J. Austral. Math. Soc. **24** (Series A) (1977), 309–311.

NOTE ON COMPACT CLOSED CATEGORIES

B. J. DAY*

(Received 8 September 1976)

Communicated by R. H. Street

Abstract

Several categorical aspects of localisation to compact closed categories and free compact closed categories are discussed.

Introduction

The concept of a symmetric compact closed category was formalised by Kelly (1972). Generally speaking a compact bicategory is a bicategory in which each 1-cell has an adjoint. The details of this article can be followed through in this generality but we discuss, for simplicity, the "one-object" symmetric case over $\mathscr{E}ns$.

By way of introduction we repeat the brief survey of Kelly (1972). A compact closed category is a symmetric monoidal category $(\mathscr{A}, \otimes, I)$ and a functor $*: \mathscr{A}^{op} \to \mathscr{A}$ and natural transformations $g_A : I \to A \otimes A*$, $h_A : A * \otimes A \to I$ such that $(1 \otimes h)(g \otimes 1) = 1 : A \to A \otimes A * \otimes A \to A$ and $(h \otimes 1)(1 \otimes g) = 1 : A * \to A * \otimes A \otimes A * \to A *$. Such a category is closed, with $[A, B] = A * \otimes B$; moreover, since adjoints are unique, we have $A \cong A**$ for all $A \in \mathscr{A}$. Conversely, a monoidal closed category is compact exactly when the canonical transformation $\kappa : [A, I] \otimes A \to [A, A]$ is an isomorphism whereupon $A* = [A, I]$, h is evaluation and g is $I \to [A, A] = [A, A \otimes I]$ followed by κ^{-1}; as a consequence we have $A \cong [[A, I], I]$.

Perhaps the simplest non-trivial example of a compact closed category is the category of finite-dimensional vector spaces over a given field.

---

* The author gratefully acknowledges the support of a Postdoctoral Research Fellowship from the Australian Research Grants Committee.

Free compact closed categories and monadicity.

Let $\mathscr{SMC}$ denote the category of small symmetric monoidal closed categories and strict symmetric monoidal closed functors. Let $\mathscr{CMC}$ denote the full subcategory of small compact closed categories.

PROPOSITION 1. *The inclusion $\mathscr{CMC} \subset \mathscr{SMC}$ has a left adjoint.*

PROOF. We assign to each $\mathscr{A} = (\mathscr{A}, \otimes, I, \cdots) \in \mathscr{SMC}$ a universal compactification $C(\mathscr{A})$ together with a projection $P : \mathscr{A} \to C(\mathscr{A})$. Consider the class K of transformations $\kappa : [A, I] \otimes B \to [A, B]$ and let $\bar{K}$ be its monoidal closure: $\bar{K} = \{A \otimes \kappa ; A \in \mathscr{A}$ and $\kappa \in K\}$. Then the effect of forming the symmetric monoidal category $\mathscr{A}(\bar{K}^{-1})$ (Day (1973)) is equivalent to inverting the members of the class S comprising the transformations $\sigma : B \otimes [A, C] \to [A, B \otimes C]$; this fact can be verified by simple coherent diagrams. It can also be seen that S is in fact monoidal and that the transformations called Ten: $[A, B] \otimes [C, D] \to [A \otimes C, B \otimes D]$ are inverted. In particular the transformations $[A, I] \otimes [B, I] \to [A \otimes B, I]$ are inverted. Thus, if we write $A*$ for the image of $[A, I]$ under the projection $P : \mathscr{A} \to \mathscr{A}(S^{-1})$, we have $A * \otimes B * \cong (A \otimes B)*$. This means that *both* the functors $\otimes : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ and $[-, -] : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{A}$ factor to make $\mathscr{A}(S^{-1})$ a compact closed category. It also induces on P the structure of an $\mathscr{SMC}$ morphism. We write $C(\mathscr{A}) = \mathscr{A}(S^{-1})$. $\square$

REMARKS. The category $C(\mathscr{A})$ can be localised further to the "cancellative compactification" $C_c(\mathscr{A})$ of $\mathscr{A}$. This is formed by inverting, in addition to S, all the trasformations $A \otimes - : [B, C] \to [A \otimes B, A \otimes C]$. This process inverts all the transformations $A \otimes [A, B] \to B$ so that a cancellative compact closed category is a compact closed category for which $e : [A, B] \otimes A \to B$ is an isomorphism. In particular $A * \otimes A \cong A \otimes A * \cong I$ in $C_c(\mathscr{A})$. The isomorphism classes of $C_c(\mathscr{A})$ form a preordered abelian group. When this preorder is replaced by the trivial preorder we obtain $K_0(\mathscr{A})$ where K_0 is left adjoint to $\mathscr{Ab} \subset \mathscr{SMC}$. Thus $K_0(\mathscr{A})$ is universal for functions $f : |\mathscr{A}| \to G$. (G an abelian group) such that: (1) $A \cong B \Rightarrow fA = fB$, (2) $f(A \otimes B) = fA + fB$, (3) $f[A, B] = fB - fA$. If $\mathscr{A}$ is the free $\mathscr{SMC}$ category on a symmetric monoidal category $\mathscr{M}$ then $K_0(\mathscr{A}) = K_0(\mathscr{M}, \otimes)$ (see Swan (1968); also see Conway (1976)).

PROPOSITION 2. *$\mathscr{CMC}$ is monadic over $\mathscr{Cat}$ and has all small limits and colimits.*

PROOF. The forgetful functor $U : \mathscr{CMC} \to \mathscr{Cat}$ has a left adjoint by monadicity of $\mathscr{SMC}$ over $\mathscr{Cat}$ (see Lambek (1969)) and Proposition 1. The same technique as used by Lambek (1969) can be used here to show that U creates coequalisers of U-split pairs and coequalisers of reflective pairs; namely treat each $\sigma \in S$ and its inverse as a functor $\mathscr{A} \times \mathscr{A} \times \mathscr{A}^{\mathrm{op}} \to \mathscr{E}ns^2$. Thus $\mathscr{CMC}$ is monadic over $\mathscr{C}at$ (by Beck's theorem; see Mac Lane (1971)) and has small colimits (by Linton (1969)). □

REMARK. Whilst $\mathscr{SMC}$ is "clubable" over $\mathscr{C}at$, $\mathscr{CMC}$ is not (see Kelly (1972)).

References

J. H. Conway (1976), *On numbers and games* (Acad. Press, New York–London).

B. J. Day (1973), 'Note on monoidal localisation', *Bull. Austral. Math. Soc.*, **8**, 1–16.

S. Eilenberg and G. M. Kelly (1966), 'Closed categories', *Proc. Conference on Categorical Algebra*, La Jolla 1965 (Springer-Verlag) 421–562.

G. M. Kelly (1972), *Many-variable functorial calculus I, Coherence in Categories* (Lecture Notes in Mathematics, **281**, Springer-Verlag), 66–105.

J. Lambek (1969), *Deductive systems and categories II, Category Theory, Homology Theory and their Applications I* (Lecture Notes in Mathematics, **86**, Springer-Verlag) 76–122.

F. E. J. Linton (1969), *Coequalisers in categories of algebras*, Seminar on Triples and Categorical Homology Theory (Lecture Notes in Mathematics, **80**, Springer-Verlag) 75–90.

S. Mac Lane (1971), *Categories for the working mathematician*, GTM5 (Springer-Verlag, New York–Heidelberg–Berlin).

R. G. Swan (1968), *Algebraic K-theory* (Lecture Notes in Mathematics, **76**, Springer-Verlag).

Department of Pure Mathematics,
University of Sydney,
N. S. W. 2006,
Australia.