}Bl_{\Sigma}Z \to Bl_{\overline{\Sigma}}Bl_{\Omega}Z$ sending $\xi^{-1}(\overline{\Sigma})$ surjectively onto the exceptional divisor of the blow up of $Bl_{\Omega}Z$ along $\overline{\Sigma}$. Finally, since $Bl_{\Omega}Z$ is only singular along $\overline{\Sigma}$ and has an A_1 singularity along $\overline{\Sigma}$, the blow up $Bl_{\overline{\Sigma}}Bl_{\Omega}Z$ is smooth. It follows that ι is a proper birational map between smooth varieties that does not contract any divisor, therefore ι is a isomorphism. □

This proposition also allows us to describe the exceptional loci of the blow up maps appearing in item (2).

Let $\widehat{\Sigma} \subset Bl_{\widetilde{\Omega}}Bl_{\Sigma}Z$ be the exceptional divisor of ξ, let $\widehat{\Omega} \subset Bl_{\widetilde{\Omega}}Bl_{\Sigma}Z$ be the exceptional divisor of ρ, and recall that $\widetilde{\Omega} \cong G$ is the inverse image of Ω under the resolution f.

COROLLARY 2.5.

(1) *$\widehat{\Sigma}$ is a $\mathbb{P}^1$-bundle over $\overline{\Sigma}$ and $\widehat{\Sigma} = Bl_{\widetilde{\Omega}}\widetilde{\Sigma}$.*

(2) *$\widehat{\Omega}$ is a $\mathbb{P}^2$-bundle over $\widetilde{\Omega}$ isomorphic to $\mathbb{P}(\mathrm{Sym}^2_G\mathcal{U})$.*

Proof. (1) By item (1) of Proposition 2.4, the restriction of ξ realizes $\widehat{\Sigma}$ as a $\mathbb{P}^1$-bundle over $\overline{\Sigma}$. Since $\widehat{\Sigma}$ is also the strict transform of $\widetilde{\Sigma}$ under ρ and $\widetilde{\Omega} \subset \widetilde{\Sigma}$, the restriction of ρ to $\widehat{\Sigma}$ can be identified with the blow up map of $\widetilde{\Sigma}$ along $\widetilde{\Omega}$. As for (2), we can argue as follows. Since $\widetilde{\Omega}$ is a smooth subvariety of codimension 3 in the smooth variety $\widetilde{Z}$, the restriction of ρ to $\widehat{\Omega}$ makes it a $\mathbb{P}^2$-bundle over $\widetilde{\Omega}$. More precisely, since $\widetilde{\Omega}$ is the zero section of $\widetilde{Z} = \mathbf{Sym}^2_G\mathcal{U}$ (see Remark 2.2), there is an isomorphism $\widehat{\Omega} \simeq \mathbb{P}(\mathrm{Sym}^2_G\mathcal{U})$. □

To compute invariants of $\widetilde{K}_v$ we need the following global versions of Proposition 2.4 and Corollary 2.5.

PROPOSITION 2.6. *Let $\overline{\Sigma}_v$ be the strict transform of Σ_v in $Bl_{\Omega_v}K_v$.*

(1) *$\overline{\Sigma}_v$ is the singular locus of $Bl_{\Omega_v}K_v$, $\overline{\Sigma}_v$ is smooth and $Bl_{\Omega_{K_v}}K_v$ has an A_1 singularity along $\overline{\Sigma}_v$.*

(2) *The projective varieties $Bl_{\widetilde{\Omega}_v}Bl_{\Sigma_v}K_v$ and $Bl_{\overline{\Sigma}_v}Bl_{\Omega_v}K_v$ are smooth and isomorphic over K_v. Hence the diagram*

$$\begin{array}{ccccc} & & Bl_{\widetilde{\Omega}_v}Bl_{\Sigma_v}K_v = Bl_{\overline{\Sigma}_v}Bl_{\Omega_v}K_v & & \\ & \overset{\rho_v}{\swarrow} & & \overset{\xi_v}{\searrow} & \\ \widetilde{K}_v = Bl_{\Sigma_v}K_v & & & & Bl_{\Omega_v}K_v \\ & \underset{f_v}{\searrow} & & \underset{\eta_v}{\swarrow} & \\ & & K_v & & \end{array}$$

where the arrows are blow ups, is commutative.

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Proof. As $\Sigma_v \backslash \Omega_v$ is smooth and K_v has an A_1 singularity along $\Sigma_v \backslash \Omega_v$, item (1) follows from Theorem 2.3(a) and Proposition 2.4(1), since the blow up is a local construction. Item (2) holds since item (2) of Proposition 2.4 also implies that the natural birational map between $Bl_{\widetilde{\Omega}_v} Bl_{\Sigma_v} K_v$ and $Bl_{\overline{\Sigma}_v} Bl_{\Omega_v} K_v$ is actually an isomorphism. □

Remark 2.7. Since Σ_v contains Ω_v as a closed subscheme, its strict transform $\overline{\Sigma}_v$ in $Bl_{\Omega_v} K_v$ is isomorphic to the blow up $Bl_{\Omega_v} \Sigma_v$. Recall that $\Sigma_v \simeq (A \times A^\vee)/\pm 1$, so that its singular locus $\mathrm{Sing}((A \times A^\vee)/\pm 1)$ is in bijective correspondence with the set of 2-torsion points $(A \times A^\vee)[2]$ of $A \times A^\vee$. It follows that there is a chain of isomorphisms

$$\overline{\Sigma}_v \simeq Bl_{\mathrm{Sing}((A\times A^\vee)/\pm 1)}((A \times A^\vee)/\pm 1) \simeq (Bl_{(A\times A^\vee)[2]}(A \times A^\vee))/\pm 1.$$

This also implies that the exceptional divisor of $Bl_{\Omega_v} \Sigma_v$, which is given by the (reduced induced) intersection of the exceptional divisor $\overline{\Omega}_v$ of $Bl_{\Omega_v} K_v$ and $\overline{\Sigma}_v$, consists of a union of 256 disjoint $\mathbb{P}^3$.

Corollary 2.8. *Let $\widehat{\Sigma}_v \subset Bl_{\widetilde{\Omega}_v} Bl_{\Sigma_v} K_v$ be the exceptional divisor of ξ, let $\widehat{\Omega}_v \subset Bl_{\widetilde{\Omega}_v} Bl_{\Sigma_v} \widetilde{K}_v$ be the exceptional divisor of ρ, let $\overline{\Omega}_v \subset Bl_{\Omega_v} K_v$ be the exceptional divisor of η, and, finally, let $\overline{\Omega}_v \cap \overline{\Sigma}_v$ denote the intersection of $\overline{\Omega}_v$ and $\overline{\Sigma}_v$ with its reduced induced structure.*

(1) *$\widehat{\Sigma}_v$ is a $\mathbb{P}^1$-bundle over $\overline{\Sigma}_v$ and $\widehat{\Sigma}_v = Bl_{\widetilde{\Omega}_v} \widetilde{\Sigma}_v$.*

(2) *$\widehat{\Omega}_v$ is a $\mathbb{P}^2$-bundle over $\widetilde{\Omega}_v$ isomorphic to $\mathbb{P}(\mathrm{Sym}^2_G \mathcal{U})$.*

Proof. This follows from item (2) of Theorem 2.3 and Corollary 2.5. □

Remark 2.9. The proof of the existence of an isomorphism between the smooth projective varieties $Bl_{\widetilde{\Omega}_v} Bl_{\Sigma_v} K_v$ and $Bl_{\overline{\Sigma}_v} Bl_{\Omega_v} K_v$ follows the original strategy used by O'Grady in [O'Gr99]. For $v = (2, 0, -2)$, he proved that a symplectic desingularization of K_v can be obtained by contracting the strict transform $\widehat{\Omega}_v$ of $\overline{\Omega}_v$ in $Bl_{\overline{\Sigma}_v} Bl_{\Omega_v} K_v$. Proposition 2.6 shows, in particular, that O'Grady's procedure gives a symplectic desingularization of K_v that is isomorphic to the Lehn–Sorger desingularization $Bl_{\Sigma_v} K_v$. The proof of Proposition 2.6 is elementary because it uses the crucial description, due to Lehn and Sorger, of the analytic type of the singularities appearing in K_v.

3. The local covering

This section is devoted to the local description of the double cover, branched along the singular locus, of O'Grady's singularity.

It is known [CM93, Corollary 6.1.6] that the fundamental group of the open orbit $\mathfrak{o}(2,2)$ is isomorphic to $\mathbb{Z}/(2)$. We wish to extend this double cover to a ramified double cover of $\overline{\mathfrak{o}(2,2)} = Z$.

To this aim, let

$$W := \{v \otimes w \,|\, \sigma(v,w) = 0\} \subset V \otimes V \quad \text{and} \quad \Delta = \{v \otimes v\} \subset W$$

be the affine cone over the incidence subvariety

$$I := \{([v],[w]) \,|\, \sigma(v,w) = 0\} \subset \mathbb{P}V \times \mathbb{P}V \subset \mathbb{P}(V \otimes V).$$

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Finally denote by

$$\Gamma \subset W$$

the vertex (i.e. the origin) of W.

Since I is smooth, the vertex Γ is also the singular locus of W. Moreover, since $I \subset \mathbb{P}(V \otimes V)$ is projectively normal, W is a normal variety.

Let

$$\tau : W \to W$$

be the involution induced by restricting the linear involution $\tau_{V\otimes V}$ on $V \otimes V$ that interchanges the two factors.

The following lemma exhibits W as the desired double cover of Z.

LEMMA 3.1. *The morphism*

$$\begin{aligned} \varepsilon : W &\longrightarrow Z, \\ v \otimes w &\longmapsto \sigma(v,\cdot)w + \sigma(w,\cdot)v \end{aligned}$$

realizes Z as the quotient W/τ. In particular, ε is a finite 2:1 morphism, the ramification locus of ε is Δ and the branch locus of ε is Σ.

Proof. We leave it to the reader to check that $\varepsilon(W) \subset Z$. For a rank-2 endomorphism $A \in Z\backslash\Sigma$, let us show that $\varepsilon^{-1}(A)$ consists of two points interchanged by τ. Let $U \subset V$ be the kernel of A, which is a Lagrangian subspace. As shown in Remark 2.1, A induces a linear map $\varphi_A \in \mathrm{Hom}(U^\vee, U) = U \otimes U$ that gives a rank-2 bilinear symmetric form on $U^\vee$ and, conversely, any symmetric bilinear form on $U^\vee$ determines a rank-2 endomorphism $A \in Z$ whose kernel is U. A rank-2 symmetric bilinear form on $U^\vee$ is determined, up to scalars, by two independent distinct isotropic vectors L_1 and L_2, hence by their kernels $\ker(L_1) \subset U$ and $\ker(L_2) \subset U$. Now it suffices to note that, for v and w spanning U and for $A = \varepsilon(v \otimes w)$, the lines $\ker(L_1)$ and $\ker(L_2)$ are the lines generated by v and w.

Since $\ker A$ and $\mathrm{Im}\, A$ are orthogonal (see Remark 2.1), if $A \in \Sigma$ is a rank-1 endomorphism or the 0 endomorphism, then there exists a unique, up to scalars, $v \in V$ such that $A = \sigma(v,\cdot)v$. This shows that $\varepsilon^{-1}(A)$ consists of a unique point, which is fixed by τ.

To show that $Z \cong W/\tau$, note that ε is τ-invariant and its fibers are the orbits of the action of τ, hence ε induces a bijective morphism $W/\tau \to Z$. Since Z is normal (see Criterion 2 of [Hes79]), this morphism is an isomorphism. □

Remark 3.2. Using Lemma 3.1, we may reprove that the fundamental group of $\mathfrak{o}(2,2) = Z\backslash\Sigma$ is isomorphic to $\mathbb{Z}/(2)$. As ε is étale on $Z\backslash\Sigma$, it suffices to show that $\varepsilon^{-1}(Z\backslash\Sigma) = W\backslash\Delta$ is simply connected. One can obtain $W\backslash\Delta$ from the smooth variety $W\backslash\{0\}$ by removing a codimension-2 subvariety, hence there is an isomorphism of fundamental groups $\pi_1(W\backslash\Delta) \simeq \pi_1(W\backslash\{0\})$. Finally the map $k : W\backslash\{0\} \to \mathbb{P}(V)$ defined by $k(v \otimes w) = [v]$ is a locally trivial fibration with fiber isomorphic to the complement of 0 in a three-dimensional vector space. Therefore k has simply connected base and fiber and $\pi_1(W\backslash\{0\}) = 0$.

The morphism ε induces double coverings of the varieties $\widetilde{Z} = Bl_\Sigma Z$, $Bl_\Omega Z$ and $Bl_{\widetilde{\Omega}} Bl_\Sigma Z = Bl_{\widetilde{\Sigma}} Bl_\Omega$. The following corollary discusses the case of $Bl_\Omega Z$.

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Corollary 3.3. *The morphism ε lifts to a finite 2:1 morphism*

$$\overline{\varepsilon} : Bl_\Gamma W \to Bl_\Omega Z,$$

whose branch locus is the strict transform $\overline{\Sigma}$ of Σ in $Bl_\Omega Z$.

Proof. The morphism ε is the restriction to W of the linear map

$$\varepsilon_{V\otimes V} : V \otimes V \to \mathfrak{sp}(V)$$

sending $v \otimes w$ to $\sigma(v,\cdot)w + \sigma(w,\cdot)v$ for any $v \otimes w \subset V \otimes V$. As $\ker \varepsilon_{V\otimes V} \cap W = 0$, the map ε induces a morphism $\mathbb{P}(\varepsilon) : I \to \mathbb{P}(Z)$ between the projectivization of W and Z. There are the identifications

$$Bl_\Gamma W = Bl_0 W = \{(\mathbb{C}\alpha, v \otimes w) \in I \times W \,|\, v \otimes w \in \mathbb{C}\alpha\}$$

and

$$Bl_\Omega Z = Bl_0 Z = \{(\mathbb{C}A, B) \in \mathbb{P}(Z) \times Z \,|\, B \in \mathbb{C}A\}.$$

It follows that $\mathbb{P}(\varepsilon) \times \varepsilon$ restricts to a map $\overline{\varepsilon} : Bl_{\{0\}} W \to Bl_\Omega Z$ and, by Lemma 3.1, $\overline{\varepsilon}$ is a finite 2:1 map whose branch locus is

$$\overline{\Sigma}_W = Bl_\Omega \Sigma = Bl_0 \Sigma = \{(\mathbb{C}A, B) \in \mathbb{P}(\Sigma) \times \Sigma \,|\, B \in \mathbb{C}A\}. \qquad \square$$

Remark 3.4. Since W is the cone over a smooth variety, both its blow up at the origin $Bl_\Gamma W$ and the exceptional divisor $\overline{\Gamma} \subset Bl_\Gamma W$ are smooth. Finally the strict transform $\overline{\Delta}$ of Δ in $Bl_\Gamma W$ is isomorphic to $Bl_\Gamma \Delta$ and, since Δ is the cone over a smooth variety, also $\overline{\Delta}$ is smooth.

The following corollary treats the case of the induced double cover of $\widetilde{Z}$.

Corollary 3.5. *Let $\pi : SC_G \mathcal{U}^{\otimes 2} \to G$ be the relative affine Segre cone parametrizing decomposable tensors in the total space of the rank-4 vector bundle $\mathcal{U}^{\otimes 2}$.*

(1) *$SC_G \mathcal{U}^{\otimes 2}$ is isomorphic to $Bl_\Delta W$.*
(2) *Using this identification, the map*

$$\widetilde{\varepsilon} : Bl_\Delta W \ (= SC_G \mathcal{U}^{\otimes 2}) \to \widetilde{Z} \ (= \mathbf{Sym}^2_G \mathcal{U}),$$

induced by symmetrization on the fibers, is a finite 2:1 morphism lifting ε, whose branch locus is $\widetilde{\Sigma}$.

Proof. (1) By definition of fiber product, $W \times_Z \widetilde{Z}$ is equal to

$$\{(v \otimes w, A, U) \in W \times Z \times G \,|\, \varepsilon(v \otimes w) = A \text{ and } v, w \in U\}$$

and, by Lemma 3.1, the fiber over U of the projection $\pi_G : W \times_Z \widetilde{Z} \to G$ is naturally isomorphic to the variety $SC\,U^{\otimes 2}$ of decomposable tensors in $U \otimes U$. It follows that $SC_G \mathcal{U}^{\otimes 2}$ is isomorphic to $W \times_Z \widetilde{Z}$.

Let us show that $W \times_Z \widetilde{Z}$ has a birational morphism to $Bl_\Delta W$. Let $\pi_W : W \times_Z \widetilde{Z} \to W$ be the projection. By the universal property of blow ups, it will suffice to show that the schematic inverse image $\pi_W^{-1}(\Delta)$ is a Cartier divisor.

For $U \in G$, the projection π_W sends the fiber $\pi_G^{-1}(U)$ isomorphically onto $SC\,U^{\otimes 2}$. Hence the schematic intersection $\pi_W^{-1}(\Delta) \cap \pi_G^{-1}(U)$ is isomorphic to the schematic intersection $\Delta \cap SC\,U^{\otimes 2}$,

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i.e. the reduced cone over a smooth conic $C \subset \mathbb{P}(U \otimes U)$ parametrizing symmetric decomposable tensors in $U \otimes U$. As, on varying $U \in G$, the intersections $\pi_W^{-1}(\Delta) \cap \pi_G^{-1}(U)$ form a locally trivial family over G, the family $\pi_W^{-1}(\Delta) \to G$ is locally trivial. Finally, as the cone over $C \subset \mathbb{P}(U \otimes U)$ is a Cartier divisor in the variety of decomposable tensors of $U \otimes U$, the scheme $\pi_W^{-1}(\Delta)$ is a Cartier divisor in $W \times_Z \widetilde{Z}$.

On the other hand, $Bl_\Delta W$ has a regular birational morphism to $W \times_Z \widetilde{Z}$ inverting the previous birational morphism.

This will follow if we prove that the ideal of $\varepsilon^{-1}(\Sigma)$ in W is the square I_Δ^2 of the ideal of Δ. In fact, in this case, the blow up $Bl_\Delta W$ equals the blow up $Bl_{\varepsilon^{-1}(\Sigma)}W$; hence the schematic inverse image of Σ in $Bl_\Delta W$ is a Cartier divisor. Therefore, as $\widetilde{Z} = Bl_\Sigma Z$, by the universal property of blow ups, we can conclude that there exists a commutative diagram

$$\begin{array}{ccc} Bl_\Delta W & \xrightarrow{\xi'} & \widetilde{Z} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[\varepsilon]{} & Z \end{array}$$

inducing the desired birational regular morphism from $Bl_\Delta W$ to $W \times_Z \widetilde{Z}$.

To determine the ideals of $\varepsilon^{-1}(\Sigma)$ and Δ in W, we recall that the involution τ is the restriction of the linear involution $\tau_{V\otimes V}$ on $V \otimes V$ which can be interpreted as the transposition on 4×4 matrices if we choose a basis for V. Moreover, the ideal of Δ in W is generated by the restrictions of the linear antiinvariant functions on $V \otimes V$ (this already holds for the ideal of Δ in the affine cone over the Segre variety $\mathbb{P}(V) \times \mathbb{P}(V) \subset \mathbb{P}(V \otimes V)$). Hence, I_Δ^2 is generated by restrictions of products of pairs of linear antiinvariant functions on $V \otimes V$ and any such product comes from a function on the quotient $(V \otimes V)/\tau_{V\otimes V}$ vanishing along the branch locus B. Since B contains the branch locus Σ of W/τ, we conclude that the ideal of $\varepsilon^{-1}(\Sigma)$ contains I_Δ^2. Equality holds because $W\backslash\{0\}$ is smooth and the fixed locus $\Delta\backslash\{0\}$ has codimension 2, hence $\varepsilon^{-1}(\Sigma)$ equals the subscheme Δ_2 defined by I_Δ^2 outside the origin. As Δ_2 is a subcone of W, it is the closure of $\Delta_2\backslash\{0\}$, therefore it is a closed subscheme of $\varepsilon^{-1}(\Sigma)$.

(2) The existence of the regular morphism $\widetilde{\varepsilon}$ lifting ε follows from (1). The branch locus of $\tilde{\varepsilon}$ is $\widetilde{\Sigma}$ because, by our description of $\tilde{\varepsilon}$, it parametrizes singular bilinear symmetric tensors (see Remark 2.2). □

Corollary 3.5 also allows us to describe the singularities of the exceptional divisor $\widetilde{\Delta}$ of the blow up $Bl_\Delta W$ of W along Δ.

Remark 3.6. We have that $Bl_\Delta W \simeq SC_G\mathcal{U}^{\otimes 2}$ is a locally trivial bundle over G with fiber the affine cone over a smooth quadric in $\mathbb{P}^3$. Hence it is smooth outside the zero section $\widetilde{\Gamma}$ and any point of $\widetilde{\Gamma}$ has a neighborhood isomorphic to the product of the affine cone over a smooth quadric and a smooth three-dimensional variety. As ε and $\widetilde{\varepsilon}$ are finite, the morphism $\widetilde{\varepsilon}$ sends the exceptional divisor $\widetilde{\Delta} \subset Bl_\Delta W$ onto the exceptional divisor $\widetilde{\Sigma} \subset \widetilde{Z}$. By the definition of $\widetilde{\varepsilon}$ in item (2) of Corollary 3.5, the divisor $\widetilde{\Delta}$ parametrizes symmetric decomposable tensors in the fibers of $\pi_G : SC_G\mathcal{U}^{\otimes 2} \to G$, hence it is a locally trivial bundle with fiber the affine cone over a smooth conic. Therefore it is smooth outside $\widetilde{\Gamma}$ and has an A_1 singularity along $\widetilde{\Gamma}$.

The following corollary completes the picture of the double covering induced by ε in the local case.

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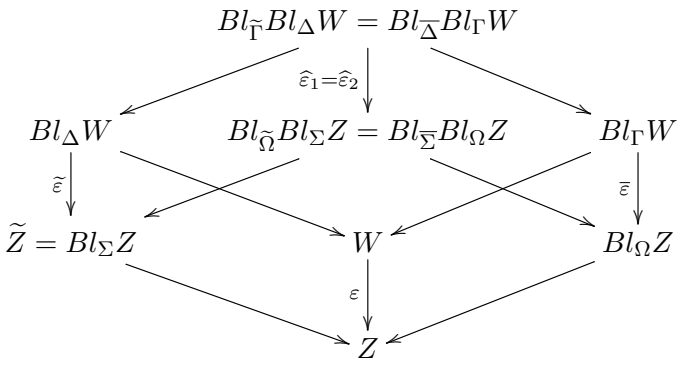
Corollary 3.7.

(1) *There exist finite degree-2 morphisms* $\widehat{\varepsilon}_1 : Bl_{\widetilde{\Gamma}}Bl_\Delta W \to Bl_{\widetilde{\Omega}}Bl_\Sigma Z$ *and* $\widehat{\varepsilon}_2 : Bl_{\overline{\Delta}}Bl_\Gamma W \to Bl_{\overline{\Sigma}}Bl_\Omega Z$, *lifting* $\widetilde{\varepsilon}$ *and* $\overline{\varepsilon}$, *whose branch loci are the strict transform of* $\widetilde{\Sigma}$ *in* $Bl_{\widetilde{\Omega}}Bl_\Sigma Z$ *and the exceptional divisor* $\widehat{\Sigma}$ *of* $Bl_{\overline{\Sigma}}Bl_\Omega Z$, *respectively.*

(2) *The varieties* $Bl_{\widetilde{\Gamma}}Bl_\Delta W$ *and* $Bl_{\overline{\Delta}}Bl_\Gamma W$ *are smooth and isomorphic over* W. *Hence there exists a commutative diagram*



where the diagonal arrows are blow ups.

Proof. (1) Recall that $Bl_\Delta W \simeq SC_G \mathcal{U}^{\otimes 2}$ and $Bl_\Sigma Z \simeq \mathbf{Sym}^2_G \mathcal{U}$ are locally trivial bundles over G, and $\widetilde{\Gamma}$ and $\widetilde{\Omega}$ are their respective zero sections. As $\widetilde{\varepsilon} : Bl_\Delta W \to Bl_\Sigma Z$ is a morphism over G, the existence of $\widehat{\varepsilon}_1 : Bl_{\widetilde{\Gamma}}Bl_\Delta W \to Bl_{\widetilde{\Omega}}Bl_\Sigma Z$ branched over the strict transform of $\widetilde{\Sigma}$ in $Bl_{\widetilde{\Omega}}Bl_\Sigma Z$ follows from the existence of a commutative diagram of the form

$$\begin{array}{ccc} Bl_0 SC\, U^{\otimes 2} & \longrightarrow & Bl_0 \operatorname{Sym}^2 U \\ \downarrow & & \downarrow \\ SC\, U^{\otimes 2} & \longrightarrow & \operatorname{Sym}^2 U \end{array}$$

where U is a two-dimensional vector space, $SC\, U^{\otimes 2} \subset U \otimes U$ is the affine cone, parametrizing decomposable tensors, over the Segre variety $\mathbb{P}(U) \times \mathbb{P}(U)$, the vertical arrows are blow ups, and the horizontal arrows are induced by symmetrization (hence their branch locus parametrizes singular symmetric tensors).

By Corollary 3.3, the branch locus of the finite 2:1 morphism $\overline{\varepsilon} : Bl_\Gamma W \to Bl_\Omega Z$ is the singular locus $\overline{\Sigma}$ of $Bl_\Omega Z$. By item (1) of Proposition 2.4, $Bl_\Omega Z$ has an A_1 singularity along $\overline{\Sigma}$ and this suffices to imply the existence of the desired finite 2:1 morphism $\widehat{\varepsilon}_2 : Bl_{\overline{\Delta}}Bl_\Gamma W \to Bl_{\overline{\Sigma}}Bl_\Omega Z$ whose branch locus is the exceptional divisor $\widehat{\Sigma}$ of $Bl_{\overline{\Sigma}}Bl_\Omega Z$.

(2) The smoothness of $Bl_{\widetilde{\Gamma}}Bl_\Delta W$ and $Bl_{\overline{\Delta}}Bl_\Gamma W$ follows from Remark 3.6 and Remark 3.4, respectively.

It remains to show that the natural birational map $j : Bl_{\widetilde{\Gamma}}Bl_\Delta W \dashrightarrow Bl_{\overline{\Delta}}Bl_\Gamma W$ extends to a biregular morphism. Using the identification $Bl_{\widetilde{\Omega}}Bl_\Sigma Z = Bl_{\overline{\Sigma}}Bl_\Omega$, $\widehat{\varepsilon}_1$ and $\widehat{\varepsilon}_2$ may be seen as finite covers of $Bl_{\widetilde{\Omega}}Bl_\Sigma Z$ and we have an equality of rational maps $\varepsilon_2 \circ j = \varepsilon_1$. It follows that the closure of the graph of j is contained in the fiber product $Bl_{\widetilde{\Gamma}}Bl_\Delta W \times_{Bl_{\widetilde{\Omega}}Bl_\Sigma Z} Bl_{\overline{\Delta}}Bl_\Gamma W$. As $\widehat{\varepsilon}_1$ and $\widehat{\varepsilon}_2$ are finite, the closure of the graph of j is finite and generically injective on the smooth factors $Bl_{\widetilde{\Gamma}}Bl_\Delta W$ and $Bl_{\overline{\Delta}}Bl_\Gamma W$. By Zariski's main theorem it is the graph of an isomorphism extending j.

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The commutativity of the diagram holds because all maps are regular and commutativity is trivial on open dense subsets. □

In the final remark of this section we discuss the behavior of the restriction of the morphisms appearing in the diagram in item (2) of Corollary 3.7 to the divisors appearing over Ω. Since this remark will not be used in the rest of the paper, some of the computations are left to the reader.

Remark 3.8. Let $\widehat{\Gamma}$ be the exceptional divisor of the blow up of $Bl_{\Delta}W$ along its singular locus $\widetilde{\Gamma}$ (see Remark 3.6). By restricting the morphisms in the upper part of the diagram in Corollary 3.7, we get the following diagram.

$$\begin{array}{ccccc} \widetilde{\Gamma} & \xleftarrow{a_1} & \widehat{\Gamma} & \xrightarrow{a_2} & I \\ {\scriptstyle a_3}\downarrow & & {\scriptstyle a_4}\downarrow & & {\scriptstyle a_5}\downarrow \\ \widetilde{\Omega} & \xleftarrow{a_6} & \widehat{\Omega} & \xrightarrow{a_7} & \overline{\Omega} \end{array}$$

As $\widetilde{\Omega}$ is contained in the branch locus of $\widetilde{\varepsilon}$, the morphism a_4 is an isomorphism and $\widetilde{\Gamma}$ and $\widetilde{\Omega}$ are isomorphic to G. By item (1) of Corollary 3.5, the exceptional divisor $\widehat{\Gamma}$ has a natural identification with $\mathbb{P}(\mathcal{U}) \times_G \mathbb{P}(\mathcal{U})$ and a_1 is the natural fibration over G. Analogously, by Remark 2.1, the divisor $\widehat{\Omega}$ is identified with $\mathbb{P}(\mathrm{Sym}^2\mathcal{U}) = \mathbb{P}(\widetilde{Z})$ and a_6 is the fibration over G.

The restriction $a_4 : \mathbb{P}(\mathcal{U}) \times_G \mathbb{P}(\mathcal{U}) \to \mathbb{P}(\mathrm{Sym}^2\mathcal{U})$ of $\widehat{\varepsilon}_{1,v}$ is the natural 2:1 morphism.

The birational morphism $a_2 : \mathbb{P}(\mathcal{U}) \times_G \mathbb{P}(\mathcal{U}) \to I \subset \mathbb{P}(V) \times \mathbb{P}(V)$ is induced by composing with the natural morphism $\mathbb{P}(\mathcal{U}) \to \mathbb{P}(V)$.

The birational morphism $a_7 : \mathbb{P}(\widetilde{Z}) \to Z$ is induced by f, and finally the finite 2:1 morphism $a_5 : I \to \overline{\Omega} = \mathbb{P}(Z)$ is the map $\mathbb{P}(\varepsilon)$ obtained from ε by projectivization (see the proof of Corollary 3.3).

4. The global covering

In this section we globalize the local double coverings of Lemma 3.7.[2] Our starting point is the following result contained in [Rap04] and [PR14]. Keeping the notation as above, let $\widetilde{\Sigma}_v \subset \widetilde{K}_v$ be the exceptional divisor of the blow up $\widetilde{K}_v = Bl_{\Sigma}K_v \to K_v$.

THEOREM 4.1 [Rap04, Theorem 3.3.1]. *The class of $\widetilde{\Sigma}_v$ in the Picard group $\mathrm{Pic}(\widetilde{K}_v)$ of $\widetilde{K}_v$ is divisible by two.*

Proof. The case of $\widetilde{K}_{(2,0,-2)}$ is dealt in [Rap04, Theorem 3.3.1]. The general case follows from [PR14, Theorem 3.1 and Remark 3.4]. □

As the Picard group of the IHS manifold $\widetilde{K}_v$ is torsion-free, there exists a unique normal projective variety $\widetilde{Y}_v$ equipped with a double cover $\widetilde{\varepsilon}_v : \widetilde{Y}_v \to \widetilde{K}_v$ branched over $\widetilde{\Sigma}_v$. This double cover allows us to construct the global analogue of the morphism ε of Lemma 3.7.

THEOREM 4.2. *There exists a unique normal projective variety Y_v equipped with a finite 2:1 morphism $\varepsilon_v : Y_v \to K_v$ whose branch locus is Σ_v. The ramified double cover induced by ε_v on a small analytic neighborhood of a point of Ω_v is isomorphic to the ramified double cover induced by $\varepsilon : W \to Z$ on a small analytic neighborhood of a point of Ω in Z.*

[2] The varieties that we construct in this section depend on the abelian surface A and the chosen v-generic polarization but, as in the previous sections, we omit this dependence to avoid cumbersome notation.

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Proof. For any $p \in \Omega_v$ there exists a small analytic neighborhood $U_{p,v}$ of $p \in K_v$ that is biholomorphic to the intersection of Z with an open ball. Hence, for any $p \in \Omega_v$ there exists a proper complex analytic space $Y_{p,v}$ and a finite 2:1 morphism $\varepsilon_{p,v} : Y_{p,v} \to U_{p,v}$ branched along $U_{p,v} \cap \Sigma_v$, which is obtained by restricting ε.

On the other hand, there exists an analytic manifold Y_v^o and a finite 2:1 morphism $\varepsilon_v^o : Y_v^o \to K_v \backslash \Omega_v$ branched along $\Sigma_v \backslash \Omega_v$. To see this, first of all note that by restricting $\widetilde{\varepsilon}_v : \widetilde{Y}_v \to \widetilde{K}_v$ we get a double covering of $\widetilde{K}_v \backslash \widetilde{\Omega}_v$ branched along $\widetilde{\Sigma}_v \backslash \widetilde{\Omega}_v$. Since K_v has an A_1 singularity along Σ_v, the exceptional divisor $\widetilde{\Sigma}_v \backslash \widetilde{\Omega}$ is a $\mathbb{P}^1$-bundle whose normal bundle has degree -2 on the fibers. It follows that $\widetilde{\varepsilon}_v^{-1}(\widetilde{\Sigma}_v \backslash \widetilde{\Omega})$ is a $\mathbb{P}^1$-bundle whose normal bundle has degree -1 on the fibers. By Nakano's theorem (see [Nak71] and [FN71/72]), $\widetilde{\varepsilon}_v^{-1}(\widetilde{K}_v \backslash \widetilde{\Omega})$ is the blow up of a complex manifold Y_v^o along a submanifold isomorphic to $\Sigma_v \backslash \Omega_v$. Moreover, since $f_v \circ \widetilde{\varepsilon}_v : \widetilde{Y}_v \to K_v$ is constant on the fibers of the $\mathbb{P}^1$ bundle $\widetilde{\varepsilon}_v^{-1}(\widetilde{\Sigma}_v \backslash \widetilde{\Omega})$, it induces the desired finite 2:1 morphism $\varepsilon_v^o : Y_v^o \to K_v \backslash \Omega_v$.

To yield the existence of $\varepsilon_v : Y_v \to K_v$, it will suffice to prove that $\varepsilon_{p,v}$ and ε_v^o induce isomorphic double covers on $U_p \backslash \{p\}$ so that they can be glued to get ε_v. Recall from Remark 3.2 that the fundamental group of $Z \backslash \Sigma$ is $\mathbb{Z}/2\mathbb{Z}$. Since the same holds for $U_{p,v} \backslash \Sigma$, the étale double covers induced by $\varepsilon_{p,v}$ and ε_v^o on $U_{p,v} \backslash \Sigma$ are isomorphic. The closure of the graph of this isomorphism in the fiber product $\varepsilon_v^{o-1}(U_{p,v} \backslash \{p\}) \times_{U_{p,v} \backslash \{p\}} \varepsilon_{p,v}^{-1}(U_p \backslash \{p\})$ is finite and bimeromorphic on the manifolds $\varepsilon_v^{o-1}(U_{p,v} \backslash \{p\})$ and $\varepsilon_{p,v}^{-1}(U_p \backslash \{p\})$, hence it is the graph of an isomorphism of double covers K_v.

The glued complex analytic space Y_v is also projective as a consequence of GAGA's principles [Gro63, Corollary 4.6], since it has a finite proper map to a projective variety. Finally Y_v is normal since W is normal and since the normality of a complex variety may be checked on the associated complex analytic space [Gro63, Proposition 2.1].

To prove uniqueness of ε_v, let $\varepsilon_v' : Y_v' \to K_v$ be a finite 2:1 morphism branched over Σ_v such that Y_v' is normal. In this case $Y_v \backslash \varepsilon_v^{-1}(\Sigma_v)$ and $Y_v' \backslash \varepsilon_v'^{-1}(\Sigma_v)$ are algebraic proper étale double covers of $K_v \backslash \Sigma_v = \widetilde{K}_v \backslash \widetilde{\Sigma}_v$. Any such cover is determined by a 2-torsion point in the Picard group $\mathrm{Pic}(\widetilde{K}_v \backslash \widetilde{\Sigma}_v)$ and a nowhere vanishing section (unique up to scalars) of the trivial line bundle. As $\widetilde{\Sigma}_v$ is irreducible and its class is divisible by 2 in the free group $\mathrm{Pic}(\widetilde{K}_v)$, there exists a unique nontrivial 2-torsion point in $\mathrm{Pic}(\widetilde{K}_v \backslash \widetilde{\Sigma}_v)$. Moreover, as K_v is normal and Σ_v has codimension 2 in K_v, a regular function on $K_v \backslash \Sigma_v$ extends to the projective variety K_v and therefore it is constant. It follows that $Y_v \backslash \varepsilon_v^{-1}(\Sigma_v)$ and $Y_v' \backslash \varepsilon_v'^{-1}(\Sigma_v)$ are isomorphic étale double covers of $K_v \backslash \Sigma_v$.

Repeating the argument in the final part of the proof of the existence, the closure of the graph of this isomorphism in the fiber product $Y_v \times_{K_v} Y_v'$ is finite and birational over the normal varieties Y_v and Y_v', hence it is the graph of an isomorphism of double covers. The local characterization of ε near points of Ω_v holds by construction. □

Theorem 4.2 allows one to prove a straightforward global version of Lemma 3.7. Let $\Delta_v \subset Y_v$ be the ramification locus (with the reduced induced structure) of ε_v and let Γ_v be the singular locus (consisting of 256 points) of Y_v. Denote by $\widetilde{\Gamma}_v$ the inverse image with reduced structure of Γ_v in $Bl_{\Delta_v} Y_v$ and denote by $\overline{\Delta}_v$ the strict transform of Δ_v in $Bl_{\Gamma_v} Y_v$.

Corollary 4.3.

(1) *The projective varieties* $Bl_{\widetilde{\Gamma}_v} Bl_{\Delta_v} Y_v$ *and* $Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} Y_v$ *are smooth and isomorphic over* Y_v.

(2) *There exist finite 2:1 morphisms* $\widetilde{\varepsilon}_v : Bl_{\Delta_v} Y_v \to \widetilde{K}_v$, $\overline{\varepsilon}_v : Bl_{\Gamma_v} Y_v \to Bl_{\Omega_v} K_v$, $\widehat{\varepsilon}_{1,v} : Bl_{\widetilde{\Gamma}_v} Bl_{\Delta_v} Y_v \to Bl_{\widetilde{\Omega}_v} Bl_{\Sigma_v} K_v$ *and* $\widehat{\varepsilon}_{2,v} : Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} Y_v \to Bl_{\overline{\Sigma}_v} Bl_{\Omega_v} K_v$ *lifting* ε_v. *Hence, there*

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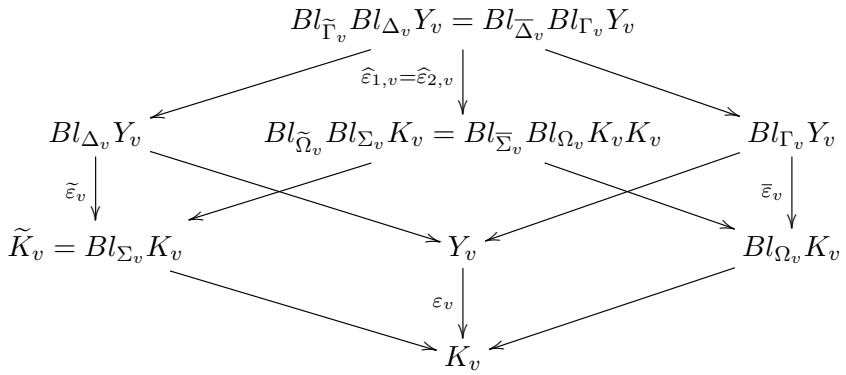
exists a commutative diagram



where the diagonal arrows are blow ups.

Proof. (1) Over the inverse images of the smooth locus of Y_v, the existence of the isomorphism is trivial, whereas over the inverse images of small Euclidean neighborhoods of the singular points of Y_v it follows from item (2) of Lemma 3.7. Since the global blow up is obtained by gluing local blow ups, $Bl_{\widetilde{\Gamma}_v} Bl_{\Delta_v} Y_v$ and $Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} Y_v$ are isomorphic over Y_v. The smoothness of $Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} Y_v$ and $Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} Y_v$ follows from the smoothness of $Bl_{\overline{\Delta}} Bl_{\Gamma} W$ and $Bl_{\overline{\Delta}} Bl_{\Gamma} W$, which was proven in Corollary 3.7. Hence item (1) holds.

(2) The existence of the liftings of the double cover ε_v over the inverse images of $K_v \backslash \Omega_v$ is clear. Over the inverse images of a small Euclidean neighborhood in K_v of a point of Ω_v, the existence of the lift follows from Theorem 2.3(a), Corollaries 3.3 and 3.5 and Lemma 3.7(1). Since the lift of a morphism to bimeromorphic varieties is unique, whenever it exists, it is possible to glue the local liftings and obtain the desired global morphism. □

Remark 4.4. In Corollary 4.3 we have shown that $Bl_{\Delta_v} Y_v$ is a double cover of $\widetilde{K}_v$ and that it is branched over $\widetilde{\Sigma}_v$. Moreover, by Corollary 3.5, the projective variety $Bl_{\Delta_v} Y_v$ is normal. Since the Picard group of the IHS manifold $\widetilde{K}_v$ is torsion-free, there exists a unique such double cover. It follows that $Bl_{\Delta_v} Y_v = \widetilde{Y}_v$ and $\widetilde{\varepsilon}_v = \widetilde{\varepsilon}_v$.

In order to describe the ramification loci of these double coverings, we need to introduce some further notation. In the following corollary we denote by $\widetilde{\Delta}_v \subset Bl_{\Delta_v} Y_v$ the exceptional divisor and by $\widehat{\Delta}_v \subset Bl_{\widetilde{\Gamma}_v} Bl_{\Delta_v} Y_v = Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} Y_v$ the strict transform of $\widetilde{\Delta}_v$ or, equivalently, the exceptional divisor of the blow up of $Bl_{\Gamma_v} Y_v$ along $\overline{\Delta}_v$.

COROLLARY 4.5.

(1) *The branch loci of* $\widetilde{\varepsilon}_v$, $\overline{\varepsilon}_v$ *and* $\widehat{\varepsilon}_{1,v}$ $(= \widehat{\varepsilon}_{2,v})$ *are* $\widetilde{\Sigma}_v$, $\overline{\Sigma}_v$ *and* $\widehat{\Sigma}_v$, *respectively.*

(2) *The ramification loci of* $\widetilde{\varepsilon}_v$, $\overline{\varepsilon}_v$ *and* $\widehat{\varepsilon}_{1,v}$ $(= \widehat{\varepsilon}_{2,v})$ *are* $\widetilde{\Delta}_v$, $\overline{\Delta}_v$ *and* $\widehat{\Delta}_v$, *respectively.*

Proof. The statements on the branch loci are determined by the analogous statement proved for the local case. Specifically, (1) follows from Corollaries 3.5 and 3.3 and item (1) of Corollary 3.7.

Since the ramification locus of ε_v is Δ_v and its branch locus is Σ_v, (2) follows from (1) and from the commutativity of the diagram in item (2) of Corollary 4.3. □

In the final part of this section on the global geometry of the double covers induced by ε_v, we compare their ramification and their branch loci and discuss the associated involutions.

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Remark 4.6. Since $\Sigma_v \simeq A \times A^\vee/\pm 1$, $\widetilde{\Sigma}_v$ has an A_1 singularity along its singular locus (see Remark 2.2). Moreover, $\overline{\Sigma}_v$ and $\widehat{\Sigma}_v$ are smooth (see Proposition 2.6) and the branch loci of ε_v, $\widetilde{\varepsilon}_v$, $\overline{\varepsilon}_v$ and $\widehat{\varepsilon}_{1,v}$ $(= \widehat{\varepsilon}_{2,v})$ are normal. Hence these double covers induce isomorphisms $\Delta_v \simeq \Sigma_v$, $\widetilde{\Delta}_v \simeq \widetilde{\Sigma}_v$, $\overline{\Delta}_v \simeq \overline{\Sigma}_v$ and $\widehat{\Delta}_v \simeq \widehat{\Sigma}_v$.

Remark 4.7. Remark 3.4 implies that $Bl_{\Gamma_v} Y_v$ and $Bl_{\widetilde{\Gamma}_v} Bl_{\Delta_v} Y_v = Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} Y_v$ are smooth, Corollary 3.5 implies that $Bl_{\Delta_v} Y_v$ is normal and, by Theorem 4.2, Y_v is normal too. Hence the finite 2:1 morphisms ε_v, $\widetilde{\varepsilon}_v$, $\overline{\varepsilon}_v$ and $\widehat{\varepsilon}_{1,v} (= \widehat{\varepsilon}_{2,v})$ induce regular involutions τ_v, $\widetilde{\tau}_v$, $\overline{\tau}_v$, $\widehat{\tau}_{1,v}$ and $\widehat{\tau}_{2,v}$ on Y_v, $Bl_{\Delta_v} Y_v$, $Bl_{\Gamma_v} Y_v$, $Bl_{\widetilde{\Gamma}_v} Bl_{\Delta_v} Y_v$ and $Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} Y_v$, respectively. Recall that K_v is normal, and hence so is $Bl_{\Omega_v} K_v$ by Proposition 2.4. As for $\widetilde{K}_v$ and $Bl_{\widetilde{\Omega}_v} Bl_{\Sigma_v} K_v = Bl_{\overline{\Sigma}_v} Bl_{\Omega_v} K_v$, they are both smooth. It follows that the morphisms ε_v, $\widetilde{\varepsilon}_v$, $\overline{\varepsilon}_v$ and $\widehat{\varepsilon}_{1,v} (= \widehat{\varepsilon}_{2,v})$ can be identified with the quotient maps of the respective involutions τ_v, $\widetilde{\tau}_v$, $\overline{\tau}_v$, $\widehat{\tau}_{1,v}$ and $\widehat{\tau}_{2,v}$.

5. The birational geometry of Y_v

In this section we describe the global geometry of Y_v, we show that it is birational to an IHS manifold of $\mathrm{K3}^{[n]}$ type, and we describe the birational map explicitly.

In the first part of the section, we consider the special case where A is a principally polarized abelian surface, whose Néron–Severi group is generated by the principal symmetric polarization Θ. As is well known, the linear system $|2\Theta|$ defines a morphism $g_{|2\Theta|} : A \to |2\Theta|^\vee \simeq \mathbb{P}^3$ whose image is the singular Kummer surface Kum_s of A, a nodal quartic surface isomorphic to the quotient $A/\pm 1$. The smooth Kummer surface S of A is the blow up of Kum_s along the singular locus $A[2]$.

We are going to show that in this case $Y_{(0,2\Theta,2)}$ is birational to the Hilbert scheme $S^{[3]}$, for which we will need the following remark on known properties of the linear system $|2\Theta|$.

Remark 5.1. (1) The locus of $|2\Theta|$ parametrizing singular curves consists of 17 irreducible divisors: the divisor R parametrizing reducible curves and, for any 2-torsion point $\alpha \in A[2]$, the divisor N_α parametrizing curves passing through α. The divisor R is isomorphic to Kum_s and a general point of R corresponds to a curve of the form $\Theta_x \cup \Theta_{-x}$, where Θ_x and Θ_{-x} meet transversally outside of $A[2]$. For every $\alpha \in A[2]$, the divisor N_α is isomorphic to $\mathbb{P}^2$ and parametrizes curves whose images in Kum_s are plane sections through the singular point $g_{|2\Theta|}(\alpha)$. The general point of N_α corresponds to a curve C that is a double cover of a quartic plane curve with precisely one node; this double covers ramifies over the node and therefore C has a node in α and no other singularity.

(2) For the natural choices in the definition of the map

$$\mathbf{a}_{(0,2\Theta,2)} : M_{(0,2\Theta,2)}(A,\Theta) \to A \times A^\vee$$

(see Introduction, formula (1.2)), the subvariety $K_{(0,2\Theta,2)} := \mathbf{a}^{-1}_{(0,2\Theta,2)}(0,0) \subset M_{(0,2\Theta,2)}(A,\Theta)$ parametrizes sheaves whose determinant is equal to $\mathcal{O}(2\Theta)$ and whose second Chern class sums up to $0 \in A$. Since $M_{(0,2\Theta,2)}(A,\Theta)$ parametrizes pure dimension-1 sheaves, there exists a regular morphism $t : K_v \to |2\Theta| \simeq \mathbb{P}^3$, called the support morphism, which to every polystable sheaf associates its Fitting subscheme (see [Lep93]). The morphism t is surjective and, since $K_{(0,2\Theta,2)}$ has a resolution that is an IHS manifold, all its fibers are three-dimensional.

(3) Since the polarization Θ is symmetric, -1^* induces an involution on the moduli space $M_{(0,2\Theta,2)}(A,\Theta)$ whose fixed locus contains the variety $K_{(0,2\Theta,2)}$. Indeed, any smooth curve $C \in |2\Theta|$ is an étale double cover of its image $g_{|2\Theta|}(C)$. The pull back to C of any degree-3 line bundle

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on $g_{|2\Theta|}(C)$ is a stable sheaf of $K_{(0,2\Theta,2)}$ which is -1^*-invariant. Moreover, the pull backs of two line bundles on $g_{|2\Theta|}(C)$ are isomorphic if and only if the two line bundles differ by a 2-torsion line bundle defining the étale double cover $C \to g_{|2\Theta|}(C)$. Hence, there exists a six-dimensional algebraic subset of $K_{(0,2\Theta,2)}$ that is fixed by the involution and hence, by closure of the fixed locus, the whole $K_{(0,2\Theta,2)}$ is fixed.

(4) If $C \in |2\Theta|$ is smooth or general in R or N_α, the general point of $t^{-1}(C)$ represents a sheaf that is locally free on its support. This holds because, for any nodal curve C, any torsion-free sheaf on C that is not locally free is the limit of locally free sheaves on C varying in a family parametrized by $\mathbb{P}^1$. Since any $\mathbb{P}^1$ has to be contracted by $\mathbf{a}_{(0,2\Theta,2)}$, it follows that it has to be contained in $K_{(0,2\Theta,2)}$ and the claim follows.

(5) The inverse image on $K_{(0,2\Theta,2)}$ of an irreducible surface contained in $|2\Theta| \simeq \mathbb{P}^3$ is irreducible. Since t is equidimensional, it suffices to prove that $t^{-1}(C)$ is irreducible for any curve C that is smooth or general in R or in N_α. If C is such a curve, the locus $t^{-1}(C)^{lf}$ parametrizing sheaves in $t^{-1}(C)$ that are locally free on their support is dense in $t^{-1}(C)$ and, moreover, -1 has at most one fixed point on C. It follows that, for any $F \in t^{-1}(C)^{lf}$, the -1 action on F can be linearized in such a way that the action is trivial on the fiber over the fixed point. By Kempf's descent lemma (see [HL10, Theorem 4.2.15]), this means that F is the pull back of a line bundle on the irreducible nodal curve $g_{|2\Theta|}(C)$. Since the generalized Jacobian of an irreducible nodal curve is irreducible, $t^{-1}(C)^{lf}$ and its closure $t^{-1}(C)$ are also irreducible.

LEMMA 5.2. *Let A be a principally polarized abelian surface, with $NS(A) = \mathbb{Z}\Theta$. Then $\widetilde{Y}_{(0,2\Theta,2)}$ is birational to the Hilbert scheme $S^{[3]}$.*

Proof. Let D be the pull back on S of a plane section of Kum_s. We are going to show that $\widetilde{Y}_{(0,\Theta,2)}$ is birational to the smooth projective moduli space $M_{(0,D,1)}$ parametrizing sheaves on S with Mukai vector $(0, D, 1)$ and which are stable with respect to a fixed $(0, D, 1)$-generic polarization. The moduli space $M_{(0,D,1)}$ is well known to be birational to $S^{[3]}$ (see [Bea99, Proposition 1.3]).

By construction, there exists an isomorphism between linear systems $\psi : |D| \to |2\Theta|$. Moreover, any sheaf $F \in M_{(0,D,1)}$, whose support is a smooth curve, may be seen as a sheaf on Kum_s and its pull back to A is a stable sheaf of $K_{(0,2\Theta,2)}$. It follows that there exists a commutative diagram

$$\begin{array}{ccc} & & \widetilde{K}_{(0,2\Theta,2)} \\ & \overset{\varphi}{\nearrow} & \downarrow{\scriptstyle f_{(0,2\Theta,2)}} \\ M_{(0,D,1)} & & K_{(0,2\Theta,2)} \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ |D| & \xrightarrow[\psi]{} & |2\Theta| \end{array} \tag{5.1}$$

where s and t are the two support morphisms. If $C \in |2\Theta|$ is a smooth curve, it is a connected étale double cover of the smooth curve $g_{|2\Theta|}(C)$, and since $g_{|2\Theta|}(C) \cap A[2] = \emptyset$, it can be considered as a curve in $|D|$. The restriction of φ on $s^{-1}(g_{|2\Theta|}(C)) \simeq \mathrm{Pic}^3(g_{|2\Theta|}(C))$ is therefore well defined and gives an étale double cover of $(t \circ f_{(0,2\Theta,2)})^{-1}(C) \simeq t^{-1}(C) \subset \mathrm{Pic}^6(C)$ (see (3) of Remark 5.1). This shows that φ is a rational map of degree 2.

In order to compare $\widetilde{Y}_{(0,2\Theta,2)}$ and $M_{(0,D,1)}$, we need to determine the branch divisor B of φ, i.e. the divisor on $\widetilde{K}_v$ where a resolution of the indeterminacy of φ is not étale. We have already seen that B has to parametrize sheaves supported on singular curves.

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Let $U \subset M_{(0,D,1)}$ be the biggest open subset where φ extends to a regular morphism. As $M_{(0,D,1)}$ and $K_{(0,2\Theta,2)}$ have trivial canonical bundle, the differential of φ is an isomorphism at any point of U. As a consequence, φ does not contract any positive dimensional subvariety and $\varphi(U) \subset \widetilde{K}_{(0,2\Theta,2)}$ is an open subset.

We claim that the open subset $\varphi(U)$ intersects any divisor of $\widetilde{K}_{(0,2\Theta,2)}$, with the possible exception of $\widetilde{\Sigma}_{(0,2\Theta,2)}$. Since we have already shown that $(t \circ f_{(0,2\Theta,2)})^{-1}(C) \in \varphi(U)$ if C is smooth, it remains to check this statement for divisors contained in $(t \circ f_{(0,2\Theta,2)})^{-1}(R)$ and $(t \circ f_{(0,2\Theta,2)})^{-1}(N_\alpha)$. As $\Sigma_{(0,2\Theta,2)} \subset t^{-1}(R)$, by (5) of Remark 5.1, the divisor $(t \circ f_{(0,2\Theta,2)})^{-1}(R)$ is the union of $\widetilde{\Sigma}_{(0,2\Theta,2)}$ and the strict transform of $t^{-1}(R)$. Finally, $(t \circ f_{(0,2\Theta,2)})^{-1}N_\alpha$ is irreducible.

The general point F of $t^{-1}(R)$ represents a line bundle supported on the general curve C of R. As C does not intersect $A[2]$, its image $g_{|2\Theta|}(C) \simeq C/\pm 1$ may be seen as a curve in D and, as in the smooth case, F descends to a line bundle on $g_{|2\Theta|}(C)$. Hence $\varphi(U)$ intersects the strict transform of $t^{-1}(R)$.

Finally, by the commutativity of diagram (5.1), φ sends an open subset of $(\psi \circ s)^{-1}(N_\alpha)$ to the irreducible divisor $(t \circ f_{(0,2\Theta,2)})^{-1}N_\alpha$. Since $(\psi \circ s)^{-1}(N_\alpha)$ cannot be contracted, $\varphi(U)$ also intersects $(t \circ f_{(0,2\Theta,2)})^{-1}N_\alpha$. This completes the proof of our claim.

Let $r : N \to \widetilde{K}_v$ be a resolution of the indeterminacy of φ, hence N is a smooth projective variety such that there exists a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{r} & \widetilde{K}_{(0,2\Theta,2)} \\ \downarrow{\scriptstyle b} & \nearrow_{\varphi} & \\ M_{(0,D,1)} & & \end{array}$$

where b is birational and induces an isomorphism between $b^{-1}(U)$ and U. Let $U' \subset \widetilde{K}_{(0,2\Theta,2)} \backslash \widetilde{\Omega}_v$ be the open subset where the fibers of ξ are zero-dimensional. Note that, since N and $\widetilde{K}_{(0,2\Theta,2)}$ are smooth, r is flat over U'. Therefore $r^{-1}(U')$ is a flat ramified double cover of U' and, since $\widetilde{K}_{(0,2\Theta,2)} \backslash U'$ has codimension at least 2 and $\mathrm{Pic}(\widetilde{K}_{(0,2\Theta,2)})$ is torsion-free, this double cover is determined by its branch divisor B, i.e. by the locus where fibers of r are length-2 non-reduced subschemes.

We already know that if $p \in U' \cap \varphi(U)$ the fiber $r^{-1}(p)$ has at least one component consisting of a reduced point of U. Hence B is a divisor contained in $U' \backslash \varphi(U)$: therefore, by our claim, either B is empty or $B = U' \cap \widetilde{\Sigma}_{(0,2\Theta,2)}$. The first case is impossible becouse U' is simply connected and N is irreducible. In the second case $r^{-1}(U')$ is the unique double cover of U' ramified over $U' \cap \widetilde{\Sigma}_{(0,2\Theta,2)}$, hence $r^{-1}(U')$ is isomorphic to $\varepsilon_v^{-1}(U') \subset \widetilde{Y}_{(0,2\Theta,2)}$. □

The following proposition generalizes Lemma 5.2, by showing that Y_v is always birational to an IHS manifold, and describes a resolution of the indeterminacy of the birational map.

Recall that the exceptional divisor $\overline{\Gamma}_v$ of $Bl_{\Gamma_v}Y_v$ consists of the disjoint union of 256 copies $I_{i,v}$ of the incidence variety $I \subset \mathbb{P}(V) \times \mathbb{P}(V)$, each of which has two natural $\mathbb{P}^2$ fibrations given by the projections onto $\mathbb{P}(V)$. For any i, we let $p_i : I_{i,v} \to \mathbb{P}(V)$ be one of the two projections. Since Y_v is locally analytically isomorphic to the cone W, the normal bundle of $I_{i,v}$ in $Bl_{\Gamma_v}Y_v$ has degree -1 on the fibers of p_i.

Therefore, by applying Nakano's contraction theorem [Nak71], there exists a complex manifold $\underline{Y}_v$ and a morphism of complex manifolds $h_v : Bl_{\Gamma_v}Y_v \to \underline{Y}_v$ whose exceptional locus is $\overline{\Gamma}_v$ and is such that the image $J_{i,v} := h_v(I_{i,v})$ of any component $\overline{\Gamma}_v$ is isomorphic to $\mathbb{P}^3$.

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Moreover, the restriction of h_v on $I_{i,v}$ equals p_i and h_v realizes $Bl_{\Gamma_v}Y_v$ as the blow up of $\underline{Y}_v$ along the disjoint union $J := h_v(\overline{\Gamma}_v)$ of the $J_{i,v}$.

PROPOSITION 5.3. *Keeping the notation as above, the complex manifold $\underline{Y}_v$ is a projective IHS manifold that is deformation equivalent to the Hilbert scheme parametrizing zero-dimensional subschemes of length* 3 *on a* K3 *surface.*

Proof. Note that the ramification locus of $\varepsilon_v : Y_v \to K_v$ has codimension 2. It follows that the canonical divisor of Y_v is trivial and the canonical divisor of $Bl_{\Gamma_v}Y_v$ is supported on $\overline{\Gamma}_v$. As the normal bundle of $I_{i,v}$ in $Bl_{\Gamma_v}Y_v$ has degree -1 on the fibers of both the $\mathbb{P}^2$ fibrations of $I_{i,v}$, by adjunction, the canonical bundle of the smooth variety $Bl_{\Gamma_v}Y_v$ is $2\sum_{i=1}^{256} I_{i,v}$.

Let r_i be a line contained in a fiber of p_i and let l_i be a line contained in a fiber of the other $\mathbb{P}^2$ fibration of $I_{i,v}$. *A priori*, it is not clear whether r_i and l_i are numerically equivalent. Nevertheless, since Y_v is projective and r_i and l_i generate the cone of effective curves on $I_{i,v}$, the set $\{r_i\}$ represents 256 $K_{Bl_{\Gamma_v}Y_v}$-negative extremal rays of the Mori cone of $Bl_{\Gamma_v}Y_v$. If r_i and l_i are equivalent, the contraction of r_i contracts $I_{i,v}$ to a point admitting a Zariski neighborhood isomorphic to a Zariski neighborhood of the ith singular point of Γ_v in the normal variety Y_v. If r_i and l_i are independent, the contraction of r_i can be identified with the Nakano contraction restricting to p_i on $I_{i,v}$.

In any case, the contraction of r_i is divisorial and, by [KM98, Corollary 3.18], it produces only $\mathbb{Q}$-factorial singularities. Hence, after 256 extremal contractions we terminate with a $\mathbb{Q}$-factorial variety with trivial canonical divisor and with terminal singularities. If $v = (0, 2\Theta, 2)$, by Lemma 5.2, the variety $M_{(0,2\Theta,2)}$ is a minimal model of the IHS manifold $S^{[3]}$ and by a theorem due to Greb, Lehn and Rollenske (see [GLR13, Proposition 6.4]) it is an IHS manifold. In particular r_i and l_i are always numerically independent and this IHS manifold is isomorphic to the Nakano contraction $\underline{Y}_v$.

To deal with the general case, recall from [PR13, Theorem 1.6] (and its proof) that the singular variety K_v can be deformed to $K_{(0,2\Theta,2)}$ using only isomorphisms induced by Fourier–Mukai transform and locally trivial deformations induced by deformation of the underlying abelian surface (see [PR13, Proposition 2.16]).

Extending the construction of Theorem 4.2 to the case of a locally trivial deformation, it is also possible to deform Y_v to $Y_{(0,2\Theta,2)}$ by a locally trivial deformation. By blowing up the subvariety consisting of singular points of all fibers in the total space of the deformation, we get that $Bl_{\Gamma_v}Y_v$ can be deformed to $Bl_{\Gamma_{(0,2\Theta,2)}}Y_{(0,2\Theta,2)}$. Up to an étale base change on the base of the deformation, we may also assume that the exceptional divisor consists of 256 connected components $\mathcal{I}_i$, each of which has two fibrations and one of them restricts to p_i on $I_{i,v}$. Applying again Nakano's theorem, the $\mathcal{I}_i$ may be contracted respecting the chosen fibration.

As a consequence, the complex manifold $\underline{Y}_v$ obtained from $Bl_{\Gamma_v}Y_v$ by contracting the r_i is deformation equivalent (via smooth deformations) to an IHS manifold $\underline{Y}_{(0,2\Theta,2)}$ that is birational to $S^{[3]}$. It remains to show that $\underline{Y}_v$ is projective. As in the case $v = (0, 2\Theta, 2)$, it suffices to show that r_i and l_i are numerically independent. This is true because parallel transport preserves numerical independence and the analogous statement has been shown to hold on $Bl_{\Gamma_{(0,2\Theta,2)}}Y_{(0,2\Theta,2)}$. □

By construction, $\underline{Y}_v$ has a regular birational morphism to Y_v contracting J to Γ_v. In the following remark we show that the involution τ_v on Y_v cannot be lifted to a regular involution on $\underline{Y}_v$.

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Remark 5.4. Since the involution $\overline{\tau}_v : Bl_{\Gamma_v} Y_v \to Bl_{\Gamma_v} Y_v$ sends Γ_v to itself, it descends to a rational involution $\tau_T : \underline{Y}_v \dashrightarrow \underline{Y}_v$ restricting to a regular involution on the complement $\underline{Y}_v \backslash J_v$ of the union of the projective spaces $J_{i,v}$ in $\underline{Y}_v$. Since, by definition of τ, the involution $\overline{\tau}_v$ exchanges the two $\mathbb{P}^2$ fibrations on $I_{i,v}$, the indeterminacy locus of τ_T is J_v. Finally, since $Bl_{\Gamma_v} Y_v \simeq Bl_{J_v} \underline{Y}_v$, the rational involution τ_T may be described as the composition of a Mukai flop along J_v and an isomorphism outside of this locus.

6. The Hodge numbers

Collecting the results of the previous sections, we finally present the new construction of $\widetilde{K}_v$ that allows us to calculate the Betti and Hodge numbers of $\widetilde{K}_v$.

To simplify notation, let us set

$$\begin{aligned} \widehat{Y}_v &:= Bl_{\widetilde{\Gamma}_v} Bl_{\Delta_v} Y_v = Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} Y_v, \\ \widehat{K}_v &:= Bl_{\widetilde{\Sigma}_v} Bl_{\Omega_v} K_v = Bl_{\overline{\Delta}_v} Bl_{\Gamma_v} K_v, \\ \overline{Y}_v &:= Bl_{\Gamma_v} Y_v. \end{aligned}$$

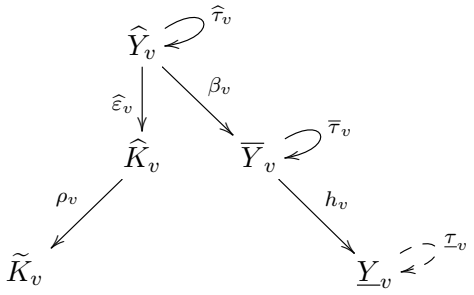
With this notation, the finite 2:1 morphism

$$\widehat{\varepsilon}_v := \widehat{\varepsilon}_{1,v} = \widehat{\varepsilon}_{2,v} : \widehat{Y}_v \to \widehat{K}_v$$

is a double cover between smooth varieties and is branched over the smooth divisor $\widehat{\Sigma}_v$ (see (1) of Corollary 4.3). Hence, $\widehat{\varepsilon}_v$ realizes $\widehat{K}_v$ as the quotient of $\widehat{Y}_v$ under the action of the associated involution $\widehat{\tau}_v : \widehat{Y}_v \to \widehat{Y}_v$.

This permits the reconstruction of $\widehat{K}_v$ starting from the IHS manifold $\underline{Y}_v$ of K3$^{[3]}$ type and using only birational modifications of smooth projective varieties and the finite 2:1 morphism $\widehat{\varepsilon}_v$.

The following commutative diagram contains all the varieties and maps that we will use.



Here β_v is the blow up map of $\overline{Y}_v$ along the smooth subvariety $\overline{\Delta}_v$, and $\overline{\tau}_v$ is the involution associated with $\overline{\tau}_v$ (see Remark 4.7).

Note that this diagram contains only maps between smooth varieties that appear in (2) of Corollary 4.3 and any diagonal map that appears is the blow up of a smooth variety along a smooth subvariety.

The IHS manifold $\underline{Y}_v$ carries a rational involution $\underline{\tau}_v$ whose indeterminacy locus J_v is the disjoint union of 256 projective three-dimensional spaces (see Remark 5.4). The rational involution $\underline{\tau}_v$ lifts to the regular involution $\overline{\tau}_v$ on the blow up $\overline{Y}_v$ of $\underline{Y}_v$ along J_v, which in turn lifts to the involution $\widehat{\tau}_v : \widehat{Y}_v \to \widehat{Y}_v$ on the blow up of $\overline{Y}_v$ along the fixed locus $\overline{\Delta}_v$ of $\overline{\tau}_v$ (see (2) of Corollary 4.5). Finally, the quotient $\widehat{K}_v$ of $\widehat{Y}_v$ modulo $\widehat{\tau}_v$ is the blow up of $\widetilde{K}_v$ along the union $\widetilde{\Omega}_v$ of 256 disjoint copies of the smooth three-dimensional quadric G.

We will start by computing the Betti numbers.

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PROPOSITION 6.1. *The odd Betti numbers of $\widetilde{K}_v$ are zero and the non-zero even ones are*

$$h^0(\widetilde{K}_v) = h^{12}(\widetilde{K}_v) = 1, \quad h^2(\widetilde{K}_v) = h^{10}(\widetilde{K}_v) = 8, \quad h^4(\widetilde{K}_v) = h^8(\widetilde{K}_v) = 199, \quad h^6(\widetilde{K}_v) = 1504.$$

Proof. By Proposition 5.3, $\underline{Y}_v$ is deformation equivalent to the Hilbert scheme parametrizing length-3 subschemes on a K3 surface, hence its odd Betti numbers are zero. By construction, $\overline{Y}_v$ is the blow up of $\underline{Y}_v$ along 256 disjoint projective spaces. Since the odd cohomology of the projective space is trivial, the same holds for $\overline{Y}_v$. By definition, $\widehat{Y}_v$ is the blow up of $\overline{Y}_v$ along $\overline{\Delta}_v$. We have already recalled that the ramification locus $\overline{\Delta}_v$ of $\overline{\varepsilon}_v$ is isomorphic to the corresponding branch locus $\overline{\Sigma}_v$, which, by Remark 4.6, is isomorphic to $(Bl_{(A\times A^\vee)[2]}(A \times A^\vee))/\pm 1$. As the odd cohomology classes of a torus are always antiinvariant under the action of ± 1, the odd Betti numbers of $\widehat{Y}_v$ are zero. Since $\rho_v \circ \widehat{\varepsilon}_v : \widehat{Y}_v \to \widetilde{K}_v$ is a regular surjective map between smooth projective varieties, the rational cohomology of $\widetilde{K}_v$ injects into the rational cohomology of $\widehat{Y}_v$. Hence the odd Betti numbers of $\widetilde{K}_v$ are zero.

We already know that $h^2(\widetilde{K}_v) = 8$ [O'Gr03] and that $\chi_{\text{top}}(\widetilde{K}_v) = 1920$ [Rap04]. The result follows using Salamon's formula [Sal96], which gives a linear relation among the Betti numbers of a $2n$-dimensional irreducible holomorphic symplectic manifold

$$2\sum_{j=1}^{2n}(-1)^j(3j^2 - n)b_{2n-j} = nb_{2n}.$$

In our case this yields

$$18b_4 + 90b_2 + 210 = 3b_6.$$

Since $b_2 = 8$ and

$$2 + 2b_2 + 2b_4 + b_6 = 1920$$

we obtain the proposition. □

The strategy to compute the Hodge numbers of $\widetilde{K}_v$ is the following. Since $\widehat{K}_v$ is the quotient of $\widehat{Y}_v$ by the action of $\widehat{\tau}_v$, the Hodge numbers of $\widehat{K}_v$ that determine the Hodge numbers of $\widetilde{K}_v$ are the $\widehat{\tau}_v$-invariant Hodge numbers of $\widehat{Y}_v$. The Hodge numbers of $\widehat{Y}_v$ can be easily computed in terms of the known Hodge numbers of the IHS manifold $\underline{Y}_v$ of $\mathrm{K3}^{[3]}$ type, and the action of $\widehat{\tau}_v$ is determined by the action of the rational involution $\underline{\tau}_v : \underline{Y}_v \dashrightarrow \underline{Y}_v$ (see Remark 5.4) on the Hodge groups of $\underline{Y}_v$. Finally, by Markman's monodromy results, this action only depends on its part on the second cohomology group, which is easy to compute.

In order to determine the Hodge numbers of $\widetilde{K}_v$, we first relate the Hodge numbers of $\widetilde{K}_v$ with the $\overline{\tau}_v$-invariant Hodge numbers of $\overline{Y}_v$. More specifically, we have the following lemmas.

LEMMA 6.2.

(1) *The following equalities of Hodge numbers hold:*

$$\begin{aligned} h^{p,q}(\widehat{K}_v) &= h^{p,q}(\widetilde{K}_v) \quad \text{if } p \neq q, \\ h^{1,1}(\widehat{K}_v) &= h^{1,1}(\widetilde{K}_v) + 256, \\ h^{2,2}(\widehat{K}_v) &= h^{2,2}(\widetilde{K}_v) + 512, \\ h^{3,3}(\widehat{K}_v) &= h^{3,3}(\widetilde{K}_v) + 512. \end{aligned}$$

(2) *The vector space $H^{p,q}(\widehat{Y}_v)^{\widehat{\tau}_v}$ of $\widehat{\tau}_v$-invariant (p,q)-forms on $\widehat{Y}_v$ is isomorphic to $H^{p,q}(\widehat{K}_v)$.*

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Proof. (1) This follows from the fact that $r : \widehat{K}_v \to \widetilde{K}_v$ is the blow up along the 256 quadrics $G_i \subset \widetilde{K}_v$, and that the cohomology of a three-dimensional quadric is one-dimensional in even degrees and zero otherwise.

(2) This holds because $\widetilde{K}_v \simeq \widehat{Y}_v/\widehat{\tau}_v$ (see Remark 4.7). □

In the following lemma we set $\binom{a}{b} := 0$ if $b > a$ or $b < 0$.

Lemma 6.3. *The $\widehat{\tau}_v$-invariant Hodge numbers of $\widehat{Y}_v$ and the $\overline{\tau}_v$-invariant Hodge numbers of $\overline{Y}_v$ are related in the following way:*

$$\begin{aligned}
&h^{p,q}(\widehat{Y}_v)^{\widehat{\tau}_v} = h^{p,q}(\overline{Y}_v)^{\overline{\tau}_v} = 0 && \text{for } p+q \text{ odd},\\
&h^{p,q}(\widehat{Y}_v)^{\widehat{\tau}_v} = h^{p,q}(\overline{Y}_v)^{\overline{\tau}_v} + \tbinom{4}{p-1}\tbinom{4}{q-1} && \text{for } p+q \text{ even and } p \neq q,\\
&h^{p,p}(\widehat{Y}_v)^{\widehat{\tau}_v} = h^{p,p}(\overline{Y}_v)^{\overline{\tau}_v} + \tbinom{4}{p-1}^2 && \text{for } p = 0,1,5,6,\\
&h^{p,p}(\widehat{Y}_v)^{\widehat{\tau}_v} = h^{p,p}(\overline{Y}_v)^{\overline{\tau}_v} + \tbinom{4}{p-1}^2 + 256 && \text{for } p = 2,3,4.
\end{aligned}$$

Proof. The morphism $\beta_v : \widehat{Y}_v \to \overline{Y}_v$ is the blow up of $\overline{Y}_v$ along a smooth subvariety isomorphic to $\overline{\Delta}_v$. By Corollary 4.5 and Remark 4.6, the variety $\overline{\Delta}_v$ is isomorphic to $Bl_{A\times A^\vee[2]}(A \times A^\vee/\pm 1)$, hence its Hodge numbers are the ± 1-invariant Hodge numbers of $Bl_{A\times A^\vee[2]}$. In other words

$$\begin{aligned}
&h^{p,q}(\overline{\Delta}_v) = 0 && \text{for } p+q \text{ odd},\\
&h^{p,q}(\overline{\Delta}_v) = h^{p,q}(A \times A^\vee) && \text{for } p+q \text{ even and } p \neq q,\\
&h^{p,p}(\overline{\Delta}_v) = h^{p,p}(A \times A^\vee) && \text{for } p = 0,4,\\
&h^{p,p}(\overline{\Delta}_v) = h^{p,p}(A \times A^\vee) + 256 && \text{for } p = 1,2,3,
\end{aligned}$$

with $h^{p,q}(A \times A^\vee) = \binom{4}{p}\binom{4}{q}$. As a consequence (see [Voi07, Theorem 7.31]), the Hodge numbers of $\widehat{Y}_v$ satisfy

$$\begin{aligned}
&h^{p,q}(\widehat{Y}_v) = h^{p,q}(\overline{Y}_v) = 0 && \text{for } p+q \text{ odd},\\
&h^{p,q}(\widehat{Y}_v) = h^{p,q}(\overline{Y}_v) + \tbinom{4}{p-1}\tbinom{4}{q-1} && \text{for } p+q \text{ even and } p \neq q,\\
&h^{p,p}(\widehat{Y}_v) = h^{p,p}(\overline{Y}_v) + \tbinom{4}{p-1}^2 && \text{for } p = 0,1,5,6,\\
&h^{p,p}(\widehat{Y}_v) = h^{p,p}(\overline{Y}_v) + \tbinom{4}{p-1}^2 + 256 && \text{for } p = 2,3,4.
\end{aligned}$$

The lemma follows, since the classes in $h^{p,q}(\widehat{Y}_v)$ that come from $\overline{\Delta}_v$ are the push forward of cohomology classes of the exceptional divisor $\widehat{\Delta}_v$ which, by Corollary 4.5, is the fixed locus of $\widehat{\tau}_v$. □

It remains to determine the $\overline{\tau}_v$-invariant Hodge numbers $h^{p,q}(\overline{Y}_v)^{\overline{\tau}_v}$ of $\overline{Y}_v$. This will be done by relating the action in cohomology of $\overline{\tau}_v$ with the monodromy operator

$$m(\underline{\tau}_v) : H^\bullet(\underline{Y}_v) \to H^\bullet(\underline{Y}_v)$$

associated to the birational involution $\overline{\tau}_v$.

To explain this relation, first let us recall some details on the definition of $m(\underline{\tau}_v)$.

Remark 6.4. By Theorem 2.5 of [Huy03] there exist smooth proper families of IHS manifolds $\underline{\mathcal{Y}}'_v \to S$ and $\underline{\mathcal{Y}}_v \to S$ over a one-dimensional disk S such that both the central fibers are isomorphic to $\underline{Y}_v$ and there exists a rational S-morphism $\underline{\mathcal{T}}_v : \underline{\mathcal{Y}}'_v \dashrightarrow \underline{\mathcal{Y}}_v$ sending $\underline{\mathcal{Y}}'_v \backslash J_v$ isomorphically to $\underline{\mathcal{Y}}_v \backslash J_v$ and restricting to $\underline{\tau}_v$ on central fibers.

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By specializing the closure in $\underline{\mathcal{Y}}'_v \times_S \underline{\mathcal{Y}}_v$ of the graph of $\underline{\mathcal{T}}_v$ over the central fiber, we obtain a pure six-dimensional cycle Υ on $\underline{Y}_v \times \underline{Y}_v$.

By definition $m(\underline{\tau}_v)$ is the Hodge ring automorphism of $H^\bullet(\underline{Y}_v)$ obtained as the associated correspondence of the cycle Υ. In our case, since $\underline{\mathcal{T}}_v$ induces an isomorphism between $\underline{\mathcal{Y}}'_v \backslash J_v$ and $\underline{\mathcal{Y}}_v \backslash J_v$ and restricts to $\underline{\tau}_v$ on central fibers, it follows that

$$\Upsilon = \Upsilon_{\underline{\tau}} + \sum_i m_i J_{i,v} \times J_{i,v},$$

where $\Upsilon_{\underline{\tau}}$ is the closure of the graph of $\underline{\tau}$ and the m_i are nonnegative integers.[3]

LEMMA 6.5.

(1) *For every i the cohomology class $[J_{i,v}] \in H^6(\underline{Y}_v)$ of $J_{i,v}$ is $m(\underline{\tau}_v)$-antiinvariant.*

(2) *The following relations between $m(\underline{\tau}_v)$-invariant Hodge numbers of $\underline{Y}_v$ and $\overline{\tau}_v$-invariant Hodge numbers of $\overline{Y}_v$ hold:*

$$\begin{aligned}
h^{p,q}(\overline{Y}_v)^{\overline{\tau}_v} &= h^{p,q}(\underline{Y}_v)^{m(\underline{\tau}_v)} && \text{for } p+q \leqslant 6 \text{ and } p \neq q,\\
h^{1,1}(\overline{Y}_v)^{\overline{\tau}_v} &= h^{1,1}(\underline{Y}_v)^{m(\underline{\tau}_v)} + 256 &&\\
h^{2,2}(\overline{Y}_v)^{\overline{\tau}_v} &= h^{2,2}(\underline{Y}_v)^{m(\underline{\tau}_v)} + 256 &&\\
h^{3,3}(\overline{Y}_v)^{\overline{\tau}_v} &= h^{3,3}(\underline{Y}_v)^{m(\underline{\tau}_v)} + 512. &&
\end{aligned}$$

Proof. (1) As the differential of the map $(h_v, h_v \circ \overline{\tau}_v) : \overline{Y}_v \to \underline{Y}_v \times \underline{Y}_v$ is everywhere injective, it induces an isomorphism $\Upsilon_{\underline{\tau}} \simeq \overline{Y}_v = Bl_{J_v}\underline{Y}_v$. By the key formula of [Ful98, Proposition 6.7], the class $[J_{i,v}]$ is an eigenvector for correspondence $[\Upsilon_{\underline{\tau}}]^*$ induced by $[\Upsilon_{\underline{\tau}}]$ on $H^6(\underline{Y}_v)$ and, moreover, the corresponding eigenvalue λ only depends on the normal bundle of $J_{i,v}$ in $\underline{Y}_v$; therefore it does not depend on i.

On the other hand, the correspondence induced by $J_{i,v} \times J_{i,v}$ on $H^6(\underline{Y}_v)$ multiplies $[J_{i,v}]$ by the degree of the third Chern class of its normal bundle in $\underline{Y}_v$. As this normal bundle is isomorphic to the cotangent bundle of $\mathbb{P}^3$, we obtain $[J_{i,v} \times J_{i,v}]^*[J_{i,v}] = -4[J_{i,v}]$.

It follows that

$$m(\underline{\tau}_v)[J_{i,v}] = (\lambda - 4m_i)[J_{i,v}].$$

As $m(\underline{\tau}_v)$ is an isomorphism on the integral cohomology, $\lambda - 4m_i = \pm 1$ and the sign cannot depend on i.

It remains to exclude that $m(\underline{\tau}_v)[J_{i,v}] = [J_{i,v}]$ for every i. In this case, letting A be the class of an ample divisor A of $\underline{\tau}$, we have

$$\int_{\underline{Y}_v} m(\underline{\tau}_v)[A]^3 \wedge [J_{i,v}] = \int_{\underline{Y}_v} [A]^3 \wedge [J_{i,v}] > 0.$$

Therefore, the line bundle associated with $m(\underline{\tau}_v)[A]$ would be positive on the $J_{i,v}$ and, as A is ample, it would have positive degree on any curve on $\underline{Y}_v$. Finally, by [Huy03, Proposition 3.2], $\underline{\tau}_v^*(A)$ would be an ample divisor and this is absurd because $\underline{\tau}_v$ does not extend to an isomorphism.

[3] Using the key formula of [Ful98, Proposition 6.7], it can be shown that $m_i = 1$ for every i.

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(2) Since $\underline{Y}_v$ is an IHS manifold of K3$^{[n]}$ type, its odd Betti numbers are trivial and the same holds for $\overline{Y}_v$ as it is isomorphic to the blow up of $\underline{Y}_v$ along J that is the disjoint union of 256 projective spaces. Hence, we only need to consider the case where $p+q$ is even.

If $p+q=6$, the exact sequences of the pairs $(\overline{Y}_v, \overline{Y}_v \backslash \overline{\Gamma}_v)$ and $(\underline{Y}_v, \underline{Y}_v \backslash J_v)$, using excision and Thom isomorphism, give rise to the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(J_v) & \xrightarrow{r_1} & H^6(\underline{Y}_v) & \xrightarrow{r_2} & H^6(\underline{Y}_v \backslash J_v) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle h_v^*} & & \downarrow & & \\ 0 & \longrightarrow & H^4(\overline{\Gamma}_v) & \xrightarrow{s_1} & H^6(\overline{Y}_v) & \xrightarrow{s_2} & H^6(\overline{Y}_v \backslash \overline{\Gamma}_v) & \longrightarrow & 0 \end{array}$$

In this diagram, r_2 and s_2 are surjective because the odd Betti numbers of J_v and $\overline{\Gamma}_v$ are zero and r_1 is injective because the classes $[J_{i,v}]$ are independent. This also implies that $H^5(\overline{Y}_v \backslash \overline{\Gamma}_v) = H^5(\underline{Y}_v \backslash J_v) = H^5(\underline{Y}_v) = 0$ and therefore s_1 is injective too.

As the intersection form of the middle cohomology of $\underline{Y}_v$ is nondegenerate, on $H^0(J_v)$ there is a splitting of Hodge structures

$$H^6(\underline{Y}_v) = H^0(J_v)^\perp \oplus H^0(J_v),$$

where $H^0(J_v)^\perp$ is the perpendicular to $H^0(J_v)$ in $H^6(\underline{Y}_v)$. Since $m(\underline{\tau}_v)$ acts as -1 on $H^0(J_v)$ and the correspondence $[J_{i,v} \times J_{i,v}]^*$ acts trivially on $H^0(J_v)^\perp$, we deduce that

$$H^{p,q}(\underline{Y}_v)^{m(\underline{\tau}_v)} = ((H^0(J_v)^\perp)^{p,q})^{[\Upsilon_{\underline{\tau}}]^*},$$

for $p+q=6$.

Since $h_v^*(H^0(J_v)^\perp)$ is included in the perpendicular $H^4(\overline{\Gamma}_v)^\perp$ to $H^4(\overline{\Gamma}_v)$ in $H^6(\overline{Y}_v)$, the injective pull back h_v^* induces an isomorphism of Hodge structures $H^4(\overline{\Gamma}_v)^\perp \simeq H^0(J_v)^\perp$. It follows that the intersection form on the middle cohomology of $\overline{Y}_v$ is nondegenerate on $H^4(\overline{\Gamma}_v)$ and there is a splitting of Hodge structures

$$H^6(\overline{Y}_v) = H^0(\overline{\Gamma}_v)^\perp \oplus H^4(\overline{\Gamma}_v).$$

Since $\overline{\tau}_v(\overline{\Gamma}_v) = \overline{\Gamma}_v$ we also deduce

$$H^{p,q}(\overline{Y}_v) = ((H^4(\overline{\Gamma}_v)^\perp)^{p,q})^{\overline{\tau}_v} \oplus H^{p-1,q-1}(\overline{\Gamma}_v)^{\overline{\tau}_v},$$

for $p+q=6$.

Moreover, the Hodge isomorphism $H^4(\overline{\Gamma}_v)^\perp \simeq H^0(J_v)^\perp$ identifies the action of $\overline{\tau}_v$ on $H^4(\overline{\Gamma}_v)^\perp$ with the action of $[\Upsilon_{\underline{\tau}}]^*$ on $H^0(J_v)^\perp$.

In fact, for any $\alpha \in H^0(J_v)^\perp$, we have $[\Upsilon_{\underline{\tau}}]^*(\alpha) = (h_{v*} \circ \overline{\tau}_v^* \circ h_v^*)(\alpha)$. As $\overline{\tau}_v^*(h_v^*(\alpha)) \in H^4(\overline{\Gamma}_v)^\perp$ and since the kernel of h_{v*} intersects trivially $H^4(\overline{\Gamma}_v)^\perp$, the class $\overline{\tau}_v^*(h_v^*(\alpha))$ is the unique class in $H^4(\overline{\Gamma}_v)^\perp$ whose push forward in $H^6(\underline{Y}_v)$ is $[\Upsilon_{\underline{\tau}}]^*(\alpha)$. Therefore

$$\overline{\tau}_v^*(h_v^*(\alpha)) = h_v^*([\Upsilon_{\underline{\tau}}]^*(\alpha)).$$

As a consequence,

$$H^{p,q}(\overline{Y}_v)^{\overline{\tau}_v} = H^{p,q}(\underline{Y}_v)^{m(\underline{\tau}_v)} \oplus H^{p-1,q-1}(\overline{\Gamma}_v)^{\overline{\tau}_v}$$

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and the result for $p+q=6$ follows because $\overline{\Gamma}_v$ consists of 256 copies of $I \subset \mathbb{P}(V)\times\mathbb{P}(V)$ on each of which $\overline{\tau}_v$ acts by exchanging the factors, and the cohomology of each component of $\overline{\Gamma}_v$ comes by restriction from the cohomology of $\mathbb{P}(V)\times\mathbb{P}(V)$.

Finally, if $p+q=2k$, and $k=1$ or $k=2$, as J_v has codimension 3 in $\underline{Y}_v$, restriction gives an isomorphism $H^{2k}(\underline{Y}_v) \simeq H^{2k}(\underline{Y}_v \backslash J_v)$ and there exists a Hodge decomposition

$$H^{2k}(\overline{Y}_v) = H^{2k}(\underline{Y}_v) \oplus H^{2k-2}(\overline{\Gamma}_v).$$

Moreover, $H^{2k}(\overline{Y}_v)$ can be seen as the subspaces of forms vanishing on $\overline{\Gamma}_v$ and it is stable under the action of $\overline{\tau}_v$. The same argument used in the case $p+q=6$ shows that the action of $\overline{\tau}_v$ on $H^{2k}(\underline{Y}_v)$ coincides with the action of $m(\underline{\tau}_v)$, and therefore

$$H^{p,q}(\overline{Y}_v)^{\overline{\tau}_v} = H^{p,q}(\underline{Y}_v)^{m(\underline{\tau}_v)} \oplus H^{p-1,q-1}(\overline{\Gamma}_v)^{\overline{\tau}_v}.$$

As the invariant subspaces for action of $\overline{\tau}_v$ on the degree-0 and the degree-2 cohomology of each component of $\overline{\Gamma}_v$ has dimension 1, this proves the lemma. □

It remains to determine the $m(\underline{\tau}_v)$-invariant Hodge numbers of $\underline{Y}_v$. It will suffice to deal with the case where A is a general principally polarized abelian surface with $NS(A)=\mathbb{Z}\Theta$ and where $v=(0,2\Theta,2)$.

LEMMA 6.6. *In this case the $m(\underline{\tau}_v)$-invariant Betti numbers and Hodge numbers of $\underline{Y}_{(0,2\Theta,2)}$ are:*

$$\begin{array}{llll}
(h^0)^{m(\underline{\tau}_v)}=1, & (h^2)^{m(\underline{\tau}_v)}=7, & (h^4)^{m(\underline{\tau}_v)}=171, & (h^6)^{m(\underline{\tau}_v)}=1178;\\
(h^{2,0})^{m(\underline{\tau}_v)}=1, & (h^{1,1})^{m(\underline{\tau}_v)}=5, & & \\
(h^{4,0})^{m(\underline{\tau}_v)}=1, & (h^{3,1})^{m(\underline{\tau}_v)}=6, & (h^{2,2})^{m(\underline{\tau}_v)}=157, & \\
(h^{6,0})^{m(\underline{\tau}_v)}=1, & (h^{5,1})^{m(\underline{\tau}_v)}=5, & (h^{4,2})^{m(\underline{\tau}_v)}=157, & (h^{3,3})^{m(\underline{\tau}_v)}=852.
\end{array}$$

Proof. We first determine the weight-2 $m(\underline{\tau}_{(0,2\Theta,2)})$-invariant Hodge numbers. By Lemmas 6.2, 6.3 and 6.5 we have

$$h^{2,0}(\underline{Y}_v)^{m(\underline{\tau}_v)} = h^{2,0}(\overline{Y}_v)^{\overline{\tau}_v} = h^{2,0}(\widehat{K}_v) = h^{2,0}(\widetilde{K}_v) = 1$$

and

$$h^{1,1}(\underline{Y}_v)^{m(\underline{\tau}_v)} = h^{1,1}(\overline{Y}_v)^{\overline{\tau}_v} - 256 = h^{1,1}(\widehat{K}_v) - 257 = h^{1,1}(\widetilde{K}_v) - 1 = 5.$$

In order to compute the invariant part of the Hodge structure of $\underline{Y}_{(0,2\Theta,2)}$, we use a result of Markman [Mar02, Example 14], which describes the action of monodromy operators on the Hilbert scheme of three points on a K3 surface S in terms of their action on the degree-2 cohomology. Specifically, Markman proves that there are isomorphisms of representations of the monodromy group of $S^{[3]}$,

$$\begin{aligned}
H^4(S^{[3]}) &= \mathrm{Sym}^2 H^2(S^{[3]}) \oplus H^2(S^{[3]}),\\
H^6(S^{[3]}) &= \mathrm{Sym}^3 H^2(S^{[3]}) \oplus \Lambda^2 H^2(S^{[3]}) \oplus \mathbb{C},
\end{aligned} \tag{6.1}$$

where $\mathbb{C}$ is a copy of the trivial representation.

If $v=2(0,\Theta,1)$, $\overline{Y}_v$ is birational to the Hilbert scheme $S^{[3]}$ and hence there exists an isomorphism of Hodge rings $k : H^\bullet(S^{[3]}) \to H^\bullet(\overline{Y}_v)$ and, moreover, the Hodge involution $k^{-1}\circ m(\underline{\tau}_v)\circ k$ is a monodromy operator on $S^{[3]}$. Moreover, the $m(\underline{\tau}_v)$-invariant Hodge numbers of $\overline{Y}_v$ coincide with the respective $k^{-1}\circ m(\underline{\tau}_v)\circ k$-invariant Hodge numbers of $S^{[3]}$. Since we know the weight-2 $m(\underline{\tau}_v)$-invariant Hodge numbers of $\overline{Y}_v$, we also know the weight-2

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$k^{-1}\circ m(\underline{\tau}_v)\circ k$-invariant Hodge numbers of $\underline{Y}_v$ and, using formulae (6.1), we can calculate all the $k^{-1}\circ m(\underline{\tau}_v)\circ k$-invariant numbers of $S^{[3]}$ and, therefore, all the $m(\underline{\tau}_v)$-invariant numbers of $\overline{Y}_v$.

To simplify the notation in the computation, set $H^{p,q}_+ := H^{p,q}(\underline{Y}_v)^{m(\underline{\tau}_v)}$. In particular, $H^{2,0}_- = H^{2,0}_- = 0$. By formulae (6.1) we obtain

$$\begin{gathered}
H^{4,0}_+ = \mathrm{Sym}^2 H^{2,0}_+, \quad H^{3,1}_+ = (H^{2,0}_+ \otimes H^{1,1}_+) \oplus H^{2,0}_+,\\
H^{2,2}_+ = (H^{2,0}_+ \otimes H^{0,2}_+) \oplus \mathrm{Sym}^2 H^{1,1}_+ \oplus \mathrm{Sym}^2 H^{1,1}_- \oplus H^{1,1}_+,\\
H^{6,0}_+ = \mathrm{Sym}^3 H^{2,0}_+, \quad H^{5,1}_+ = \mathrm{Sym}^2 H^{2,0}_+ \otimes H^{1,1}_+,\\
H^{4,2}_+ = (\mathrm{Sym}^2 H^{2,0}_+ \otimes H^{0,2}_+) \oplus (H^{2,0}_+ \otimes \mathrm{Sym}^2 H^{1,1}_+) \oplus (H^{2,0}_+ \otimes \mathrm{Sym}^2 H^{1,1}_+) \oplus (H^{2,0}_+ \otimes H^{1,1}_+),\\
H^{3,3}_+ = (H^{2,0}_+ \otimes H^{1,1}_+ \otimes H^{2,0}_-) \oplus \mathrm{Sym}^3 H^{1,1}_+ \oplus (H^{1,1}_+ \otimes \mathrm{Sym}^2 H^{1,1}_-)\\
\oplus (H^{2,0}_+ \otimes H^{0,2}_+) \oplus \Lambda^2 H^{1,1}_+ \oplus \Lambda^2 H^{1,1}_- \oplus \mathbb{C},
\end{gathered}$$

which give the invariant Hodge numbers. Finally, the invariant Betti numbers are determined from the invariant Hodge numbers. □

Now, a straightforward computation gives the Hodge numbers of O'Grady's six-dimensional IHS manifold.

Theorem 6.7. *Let $\widetilde{K}$ be an IHS manifold of type OG6. The odd Betti numbers of $\widetilde{K}$ are zero, its even Betti numbers are*

$$b_0 = 1, \quad b_2 = 8, \quad b_4 = 199, \quad b_6 = 1504, \quad b_8 = 199, \quad b_{10} = 8, \quad b_{12} = 1,$$

and its non-zero Hodge numbers are collected in the following table:

$$\begin{array}{ccccccc}
 & & & H^{0,0}=1 & & & \\
 & & H^{2,0}=1 & H^{1,1}=6 & H^{0,2}=1 & & \\
 & H^{4,0}=1 & H^{3,1}=12 & H^{2,2}=173 & H^{1,3}=12 & H^{0,4}=1 & \\
H^{6,0}=1 & H^{5,1}=6 & H^{4,2}=173 & H^{3,3}=1144 & H^{2,4}=173 & H^{1,5}=6 & H^{0,6}=1\\
 & H^{6,2}=1 & H^{5,3}=12 & H^{4,4}=173 & H^{3,5}=12 & H^{2,6}=1 & \\
 & & H^{6,4}=1 & H^{5,5}=6 & H^{4,6}=1 & & \\
 & & & H^{6,6}=1. & & &
\end{array}$$

Proof. As Hodge and Betti numbers are stable under smooth Kähler deformations, it will suffice to deal with the case where $\widetilde{K} = \widetilde{K}_{(0,2\Theta,2)}$ and the underlying abelian surface A is a general abelian surface, whose Neron–Severi group is generated by the principal polarization Θ. In this case, Lemmas 6.2, 6.3, 6.5 and 6.6 imply the result. □

Furthermore, the knowledge of the Hodge numbers is enough to compute the Chern numbers, as shown by Sawon [Saw99]. We have the following corollary.

Corollary 6.8. *Let $\widetilde{K}$ be a manifold of OG6 type. Then $\int_{\widetilde{K}} c_2(\widetilde{K})^3 = 30\,720$, $\int_{\widetilde{K}} c_2(\widetilde{K})c_4(\widetilde{K}) = 7680$ and $\int_{\widetilde{K}} c_6(\widetilde{K}) = \chi_{\mathrm{top}}(\widetilde{K}) = 1920$.*

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Proof. Let $\chi^p(\widetilde{K}) = \sum(-1)^q h^{p,q}(\widetilde{K})$. In our case we have $\chi^0(\widetilde{K}) = 4$, $\chi^1(\widetilde{K}) = -24$ and $\chi^2(\widetilde{K}) = 348$. As shown in [Saw99, Appendix B], we have

$$\int_{\widetilde{K}} c_2(\widetilde{K})^3 = 7272\chi^0(\widetilde{K}) - 184\chi^1(\widetilde{K}) - 8\chi^2(\widetilde{K}),$$
$$\int_{\widetilde{K}} c_2(\widetilde{K})c_4(\widetilde{K}) = 1368\chi^0(\widetilde{K}) - 208\chi^1(\widetilde{K}) - 8\chi^2(\widetilde{K}),$$
$$\int_{\widetilde{K}} c_6(\widetilde{K}) = 36\chi^0(\widetilde{K}) - 16\chi^1(\widetilde{K}) + 4\chi^2(\widetilde{K}).$$

A direct computation yields our claim. □

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