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# ON THE DENSITY OF BOUNDED BASES

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*Abstract* For a nonempty set $A$ of integers and an integer $n$, let $r_A(n)$ be the number of representations of $n$ in the form $n = a + a'$, where $a \leqslant a'$ and $a, a' \in A$, and $d_A(n)$ be the number of representations of $n$ in the form $n = a - a'$, where $a, a' \in A$. The binary support of a positive integer $n$ is defined as the subset $S(n)$ of nonnegative integers consisting of the exponents in the binary expansion of $n$, i.e., $n = \sum_{i\in S(n)} 2^i$, $S(-n) = -S(n)$ and $S(0) = \emptyset$. For real number $x$, let $A(-x, x)$ be the number of elements $a \in A$ with $-x \leqslant a \leqslant x$. The famous Erdős-Turán Conjecture states that if $A$ is a set of positive integers such that $r_A(n) \geqslant 1$ for all sufficiently large $n$, then $\limsup_{n\to\infty} r_A(n) = \infty$. In 2004, Nešetřil and Serra initially introduced the notation of "bounded" property and confirmed the Erdős-Turán conjecture for a class of *bounded* bases. They also proved that, there exists a set $A$ of integers satisfying $r_A(n) = 1$ for all integers $n$ and $|S(x) \bigcup S(y)| \leqslant 4|S(x + y)|$ for $x, y \in A$. On the other hand, Nathanson proved that there exists a set $A$ of integers such that $r_A(n) = 1$ for all integers $n$ and $2\log x/\log 5 + c_1 \leqslant A(-x, x) \leqslant 2\log x/\log 3 + c_2$ for all $x \geqslant 1$, where $c_1, c_2$ are absolute constants. In this paper, following these results, we prove that, there exists a set $A$ of integers such that: $r_A(n) = 1$ for all integers $n$ and $d_A(n) = 1$ for all positive integers $n$, $|S(x) \bigcup S(y)| \leqslant 4|S(x + y)|$ for $x, y \in A$ and $A(-x, x) > (4/\log 5)\log\log x + c$ for all $x \geqslant 1$, where $c$ is an absolute constant. Furthermore, we also construct a family of arbitrarily spare such sets $A$.

*Keywords:* binary support; bounded basis; representation function; density

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## 1. Introduction

For nonempty sets $A, B$ of integers, define

$$A + A = \{a + a' : a, a' \in A\} \quad \text{and} \quad A - A = \{a - a' : a, a' \in A\}.$$

Let $\mathbb{Z}$ be the set of integers and $\mathbb{N}$ the set of positive integers. For any integer $n$, let $r_A(n)$ be the number of representations of $n$ in the form $n = a + a'$, where $a \leqslant a'$ and $a, a' \in A$, and $d_A(n)$ be the number of representations of $n$ in the form $n = a - a'$, where $a, a' \in A$. Clearly, $d_A(-n) = d_A(n)$ for any positive integer $n$. Let $|A|$ be the cardinality of the set $A$ and $\max A$ be the maximal element in $A$. For a real number $x$, denote $|x|$ by the absolute value of $x$, $\lfloor x \rfloor$ by the largest integer no larger than $x$, $A + x = \{a + x : a \in A\}$, and $A(-x, x)$ by the number of elements $a \in A$ with $-x \leqslant a \leqslant x$.

The famous Erdős-Turán Conjecture [3] states that if $A$ is a set of positive integers such that $r_A(n) \geqslant 1$ for all sufficiently large $n$, then

$$\limsup_{n\to\infty} r_A(n) = \infty.$$

In 2004, Nešetřil and Serra [6] initially introduced the notation of "bounded" property. For a positive integer $n$, denote the *binary support* of $n$ by the subset $S(n)$ of nonnegative integers consisting of the exponents in the binary expansion of $n$, i.e., $n = \sum_{i\in S(n)} 2^i$, and $S(-n) = -S(n)$. Define $S(0) = \emptyset$. A set $A$ of integers is called *bounded* if there is a function $f : \mathbb{N}\bigcup\{0\} \to \mathbb{N}\bigcup\{0\}$ such that $f(0) = 0$ and for each $n \in A + A$ there exists a pair $x, y \in A$ with

$$n = x + y, \qquad |S(x)\bigcup S(y)| \leqslant f(|S(n)|).$$

Obviously, if $A$ is a set of positive integers and the binary expansion of each element in $A$ has no two consecutive 1's, then $A$ is a bounded set with $f(n) = n$. Nešetřil and Serra [6] confirmed the Erdős-Turán conjecture for a class of "bounded" bases.

For a set $A$ of integers, $A$ is *a basis* for $\mathbb{Z}$ if $r_A(n) \geqslant 1$ for all integers $n$ and *a unique representation basis* for $\mathbb{Z}$ if $r_A(n) = 1$ for all integers $n$. For the unique representation basis for $\mathbb{Z}$, by considering the *bounded* property, Nešetřil and Serra [6] also obtained the following result:

**Theorem A. ([6, Theorem 5]).** *There is a bounded basis $A$ of $\mathbb{Z}$ satisfying $r_A(n) = 1$ for each $n \in \mathbb{Z}$.*

*Recently, the author [4] generalized the above result by adding the restriction that $d_A(n) = 1$ for all positive integers $n$. On the other hand, research on the density of basis also attracts much interest from experts. In 2003, Nathanson [5] considered the existence of unique representation basis $A$ with logarithmic growth, that is:*

**Theorem B. ([5, Theorem 2]).** *There is a unique representation basis $A$ for $\mathbb{Z}$ such that*

$$\frac{2\log x}{\log 5} + 2(1 - \frac{\log 3}{\log 5}) \leqslant A(-x, x) \leqslant \frac{2\log x}{\log 3} + 2$$

*for all $x \geqslant 1$.*

*Afterwards, Xiong and Tang [7] extended Theorem B by considering the structure of difference, and constructed a unique representation basis $A$ of integers such that $d_A(n) = 1$ for all positive integers $n$ and*

$$\frac{4(\log x - \log 2)}{\log 15} - 1 < A(-x, x) < \frac{4(\log x - \log 2)}{\log 3} + 7$$

*for all $x > 1$.*

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In this paper, based on the above results, we incorporate the *bounded* property and prove that:

**Theorem 1.1.** *There exists a bounded basis $A$ of integers such that $r_A(n) = 1$ for all integers $n$ and $d_A(n) = 1$ for all positive integers $n$, and*

$$A(-x, x) > \frac{4}{\log 5} \log\log x + c \ \text{ for all } \ x \geqslant 1,$$

*where $c$ is an absolute constant.*

On the other hand, similar to [5] and [7], we also obtain the following result:

**Theorem 1.2.** *Let $f(x)$ be a function such that $\lim_{x\to\infty} f(x) = \infty$. Then there exists a bounded basis $A$ of integers such that $r_A(n) = 1$ for all integers $n$ and $d_A(n) = 1$ for all positive integers $n$, and*

$$A(-x, x) \leqslant f(x) \ \text{ for all sufficiently large } \ x.$$

Furthermore, noting that if $r_A(n) = 2$ for infinitely many integers, then $d_A(n) \geqslant 2$ for infinitely many integers $n$, Cilleruelo and Nathanson [2] posed the following problem:
**Cilleruelo-Nathanson Problem.** Give general conditions for functions $f_1$ and $f_2$ to assure that there exists a set $A$ such that $d_A(n) \equiv f_1(n)$ and $r_A(n) \equiv f_2(n)$. Is the condition $\liminf_{u\to\infty} f_1(u) \geqslant 2$ and $\liminf_{|u|\to\infty} f_2(u) \geqslant 2$ sufficient?

In 2011, Y.G. Chen and the author [1] answered this problem affirmatively. In this paper, we also consider the *bounded* property and obtain that:

**Theorem 1.3.** *If two functions $f_1 : \mathbb{N} \to \mathbb{N}$ and $f_2 : \mathbb{Z} \to \mathbb{N}$ satisfy that $\liminf_{u\to\infty} f_1(u) \geqslant 2$ and $\liminf_{|u|\to\infty} f_2(u) \geqslant 2$, then there exists a bounded set $A$ of integers such that $d_A(n) = f_1(n)$ for all $n \in \mathbb{N}$ and $r_A(n) = f_2(n)$ for all $n \in \mathbb{Z}$.*

## 2. Proof of Theorem 1.1 and Theorem 1.2

The main idea is from [5]-[7]. During the induction process, we focus on the choice of critical values. Denote $\sigma(n)$ by

$$S(\sigma(n)) = \{i \in S(n) : i - 1 \notin S(n)\} \ \text{ for positive integer } \ n$$

and

$$S(\sigma(n)) = \{i \in S(n) : i + 1 \notin S(n)\} \ \text{ for negative integer } \ n.$$

It easily follows from the definition of $\sigma(n)$ that $|S(n + \sigma(n))| = |S(\sigma(n))| \leqslant |S(n)|$, $S(\sigma(n))$ and $S(n + \sigma(n))$ has no two consecutive integers.

**Lemma 2.1. ([4, Lemma 2.1]).** *Let $x, y, z$ be integers with $yz > 0$ such that*

*(i)* $|S(|y|)| \leqslant |S(|z|)|$;

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*(ii)* $a > b$ *for any* $a \in S(|z|)$ *and* $b \in S(|x|) \bigcup S(|y|)$*;*
*(iii)* *each of* $S(x)$*,* $S(y)$ *and* $S(z)$ *has no two consecutive integers.*

*Then*

$$|S(x) \bigcup S(y+z)| \leqslant 4|S(x+y+z)|.$$

**Proof of Theorem 1.1.** We will construct finite sets of integers $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k \subseteq \cdots$ such that for any positive integer $k$, we have:

(i) $|A_k| = 4k+3$;
(ii) $r_{A_k}(n) \leqslant 1$ for all $n \in \mathbb{Z}$ and $d_{A_k}(n) \leqslant 1$ for all $n \in \mathbb{N}$;
(iii) $r_{A_k}(n) = 1$ for all $n \in \mathbb{Z}$ with $|n| \leqslant \lfloor \frac{k}{2} \rfloor$ and $d_{A_k}(n) = 1$ for all $n \in \mathbb{N}$ with $1 \leqslant n \leqslant k$;
(iv) $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in A_k$;
(v) the binary support of each element in $A_k$ has no two consecutive integers;
(vi) $d_k < 172 d_{k-1}^5$, where $d_k = \max\{|a| : a \in A_k\}$ and $d_0 = 1$.

Let $A_1 = \{-32, -10, 0, 9, 33, 128, 129\}$. Then $d_1 = 129$, $r_{A_1}(0) = 1$, $r_{A_1}(1) = r_{A_1}(-1) = d_{A_1}(1) = 1$, $r_{A_1}(n) \leqslant 1$ for all $n \in \mathbb{Z}$ and $d_{A_1}(n) \leqslant 1$ for all $n \in \mathbb{N}$, $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in A_1$, and the binary support of each element in $A_1$ has no two consecutive integers. Thus, (i)-(vi) hold for $k = 1$.

Assume that we have already obtained a set $A_k$ of integers satisfying (i)-(vi) for some positive integer $k$. By the definition of $d_k$ we know that $A_k \subseteq [-d_k, d_k]$. Since $r_{A_k}(0) \leqslant 1$, we have $d_k \in A_k$ and $-d_k \notin A_k$, or $-d_k \in A_k$ and $d_k \notin A_k$. Thus, $A_k + A_k \subseteq [-2d_k + 2, 2d_k]$ or $[-2d_k, 2d_k - 2]$. In any case, we have $A_k - A_k \subseteq [-2d_k+1, 2d_k-1]$. Write

$$u_k = \min\{|n| : n \notin A_k + A_k\}, \quad v_k = \min\{n > 0 : n \notin A_k - A_k\}.$$

It follows that

$$2 \leqslant u_k \leqslant 2d_k - 1, \quad 2 \leqslant v_k \leqslant 2d_k.$$

Let

$$a_k = \lceil \max\{\log_2 \left(\frac{3d_k + 1 - \sigma(u_k)}{4^{|S(u_k)|-1}}\right), \log_2 d_k + 1, \max S(u_k) + 3\} \rceil, \tag{2.1}$$

where $\lceil x \rceil$ is the least integer no less than $x$. Then

$$\sigma(u_k) + 2^{a_k} 4^{|S(u_k)|-1} \geqslant 3d_k + 1.$$

Take

$$x_k = u_k + \sigma(u_k) + 2^{a_k} + 2^{a_k+2} + 2^{a_k+4} + \cdots + 2^{a_k + 2(|S(u_k)|-1)}. \tag{2.2}$$

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Thus, $x_k - u_k \geqslant 3d_k + 1$. Furthermore,

$$x_k = u_k + \sigma(u_k) + 2^{a_k}\frac{4^{|S(u_k)|} - 1}{3} < 4d_k + \frac{1}{3}4^{|S(u_k)|}2^{a_k}, \tag{2.3}$$

where the last inequality is based on the facts that $\sigma(u_k) \leqslant u_k$ and $u_k \leqslant 2d_k - 1$. If $a_k = \lceil \log_2\left(\frac{3d_k+1-\sigma(u_k)}{4^{|S(u_k)|-1}}\right)\rceil$, then

$$4d_k + \frac{1}{3}4^{|S(u_k)|}2^{a_k} < 4d_k + \frac{1}{3}4^{|S(u_k)|}(2\cdot\frac{3d_k}{4^{|S(u_k)|-1}}) = 12d_k.$$

If $a_k = \lceil \log_2 d_k + 1\rceil$, then by $|S(u_k)| \leqslant \max S(u_k) + 1$ and $2^{\max S(u_k)} \leqslant u_k$ we know that

$$\begin{aligned}4d_k + \frac{1}{3}4^{|S(u_k)|}2^{a_k} &< 4d_k + \frac{1}{3}4^{|S(u_k)|}(2\cdot 2^{\log_2 d_k+1})\\ &\leqslant 4d_k + \frac{4}{3}d_k 4^{\max S(u_k)+1} \leqslant 4d_k + \frac{16}{3}d_k u_k^2\\ &\leqslant 4d_k + \frac{64}{3}d_k^3 < 22d_k^3.\end{aligned}$$

If $a_k = \max S(u_k) + 3$, then

$$\begin{aligned}4d_k + \frac{1}{3}4^{|S(u_k)|}2^{a_k} &= 4d_k + \frac{8}{3}4^{|S(u_k)|}2^{\max S(u_k)} \leqslant 4d_k + \frac{32}{3}2^{3\max S(u_k)}\\ &\leqslant 4d_k + \frac{32}{3}u_k^3 \leqslant 86d_k^3.\end{aligned}$$

In any case,

$$4d_k + \frac{1}{3}4^{|S(u_k)|}2^{a_k} \leqslant 86d_k^3. \tag{2.4}$$

It infers from (2.1) and (2.3) that

$$x_k < 86d_k^3.$$

Let

$$b_k = \lceil \max\{\log_2\left(\frac{3x_k + 2u_k - \sigma(v_k)}{4^{|S(v_k)|-1}}\right), \max S(v_k) + 3, a_k + 2|S(u_k)| - 1\}\rceil. \tag{2.5}$$

Then

$$\sigma(v_k) + 2^{b_k}4^{|S(v_k)|-1} \geqslant 3x_k + 2u_k.$$

Take

$$y_k = \sigma(v_k) + 2^{b_k} + 2^{b_k+2} + 2^{b_k+4} + \cdots + 2^{b_k+2(|S(v_k)|-1)}. \tag{2.6}$$

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Thus, $y_k \geqslant 3x_k + 2u_k$. Furthermore,

$$y_k + v_k = v_k + \sigma(v_k) + 2^{b_k}\frac{4^{|S(v_k)|} - 1}{3} < 4d_k + \frac{1}{3}4^{|S(v_k)|}2^{b_k}, \tag{2.7}$$

where the last inequality is based on the facts that $\sigma(v_k) \leqslant v_k$ and $v_k \leqslant 2d_k$. If $b_k = \lceil\log_2\left(\frac{3x_k+2u_k-\sigma(v_k)}{4^{|S(v_k)|-1}}\right)\rceil$, then by $x_k < 86d_k^3$ we know that

$$4d_k + \frac{1}{3}4^{|S(v_k)|}2^{b_k} < 4d_k + \frac{1}{3}4^{|S(v_k)|}(2\cdot\frac{3x_k + 2u_k}{4^{|S(v_k)|-1}}) = 4d_k + \frac{8}{3}(3x_k + 2u_k) < 689d_k^3.$$

If $b_k = \max S(v_k)+3$, then we could deduce from $|S(v_k)| \leqslant \max S(v_k)+1$ and $2^{\max S(v_k)} \leqslant v_k$ that

$$\begin{aligned} 4d_k + \frac{1}{3}4^{|S(v_k)|}2^{b_k} &= 4d_k + \frac{8}{3}4^{|S(v_k)|}2^{\max S(v_k)} \leqslant 4d_k + \frac{32}{3}2^{3\max S(v_k)} \\ &\leqslant 4d_k + \frac{32}{3}v_k^3 \leqslant 86d_k^3. \end{aligned}$$

If $b_k = a_k + 2|S(u_k)| - 1$, then it infers from (2.4) that

$$\begin{aligned} 4d_k + \frac{1}{3}4^{|S(v_k)|}2^{b_k} &= 4d_k + \frac{1}{3}4^{|S(u_k)|}2^{a_k}\cdot\frac{1}{2}4^{|S(v_k)|} \leqslant 86d_k^3\cdot\frac{1}{2}4^{|S(v_k)|} \\ &\leqslant 86d_k^3\cdot\frac{1}{2}(2d_k)^2 = 172d_k^5. \end{aligned}$$

In any case,

$$4d_k + \frac{1}{3}4^{|S(v_k)|}2^{b_k} \leqslant 172d_k^5.$$

It infers from (2.5) and (2.7) that

$$y_k + v_k < 172d_k^5.$$

To sum up,

$$3d_k < x_k - u_k < x_k < y_k < y_k + v_k < 172d_k^5. \tag{2.8}$$

Now we divide into the following two cases according to $u_k \notin A_k + A_k$ or $u_k \in A_k + A_k$.

**Case 1.** $u_k \notin A_k + A_k$.

Let

$$B_{k+1} = A_k \bigcup \{x_k, -x_k + u_k\} \quad \text{and} \quad A_{k+1} = B_{k+1} \bigcup \{y_k, y_k + v_k\}.$$

It follows from $x_k > x_k - u_k > 3d_k$, $3x_k + 2u_k \leqslant y_k < y_k + v_k$ and the definitions of $d_k$, $x_k$, $y_k$, we know that $r_{A_{k+1}}(u_k) = d_{A_{k+1}}(v_k) = 1$, $r_{A_{k+1}}(n) \leqslant 1$ for all $n \in \mathbb{Z}$ and $d_{A_{k+1}}(n) \leqslant$

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1 for all $n \in \mathbb{N}$. Thus, (ii) holds. By $a_k \geqslant \max S(u_k) + 3$, $b_k \geqslant \max S(v_k) + 3$ and the definition of $A_{k+1}$ we know that (i) and (v) hold. We will prove that $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in A_{k+1}$. If $x = y$, then

$$|S(x) \bigcup S(y)| = |S(x)| = |S(2x)| = |S(x+y)| \leqslant 4|S(x+y)|.$$

So we only need to consider $x \neq y$.

Firstly, we will prove that $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in B_{k+1}$ with $x \neq y$. Noting that

$$\begin{aligned} |S(x_k)| &\leqslant |S(u_k + \sigma(u_k))| + \left|S\left(2^{a_k} + 2^{a_k+2} + 2^{a_k+4} + \cdots + 2^{a_k+2(|S(u_k)|-1)}\right)\right| \\ &= |S(\sigma(u_k))| + |S(u_k)| \leqslant 2|S(u_k)| \end{aligned}$$

and

$$\begin{aligned} |S(-x_k + u_k)| &= \left|S\left(\sigma(u_k) + 2^{a_k} + 2^{a_k+2} + 2^{a_k+4} + \cdots + 2^{a_k+2(|S(u_k)|-1)}\right)\right| \\ &\leqslant |S(\sigma(u_k))| + |S(u_k)| \leqslant 2|S(u_k)|, \end{aligned}$$

we have

$$|S(x_k) \bigcup S(-x_k + u_k)| \leqslant 4|S(u_k)| = 4|S(x_k + (-x_k + u_k))|.$$

Let $x \in A_k$. By (2.1) we have $a_k \geqslant \log_2 d_k + 1$, then $a_k > \max S(|x|)$. Also by (2.1), we know that $a_k \geqslant \max S(u_k) + 3$. Taking $y_1 = u_k + \sigma(u_k)$ and $z_1 = 2^{a_k} + 2^{a_k+2} + 2^{a_k+4} + \cdots + 2^{a_k+2(|S(u_k)|-1)}$ in Lemma 2.1, we have

$$|S(x) \bigcup S(x_k)| = |S(x) \bigcup S(y_1 + z_1)| \leqslant 4|S(x + y_1 + z_1)| = 4|S(x + x_k)|.$$

Taking $y_2 = -\sigma(u_k)$ and $z_2 = -(2^{a_k} + 2^{a_k+2} + 2^{a_k+4} + \cdots + 2^{a_k+2(|S(u_k)|-1)})$ in Lemma 2.1, we have

$$|S(x) \bigcup S(-x_k + u_k)| = |S(x) \bigcup S(y_2 + z_2)| \leqslant 4|S(x + y_2 + z_2)| = 4|S(x - x_k + u_k)|.$$

Thus, $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in B_{k+1}$ with $x \neq y$.

Now, we will prove that $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in A_{k+1}$ with $x \neq y$. It follows from (2.6) that

$$\begin{aligned} |S(y_k)| &= \left|S\left(\sigma(v_k) + 2^{b_k} + 2^{b_k+2} + 2^{b_k+4} + \cdots + 2^{b_k+2(|S(v_k)|-1)}\right)\right| \\ &\leqslant |S(\sigma(v_k))| + \left|S\left(2^{b_k} + 2^{b_k+2} + 2^{b_k+4} + \cdots + 2^{b_k+2(|S(v_k)|-1)}\right)\right| \\ &= |S(\sigma(v_k))| + |S(v_k)| \leqslant 2|S(v_k)| \end{aligned}$$

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and

$$|S(y_k + v_k)| = \left|S\left(v_k + \sigma(v_k) + 2^{b_k} + 2^{b_k+2} + 2^{b_k+4} + \cdots + 2^{b_k+2(|S(v_k)|-1)}\right)\right|$$
$$\leqslant |S(v_k + \sigma(v_k))| + |S(v_k)| \leqslant 2|S(v_k)|,$$

we have

$$|S(y_k) \bigcup S(y_k + v_k)| \leqslant 4|S(v_k)|.$$

By $b_k \geqslant \max S(v_k) + 3$ and

$$S(2y_k + v_k) = S(2\sigma(v_k) + v_k + 2\left(2^{b_k} + 2^{b_k+2} + 2^{b_k+4} + \cdots + 2^{b_k+2(|S(v_k)|-1)}\right)),$$

we know that

$$|S(2y_k + v_k)| = |S(2\sigma(v_k) + v_k)| + \left|S\left(2^{b_k} + 2^{b_k+2} + 2^{b_k+4} + \cdots + 2^{b_k+2(|S(v_k)|-1)}\right)\right|$$
$$\geqslant \left|S\left(2^{b_k} + 2^{b_k+2} + 2^{b_k+4} + \cdots + 2^{b_k+2(|S(v_k)|-1)}\right)\right| = |S(v_k)|.$$

Thus,

$$|S(y_k) \bigcup S(y_k + v_k)| \leqslant 4|S(2y_k + v_k)|.$$

Let $x \in B_{k+1}$. By (2.5) we have $b_k \geqslant a_k + 2|S(u_k)| - 1$, namely, $b_k > a_k + 2(|S(u_k)| - 1)$. Then $b_k > \max S(|x|)$. Also by (2.5), we know that $b_k \geqslant \max S(v_k)+3$. Taking $y_1 = \sigma(v_k)$ and $z_1 = 2^{b_k} + 2^{b_k+2} + 2^{b_k+4} + \cdots + 2^{b_k+2(|S(v_k)|-1)}$ in Lemma 2.1, we have

$$|S(x) \bigcup S(y_k)| = |S(x) \bigcup S(y_1 + z_1)| \leqslant 4|S(x + y_1 + z_1)| = 4|S(x + y_k)|.$$

Taking $y_2 = \sigma(v_k)+v_k$ and $z_2 = 2^{b_k}+2^{b_k+2}+2^{b_k+4}+\cdots+2^{b_k+2(|S(v_k)|-1)}$ in Lemma 2.1, we have

$$|S(x) \bigcup S(y_k + v_k)| = |S(x) \bigcup S(y_2 + z_2)| \leqslant 4|S(x + y_2 + z_2)| = 4|S(x + y_k + v_k)|.$$

To sum up, $|S(x) \bigcup S(y)| \leqslant 4|S(x + y)|$ for $x, y \in A_{k+1}$.

**Case 2.** $u_k \in A_k + A_k$. Then $-u_k \notin A_k + A_k$.

Let

$$B_{k+1} = \{x_k, -x_k - u_k\} \quad \text{and} \quad A_{k+1} = \{y_k, y_k + v_k\},$$

where $x_k$ and $y_k$ are defined in (2.2) and (2.6). Similar to Case 1, we know that $A_{k+1}$ satisfies (i)-(ii), (iv)-(v) and $r_{A_{k+1}}(-u_k) = d_{A_{k+1}}(v_k) = 1$.

In both cases, it follows from (2.8) and the construction of $A_{k+1}$ that $d_{k+1} < 172d_k^5$. Thus, (vi) holds.

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Now we will prove that (iii) holds. (The proof of (iii) is the same as in [7, Theorem 1.1], we also give the details for the sake of completeness). If $u_k \notin A_k + A_k$, then by Case 1 we know that $u_k \in A_{k+1} + A_{k+1}$, thus, $u_{k+2} \geqslant u_{k+1} > u_k$ if $-u_k \in A_{k+1} + A_{k+1}$. Otherwise, if $-u_k \notin A_{k+1} + A_{k+1}$, then $u_{k+1} = u_k \in A_{k+1} + A_{k+1}$ and $-u_{k+1} \in A_{k+2} + A_{k+2}$ by Case 2, thus, $u_{k+2} > u_{k+1} = u_k$. If $u_k \in A_k + A_k$, then by Case 2 we know that $-u_k \in A_{k+1} + A_{k+1}$. It follows from $u_k \in A_k + A_k \subseteq A_{k+1} + A_{k+1}$ that $u_{k+2} \geqslant u_{k+1} > u_k$. In both cases, $u_{k+2} > u_k$. It follows from $u_2 \geqslant 2$ that $u_{2k} \geqslant u_2 + k - 1 \geqslant k + 1$. Thus, for any positive integer $k$ we have

$$\{-k, \cdots, -1, 0, 1, \cdots, k\} \subseteq A_{2k} + A_{2k}.$$

Similarly, $v_k < v_{k+1}$. It infers from $v_1 \geqslant 2$ that $v_k \geqslant k + 1$. Thus, for any positive integer $k$ we have

$$\{-k, \cdots, -1, 0, 1, \cdots, k\} \subseteq A_k - A_k.$$

Namely, $r_{A_{k+1}}(n) = 1$ for all $n \in \mathbb{Z}$ with $|n| \leqslant \lfloor \frac{k+1}{2} \rfloor$ and $d_{A_{k+1}}(n) = 1$ for all $n \in \mathbb{N}$ with $1 \leqslant n \leqslant k + 1$. Thus, (iii) holds.

Let

$$A = \bigcup_{k=1}^{\infty} A_k.$$

Then $r_A(n) = 1$ for all $n \in \mathbb{Z}$ and $d_A(n) = 1$ for all $n \in \mathbb{N}$, $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in A$. Furthermore, we could deduce from (vi) and $d_0 = 1$ that

$$d_k \leqslant c_1^{5^k}, \quad \text{where} \quad c_1 = \sqrt[4]{172}.$$

For sufficiently large $x$, there exists a positive integer $k$ such that $d_k \leqslant x < d_{k+1}$. It follows from $4k + 3 \leqslant A(-x, x) \leqslant 4k + 7$ that

$$A(-x, x) > \frac{4}{\log 5} \log\log x + c, \quad \text{where} \;\; c \;\; \text{is an absolute constant.}$$

This completes the proof of Theorem 1.1. □

**Proof of Theorem 1.2.** In the proof of Theorem 1.1, the only constraint on the choice of $x_k$ (resp. $y_k$) is the size of the value $a_k$ (resp. $b_k$). The following proof is similar to [5, Theorem 1] and [7, Theorem 1.1]. We apply the method of Theorem 1.1 by replacing $a_k$ with $s_k (\geqslant a_k)$. Namely, take

$$x_k = u_k + \sigma(u_k) + 2^{s_k} + 2^{s_k+2} + 2^{s_k+4} + \cdots + 2^{s_k + 2(|S(u_k)|-1)}. \tag{2.9}$$

Given a function $f(x)$ tending to infinity, we shall take induction on $k$ to construct a non-decreasing sequence of integers $\{h_k\}_{k=1}^{\infty}$ such that $A(-x, x) \leqslant f(x)$ for all integers $x$

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with $h_1 \leqslant x \leqslant d_k$. Firstly, choose $h_1 \geqslant d_1$ so that $f(x) \geqslant 11$ for $x \geqslant h_1$. Then

$$A(-x, x) \leqslant 11 \leqslant f(x) \quad \text{for} \quad h_1 \leqslant x \leqslant d_2.$$

Suppose that for some integer $k \geqslant 2$, we have already selected an integer $h_{k-1} \geqslant d_{k-1}$ such that

$$f(x) \geqslant 4k+3 \quad \text{for} \quad x \geqslant h_{k-1}, \quad A(-x, x) \leqslant f(x) \quad \text{for} \quad h_1 \leqslant x \leqslant d_k.$$

Noting that $f(x)$ tends to infinity, there exist positive integers $h_k$ and $s_{k+1}$ with $h_k \geqslant d_k$ and $h_k < x_{k+1} - u_{k+1}$ (taking large $s_{k+1}$ in (2.9)) such that $f(x) \geqslant 4k+7$ for $x \geqslant h_k$. It follows that

$$A(-x, x) \leqslant 4k+7 \leqslant f(x) \quad \text{for} \quad h_k \leqslant x \leqslant d_{k+1}.$$

For $d_k \leqslant x \leqslant h_k$, we could deduce from the construction of $A_{k+1} \setminus A_k$ and the fact $h_k < x_{k+1} - u_{k+1}$ that

$$A(-x, x) = A_k(-x, x) = 4k+3 \leqslant f(x) \quad \text{for} \quad d_k \leqslant x \leqslant h_k.$$

To sum up,

$$A(-x, x) \leqslant f(x) \quad \text{for} \quad d_k \leqslant x \leqslant d_{k+1}.$$

By the induction hypothesis we know that $A(-x, x) \leqslant f(x)$ for $h_1 \leqslant x \leqslant d_{k+1}$. It follows that

$$A(-x, x) \leqslant f(x) \quad \text{for all} \quad x \geqslant h_1.$$

This completes the proof of Theorem 1.2.

## 3. Proof of Theorem 1.3

To give the proof of Theorem 1.3, we need the following preliminary lemmas. The idea is from [1, Theorem 1.2], [4, Theorem 1.1] and [6, Theorem 5].

**Lemma 3.1.** *Let $f_1 : \mathbb{N} \to \mathbb{N}$ and $f_2 : \mathbb{Z} \to \mathbb{N}$ be two functions such that*

$$\liminf_{u \to \infty} f_1(u) \geqslant 2 \quad \text{and} \quad \liminf_{|u| \to \infty} f_2(u) \geqslant 2. \tag{3.1}$$

*Let $B \subseteq \mathbb{Z}$ be a finite set with $|B| \geqslant 2$ such that:*

*(i) $d_B(n) \leqslant f_1(n)$ for all $n \in \mathbb{N}$ and $r_B(n) \leqslant f_2(n)$ for all $n \in \mathbb{Z}$;*
*(ii) $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in B$;*

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*(iii) the binary support of each element in $B$ has no two consecutive integers.*

*If $k$ is a positive integer with $d_B(k) < f_1(k)$, then there exists a finite set $D$ with $B \subseteq D \subseteq \mathbb{Z}$ such that:*

*(iv) $d_D(k) = d_B(k) + 1$;*

*(v) $d_D(n) \leqslant f_1(n)$ for all $n \in \mathbb{N}$ and $r_D(n) \leqslant f_2(n)$ for all $n \in \mathbb{Z}$;*

*(vi) $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in D$;*

*(vii) the binary support of each element in $D$ has no two consecutive integers.*

**Proof.** Let $B = \{b_1, b_2, \cdots, b_s\}$, where $b_1 < b_2 < \cdots < b_s$. Let $m = 2\max_{1\leqslant j\leqslant s} |b_j| + k$. By (3.1), we could choose a subset $U_k$ of positive integers such that:

(1) $|U_k| = |S(k)|$;
(2) $\min U_k > k + 3\max ||B||$, where $||B|| = \{|b| : b \in B\}$;
(3) $U_k$ has no two consecutive integers;
(4) $f_1(n) \geqslant 2$ and $f_2(n) \geqslant 2$ for all integers $n \in [b-m, b+m] \bigcap \mathbb{Z}$, where $b = \sigma(k) + \sum_{i\in U_k} 2^i$.

Let

$$D = B \bigcup \{b, b+k\}.$$

Then

$$D + D = \{2b, 2b+k, 2b+2k\} \bigcup \{B+B\} \bigcup \{B+b\} \bigcup \{B+b+k\}$$

and

$$D - D = \pm\{\{k\} \bigcup \{B-B\} \bigcup \{B-b\} \bigcup \{B-b-k\}\}.$$

We could deduce from (i)-(iii), the definition of $D$ and the fact $b > 3\max ||B|| + k$ that $d_D(k) = d_B(k) + 1$, and the binary support of each element in $D$ has no two consecutive integers. Furthermore, $r_D(2b) = r_D(2b+k) = r_D(2b+2k) = 1$. It also follows from

$$b - m < b + b_1 < b + b_2 < \cdots < b + b_s < b + m,$$
$$b - m < b + k + b_1 < b + k + b_2 < \cdots < b + k + b_s < b + m$$

and (4) that $r_D(n) \leqslant 2 \leqslant f_2(n)$ for each $n \in \{B+b\} \bigcup \{B+b+k\}$. Noting that the sets $\{2b, 2b+k, 2b+2k\}$, $B+B$, $B+b$ and $B+b+k$ are pairwise disjoint, we know that $r_D(n) \leqslant f_2(n)$ for all integers $n$. Similarly, by

$$b - m < b - b_s < b - b_{s-1} < \cdots < b - b_1 < b + m,$$
$$b - m < b + k - b_s < b + k - b_{s-1} < \cdots < b + k - b_1 < b + m,$$

and (4) we have $d_D(n) \leqslant 2 \leqslant f_1(n)$ for each $n \in \{B-b\} \bigcup \{B-b-k\}$. Noting that the sets $B-B$, $B-b$ and $B-b-k$ are pairwise disjoint, we know that $d_D(n) \leqslant f_1(n)$ for all positive integers $n$. By the same proof as in Theorem 1.1, we know that $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in D$.

This completes the proof of Lemma 3.1. □

**Lemma 3.2.** *Let $f_1 : \mathbb{N} \to \mathbb{N}$ and $f_2 : \mathbb{Z} \to \mathbb{N}$ be two functions such that (3.1) holds. Let $B \subseteq \mathbb{Z}$ be a finite set with $|B| \geqslant 2$ such that:*

*(i)* $d_B(n) \leqslant f_1(n)$ *for all* $n \in \mathbb{N}$ *and* $r_B(n) \leqslant f_2(n)$ *for all* $n \in \mathbb{Z}$*;*
*(ii)* $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ *for* $x, y \in B$*;*
*(iii) the binary support of each element in* $B$ *has no two consecutive integers.*
*If* $k$ *is an integer with* $r_B(k) < f_2(k)$*, then there exists a finite set* $D$ *with* $B \subseteq D \subseteq \mathbb{Z}$ *such that:*
*(iv)* $r_D(k) = r_B(k) + 1$*;*
*(v)* $d_D(n) \leqslant f_1(n)$ *for all* $n \in \mathbb{N}$ *and* $r_D(n) \leqslant f_2(n)$ *for all* $n \in \mathbb{Z}$*;*
*(vi)* $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ *for* $x, y \in D$*;*
*(vii) the binary support of each element in* $D$ *has no two consecutive integers.*

**Proof.** Let $B = \{b_1, b_2, \cdots, b_s\}$, where $b_1 < b_2 < \cdots < b_s$. Let $m = 2\max_{1 \leqslant j \leqslant s} |b_j| + |k|$. By (3.1), we could choose a subset $U_k$ of positive integers such that:

(1) $|U_k| = |S(k)|$;
(2) $\min U_k > |k| + 3\max ||B||$;
(3) $U_k$ has no two consecutive integers;
(4) $f_1(n) \geqslant 2$ and $f_2(n) \geqslant 2$ for all integers $n \in [b-m, b+m] \bigcap \mathbb{Z}$, where

$$b = \begin{cases} k + \sigma(k) + \sum_{i \in U_k} 2^i, & \text{if } k > 0, \\ \sum_{i \in U_k} 2^i, & \text{if } k = 0, \\ k + \sigma(k) + \sum_{i \in U_k} 2^{-i}, & \text{if } k < 0. \end{cases}$$

Let

$$D = B \bigcup \{b, -b+k\}.$$

Then

$$D + D = \{k, 2b, -2b+2k\} \bigcup (B+B) \bigcup (B+b) \bigcup (B-b+k)$$

and

$$D - D = \pm\{\{2b-k\} \bigcup (B-B) \bigcup (B-b) \bigcup (B+b-k)\}.$$

We could deduce from (i)-(iii), the definition of $D$ and the fact $b > 3\max ||B|| + k$ that $r_D(k) = r_B(k) + 1$, and the binary support of each element in $D$ has no two consecutive integers. Furthermore, $r_D(2b) = r_D(-2b+2k) = 1$. It also follows from

$$b - m < b + b_1 < b + b_2 < \cdots < b + b_s < b + m,$$
$$b - m < -b + k + b_1 < -b + k + b_2 < \cdots < -b + k + b_s < b + m$$

and (4) that $r_D(n) \leqslant 2 \leqslant f_2(n)$ for each $n \in \{B+b\} \bigcup \{B-b+k\}$. Noting that the sets $\{2b, -2b+2k\}$, $B+B$, $B+b$ and $B-b+k$ are pairwise disjoint, we know that

$r_D(n) \leqslant f_2(n)$ for all integers $n$. Similarly, by

$$-2b-k < b-m < b-b_s < b-b_{s-1} < \cdots < b-b_1 < b+m < 2b+k,$$
$$b-m < b-k+b_1 < b-k+b_2 < \cdots < b-k+b_s < b+m$$

and (4) we have $d_D(n) \leqslant 2 \leqslant f_1(n)$ for each $n \in \{B-b\} \bigcup \{B+b-k\}$. Noting that the sets $\{2b-k\}$, $B-B$, $B-b$ and $B+b-k$ are pairwise disjoint, we know that $d_D(n) \leqslant f_1(n)$ for all positive integers $n$. By the same proof as in Theorem 1.1, we know that $|S(x) \bigcup S(y)| \leqslant 4|S(x+y)|$ for $x, y \in D$.

This completes the proof of Lemma 3.2. □

**Remark 3.3.** During the proof of Lemma 3.1 and Lemma 3.2, since we do not need accurate quantitative estimation for $d_k$, we just choose sufficiently large $b$ in each stage.

**Proof of Theorem 1.3.** Theorem 1.3 follows from Lemma 3.1 and Lemma 3.2. The proof is similar to Theorem 1.1, we omit the detail here.

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