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ON BOUNDEDNESS OF THE WEIGHTED BERGMAN PROJECTIONS ON THE LIPSCHITZ SPACES

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In this paper we study the boundedness of the weighted Bergman projections on the weighted subspaces of Bergman spaces and the Lipschitz spaces on the unit ball and the unit polydisc.

1. Introduction

Let B_n and D^n be the unit ball and the unit polydisc in $\mathbb{C}^n$, respectively. Let $-1 < \gamma < \infty$ and $0 < p < \infty$. Let $L^p_\gamma(B_n)$ and $L^p_\gamma(D^n)$ be L^p-spaces with respect to the weighted volume measures

$$dV_\gamma(z) = \left(1 - |z|^2\right)^\gamma dV(z), \quad \prod_{j=1}^{n}\left(1 - |z_j|^2\right)^\gamma dV(z),$$

on B_n and D^n, respectively. Let $A^p_\gamma(B_n)$ and $A^p_\gamma(D^n)$ be subspaces of $L^p_\gamma(B_n)$ and $L^p_\gamma(D^n)$ consisting of functions which are holomorphic on B_n and D^n, respectively. They are called the weighted Bergman spaces. We define

$$(1.1) \qquad P_\gamma f(z) = C_{n,\gamma}\int_{B_n} \frac{f(\zeta)}{\left(1 - \overline{\zeta}\cdot z\right)^{n+1+\gamma}} \left(1 - |\zeta|^2\right)^\gamma dV(\zeta), \quad z \in B_n,$$

where

$$C_{n,\gamma} = \frac{n!}{\pi^n}\frac{\Gamma(n+1+\gamma)}{\Gamma(n+1)\Gamma(\gamma+1)}.$$

For the unit polydisc we define

$$(1.2) \qquad P_\gamma f(z) = C_{n,\gamma}\int_{D^n} f(\zeta)\prod_{j=1}^{n}\frac{\left(1 - |\zeta_j|^2\right)^\gamma}{\left(1 - \overline{\zeta_j} z_j\right)^{\gamma+2}} dV(\zeta), \quad z \in D^n,$$

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where

$$C_{n,\gamma} = \left(\frac{\gamma+1}{\pi}\right)^n.$$

They are orthogonal projections on $L^2_\gamma(B_n)$ and $L^2_\gamma(D^n)$ onto $A^2_\gamma(B_n)$ and $A^2_\gamma(D^n)$, respectively. They are called the weighted Bergman projections on B_n and D^n, respectively.

In this paper we study the boundedness of the weighted Bergman projections on the weighted subspaces of Bergman spaces and the Lipschitz spaces.

2. $L^{p,\alpha}_\gamma$ BOUNDEDNESS

For $0 < p < \infty$, $-1 < \gamma < \infty$ and $\alpha > 0$, $L^{p,\alpha}_\gamma(B_n)$ is defined to be the class of those $f \in L^p_\gamma(B_n)$ for which

$$\sup_{z\in B_n} |f(z)|\left(1-|z|^2\right)^\alpha < \infty.$$

For $f \in L^{p,\alpha}_\gamma(B_n)$, we define

$$\|f\|_{L^{p,\alpha}_\gamma(B_n)} := \max\left(\|f\|_{L^p_\gamma(B_n)}, \sup_{z\in B_n} |f(z)|\left(1-|z|^2\right)^\alpha\right).$$

Then the weighted subspace $L^{p,\alpha}_\gamma(B_n)$ of $L^p_\gamma(B_n)$ is a Banach space with the norm $\|\cdot\|_{L^{p,\alpha}_\gamma(B_n)}$ when $1 \leqslant p < \infty$. Let $A^{p,\alpha}_\gamma(B_n)$ be the subspace of $L^{p,\alpha}_\gamma(B_n)$ consisting of functions which are holomorphic on B_n. We note that

$$\|f\|^p_{L^p_\gamma(B_n)} \leqslant \left(\sup_{z\in B_n} |f(z)|\left(1-|z|^2\right)^\alpha\right)^p \int_{B_n} \left(1-|z|^2\right)^{\gamma-\alpha p} dV(z).$$

Thus for $f \in A^{p,\alpha}_\gamma(B_n)$ it follows that

$$\|f\|_{L^{p,\alpha}_\gamma(B_n)} \approx \sup_{z\in B_n} |f(z)|\left(1-|z|^2\right)^\alpha \quad \text{for} \quad \alpha < \frac{(n+\gamma)}{p}.$$

We can see that ([7, 2]) for $0 < p < \infty$ and $-1 < \gamma < \infty$

$$f(z) = \mathcal{O}\left(\frac{1}{\left(1-|z|^2\right)^{(n+1+\gamma)/p}}\right) \quad \text{for} \quad f \in A^p_\gamma(B_n).$$

Hence $A^{p,\alpha}_\gamma(B_n) = A^p_\gamma(B_n)$ for $\alpha \geqslant (n+1+\gamma)/p$.

For the polydisc we define $L^{p,\alpha}_\gamma(D^n)$ by the class of those $f \in L^p_\gamma(D^n)$ for which

$$\sup_{z\in D^n} |f(z)| \prod_{j=1}^n \left(1-|z_j|^2\right)^\alpha < \infty.$$

For $f \in L^{p,\alpha}_\gamma(D^n)$, we define

$$\|f\|_{L^{p,\alpha}_\gamma(D^n)} := \max\Bigl(\|f\|_{L^p_\gamma(D^n)},\ \sup_{z\in D^n}|f(z)|\prod_{j=1}^{n}\bigl(1-|z_j|^2\bigr)^\alpha\Bigr).$$

Let $A^{p,\alpha}_\gamma(D^n)$ be the subspace of $L^{p,\alpha}_\gamma(D^n)$ consisting of functions which are holomorphic on D^n. By the representation (1.2), Hölder's inequality, and (i) of Lemma 2.1, we can see that

$$f(z) = \mathcal{O}\Bigl(\frac{1}{\prod_{j=1}^{n}\bigl(1-|z_j|^2\bigr)^{(2+\gamma)/p}}\Bigr) \quad \text{for} \quad f \in A^p_\gamma(D^n).$$

Hence $A^{p,\alpha}_\gamma(D^n) = A^p_\gamma(D^n)$ for $\alpha \geqslant (2+\gamma)/p$.

For an account of the known results on these spaces, see [**4, 6**].

Lemma 2.1. ([**7**]) *For $z \in B_n$, c real, $t > -1$, define*

$$J_{c,t}(z) = \int_{B_n}\frac{\bigl(1-|\zeta|^2\bigr)^t}{|1-\overline{\zeta}\cdot z|^{n+1+t+c}}\,dV(\zeta).$$

where $dV(\zeta)$ is the volume measure.

(i) *When $c > 0$, then*

$$J_{c,t}(z) \approx \bigl(1-|z|^2\bigr)^{-c}.$$

(ii) *When $c = 0$, then*

$$J_{0,t}(z) \approx \log\frac{1}{1-|z|^2}.$$

The notation $a(z) \approx b(z)$ means that the ratio $a(z)/b(z)$ has a positive finite limit as $|z| \to 1$.

In [**1**] we can see that the weighted Bergman projection P_γ maps $L^p_\gamma(B_n)$ onto $A^p_\gamma(B_n)$, boundedly, for $1 < p < \infty$ and $\gamma > -1$. In this section we consider the boundedness of P_γ on weighted subspaces $L^{p,\alpha}_\gamma$ of L^p_γ.

Theorem 2.2. *For $1 \leqslant p < \infty$, $\gamma > -1$, and $0 < \alpha < \gamma + 1$, the weighted Bergman projection P_γ maps $L^{p,\alpha}_\gamma(B_n)$ onto $A^{p,\alpha}_\gamma(B_n)$, boundedly.*

Proof: From (1.1) we have

$$|P_\gamma f(z)| \lesssim \int_{B_n}|f(\zeta)|\frac{\bigl(1-|\zeta|^2\bigr)^\gamma}{|1-\overline{\zeta}\cdot z|^{n+1+\gamma}}\,dV(\zeta)$$
$$\leqslant \sup_{\zeta\in B_n}|f(\zeta)|\bigl(1-|\zeta|^2\bigr)^\alpha\int_{B_n}\frac{\bigl(1-|\zeta|^2\bigr)^{\gamma-\alpha}}{|1-\overline{\zeta}\cdot z|^{n+1+\gamma}}\,dV(\zeta).$$

By (i) of Lemma 2.1, the right side integral of the last inequality is bounded by $1/(1-|z|^2)^\alpha$. Thus we have

$$|P_\gamma f(z)|(1-|z|^2)^\alpha \lesssim \sup_{\zeta\in B_n} |f(\zeta)|(1-|\zeta|^2)^\alpha, \quad z\in B_n. \tag{2.1}$$

First we consider the case $1<p<\infty$. In [**1**] we can see that

$$\|P_\gamma f\|_{L^p_\gamma(B_n)} \lesssim \|f\|_{L^p_\gamma(B_n)} \quad \text{for} \quad 1<p<\infty. \tag{2.2}$$

By (2.1) and (2.2), we get the result for the case $1<p<\infty$.

Now we consider the case $p=1$. By (2.1), it follows that

$$\begin{aligned} \|P_\gamma f\|_{L^1_\gamma(B_n)} &= \int_{B_n} |P_\gamma f(z)|(1-|z|^2)^\gamma dV(z) \\ &\lesssim \int_{B_n} (1-|z|^2)^{\gamma-\alpha} dV(z). \end{aligned} \tag{2.3}$$

Since $0<\alpha-\gamma<1$, the last integral is bounded by the constant depending on γ,α, and n. By (2.1) and (2.3), we get the result for the case $p=1$. Therefore the result holds for all cases $1\leqslant p<\infty$. ∎

Theorem 2.3. *For $1\leqslant p<\infty$, $\gamma>-1$, and $0<\alpha<\gamma+1$, the weighted Bergman projection P_γ maps $L^{p,\alpha}_\gamma(D^n)$ onto $A^{p,\alpha}_\gamma(D^n)$, boundedly.*

Proof: In [**3**] we can see that

$$\|P_\gamma f\|_{L^p_\gamma(D^n)} \lesssim \|f\|_{L^p_\gamma(D^n)} \quad \text{for} \quad 1<p<\infty.$$

By the similar method as the proof of Theorem 2.2, we can get the result. ∎

3. Hölder boundedness

In order to prove that a function belongs to a Lipschitz space Λ_α we shall use the following Hardy-Littlewood type lemma.

Lemma 3.1. *Let $\Omega \Subset \mathbb{C}^n$ be a domain with piecewise smooth boundary. Suppose $f\in C^1(\Omega)$ and that for some $0<\alpha<1$ there is a constant C, such that*

$$|\nabla f(z)| \leqslant C\,\delta_\Omega(z)^{\alpha-1} \quad \text{for all} \quad z\in\Omega,$$

where $\delta_\Omega(z)$ is the distance function for Ω. Then $f\in\Lambda_\alpha(\Omega)$.

The proof of the above lemma and of more general results about the Lipschitz spaces can be found in [**5**].

THEOREM 3.2. *Suppose $0 < \alpha < 1$. Then the weighted Bergman projection P_γ maps $\Lambda_\alpha(B_n)$ onto $\Lambda_\alpha(B_n)$, boundedly.*

PROOF: By symmetry, for $z = (z_1, \cdots, z_n) \in B_n$, it suffices to treat the case $j = 1$, that is,

$$\left|\frac{\partial}{\partial z_1} P_\gamma f(z)\right| \lesssim |f|_{\Lambda_\alpha(B_n)} \left(1 - |z|^2\right)^{\alpha - 1}. \tag{3.1}$$

By (1.1), we have

$$\begin{aligned}\frac{\partial}{\partial z_1} P_\gamma f(z) &= C_{n,\gamma} \int_{B_n} \frac{f(\zeta)\overline{\zeta}_1}{\left(1 - \overline{\zeta} \cdot z\right)^{n+\gamma+2}} \left(1 - |\zeta|^2\right)^\gamma \, dV(\zeta) \\ &= C_{n,\gamma} \int_{B_n} \frac{\left(f(\zeta) - f(z)\right)\overline{\zeta}_1}{\left(1 - \overline{\zeta} \cdot z\right)^{n+\gamma+2}} \left(1 - |\zeta|^2\right)^\gamma \, dV(\zeta) \\ &\qquad + C_{n,\gamma} \int_{B_n} \frac{f(z)\overline{\zeta}_1}{\left(1 - \overline{\zeta} \cdot z\right)^{n+\gamma+2}} \left(1 - |\zeta|^2\right)^\gamma \, dV(\zeta) \\ &= I(z) + II(z).\end{aligned}$$

Since

$$C_{n,\gamma} \int_{B_n} \frac{1}{\left(1 - \overline{\zeta} \cdot z\right)^{n+1+\gamma}} \left(1 - |\zeta|^2\right)^\gamma \, dV(\zeta) = 1,$$

we have, by differentiating the integral above with respect to z_1,

$$C_{n,\gamma} \int_{B_n} \frac{\overline{\zeta}_1}{\left(1 - \overline{\zeta} \cdot z\right)^{n+2+\gamma}} \left(1 - |\zeta|^2\right)^\gamma \, dV(\zeta) = 0,$$

and we then have $II(z) = 0$.

Now $|\zeta - z|/|1 - \overline{\zeta} \cdot z| < 1$, and using the property (i) of Lemma 2.1, we have

$$\begin{aligned}|I(z)| &\lesssim \int_{B_n} \frac{|\zeta - z|^\alpha |f|_{\Lambda_\alpha}}{|1 - \overline{\zeta} \cdot z|^{n+\gamma+2}} \left(1 - |\zeta|^2\right)^\gamma \, dV(\zeta) \\ &\leqslant |f|_{\Lambda_\alpha(B_n)} \int_{B_n} \frac{\left(1 - |\zeta|^2\right)^\gamma}{|1 - \overline{\zeta} \cdot z|^{n+\gamma+2-\alpha}} \, dV(\zeta) \\ &\lesssim |f|_{\Lambda_\alpha(B_n)} \frac{1}{\left(1 - |z|^2\right)^{1-\alpha}}.\end{aligned}$$

Thus we get (3.1). ∎

We consider the case of the unit polydisc, it can be treated in the same way as in the proof of Theorem 3.2.

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Theorem 3.3. *Suppose $0 < \beta < \alpha < 1$. Then the weighted Bergman projection P_γ maps $\Lambda_\alpha(D^n)$ onto $\Lambda_\beta(D^n)$, boundedly.*

Proof: By (1.2), we have, by the same process as in the proof of Theorem 3.2,

$$\frac{\partial}{\partial z_1} P_\gamma f(z) = C_{n,\gamma} \int_{D^n} f(\zeta) \frac{(\gamma+2)\overline{\zeta}_1 (1-|\zeta_1|^2)^\gamma}{(1-\overline{\zeta}_1 z_1)^{\gamma+3}} \prod_{j=2}^n \frac{(1-|\zeta_j|^2)^\gamma}{(1-\overline{\zeta_j} z_j)^{\gamma+2}}\, dV(\zeta)$$
$$= C_{n,\gamma}(\gamma+2) \int_{D^n} \frac{\overline{\zeta}_1 (f(\zeta) - f(z_1, \zeta_2, \cdots, \zeta_n))(1-|\zeta_1|^2)^\gamma}{(1-\overline{\zeta}_1 z_1)^{\gamma+3}}$$
$$\times \prod_{j=2}^n \frac{(1-|\zeta_j|^2)^\gamma}{(1-\overline{\zeta_j} z_j)^{\gamma+2}}\, dV(\zeta).$$

Then, by (i) and (ii) of Lemma 2.1, we have

(3.2)
$$\left|\frac{\partial}{\partial z_1} P_\gamma f(z)\right| \lesssim |f|_{\Lambda_\alpha(D^n)} \int_{D^n} \frac{|\zeta_1 - z_1|^\alpha (1-|\zeta_1|^2)^\gamma}{|1-\overline{\zeta}_1 z_1|^{\gamma+3}} \prod_{j=2}^n \frac{(1-|\zeta_j|^2)^\gamma}{|1-\overline{\zeta_j} z_j|^{\gamma+2}}\, dV(\zeta)$$
$$\lesssim |f|_{\Lambda_\alpha(D^n)} \frac{1}{(1-|z_1|^2)^{1-\alpha}} \prod_{j=2}^n \log \frac{1}{1-|z_j|^2}.$$

Let $0 < \varepsilon < \alpha$. Then it follows that

(3.3)
$$\frac{1}{(1-|z_1|^2)^{1-\alpha}} \prod_{j=2}^n \log \frac{1}{1-|z_j|^2} \lesssim \frac{1}{\min_{1 \leqslant j \leqslant n} (1-|z_j|^2)^{1-\alpha+\varepsilon}}$$
$$\lesssim \frac{1}{\delta_{D^n}(z)^{1-\alpha+\varepsilon}}.$$

By (3.2) and (3.3), we have

$$\left|P_\gamma f(z)\right| \lesssim |f|_{\Lambda_\alpha(D^n)} \frac{1}{\delta_{D^n}(z)^{1-\alpha+\varepsilon}}.$$

Thus we get the result. ∎

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