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On Quotients of Non-Archimedean Köthe Spaces

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Abstract. We show that there exists a non-archimedean Fréchet-Montel space W with a basis and with a continuous norm such that any non-archimedean Fréchet space of countable type is isomorphic to a quotient of W. We also prove that any non-archimedean nuclear Fréchet space is isomorphic to a quotient of some non-archimedean nuclear Fréchet space with a basis and with a continuous norm.

Introduction

In this paper all linear spaces are over a non-archimedean non-trivially valued field $\mathbb{K}$ which is complete under the metric induced by the valuation $|\cdot|: \mathbb{K} \to [0, \infty)$. For fundamentals of locally convex Hausdorff spaces (lcs) and normed spaces we refer to [4, 5, 6].

In [9, 10] we investigated closed subspaces in Fréchet spaces of countable type. In this paper we study quotients of Fréchet spaces of countable type.

By a *Köthe space* we mean a Fréchet space with a basis and with a continuous norm. First, we prove that any Fréchet space of countable type is isomorphic to a quotient of some Köthe space V (Theorem 3 and Corollary 4) and any Köthe space is isomorphic to a quotient of some Köthe–Montel space (Theorem 5). Thus any Fréchet space of countable type is isomorphic to a quotient of some Köthe–Montel space W (Corollary 6).

Next, we show that any nuclear Fréchet space is isomorphic to a quotient of some nuclear Köthe space (Theorem 7), but there is no nuclear Fréchet space X such that any nuclear Köthe space is isomorphic to a quotient of X (Theorem 10 and Corollary 12).

Preliminaries

The linear span of a subset A of a linear space E is denoted by $\operatorname{lin} A$.

Let E, F be locally convex spaces. A map $T: E \to F$ is called an *isomorphism* if T is linear, injective, surjective and the maps T, T^{-1} are continuous. E is *isomorphic* to F if there exists an isomorphism $T: E \to F$.

A *seminorm* on a linear space E is a function $p: E \to [0, \infty)$ such that $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{K}, x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. A seminorm p on E is a *norm* if $\ker p = \{0\}$.

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The set of all continuous seminorms on a metrizable lcs E is denoted by $\mathcal{P}(E)$. A non-decreasing sequence $(p_k) \subset \mathcal{P}(E)$ is a *base* in $\mathcal{P}(E)$ if for every $p \in \mathcal{P}(E)$ there exists $k \in \mathbb{N}$ with $p \leq p_k$. A sequence (p_k) of norms on E is a *base of norms* in $\mathcal{P}(E)$ if it is a base in $\mathcal{P}(E)$.

Any metrizable lcs E possesses a base (p_k) in $\mathcal{P}(E)$.

A metrizable lcs E is of *finite type* if $\dim(E/\ker p) < \infty$ for any $p \in \mathcal{P}(E)$, and of *countable type* if E contains a linearly dense countable set.

A *Fréchet space* is a metrizable complete lcs. Any infinite-dimensional Fréchet space of finite type is isomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}}$ of all sequences in $\mathbb{K}$ with the topology of pointwise convergence (see [2, Theorem 3.5]).

Let (x_n) be a sequence in a Fréchet space E. The series $\sum_{n=1}^{\infty} x_n$ is convergent in E if and only if $\lim x_n = 0$.

A sequence (x_n) in an lcs E is a *basis* in E if each $x \in E$ can be written uniquely as $x = \sum_{n=1}^{\infty} \alpha_n x_n$ with $(\alpha_n) \subset \mathbb{K}$. If additionally the coefficient functionals $f_n \colon E \to \mathbb{K}, x \to \alpha_n$, $(n \in \mathbb{N})$ are continuous, then (x_n) is a *Schauder basis* in E. As in the real and complex case any basis in a Fréchet space is a Schauder basis (see [3, Corollary 4.2]).

A *Banach space* is a normable Fréchet space. Any infinite-dimensional Banach space E of countable type is isomorphic to the Banach space c_0 of all sequences in $\mathbb{K}$ converging to zero with the sup-norm [5, Theorem 3.16].

Let p be a seminorm on a linear space E and $t \in (0, 1)$. A sequence (x_n) in E is *t-orthogonal* with respect to p if $p(\sum_{i=1}^{n} \alpha_i x_i) \geq t \max_{1 \leq i \leq n} p(\alpha_i x_i)$ for all $n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{K}$.

A sequence (x_n) in an lcs E is 1-*orthogonal* with respect to $(p_k) \subset \mathcal{P}(E)$ provided $p_k(\sum_{i=1}^{n} \alpha_i x_i) = \max_{1 \leq i \leq n} p_k(\alpha_i x_i)$ for all $k, n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{K}$.

Every basis (x_n) in a Fréchet space E is 1-orthogonal with respect to some basis (p_k) in $\mathcal{P}(E)$ [2, Proposition 1.7].

Let $B = (b_{n,k})$ be an infinite real matrix with $0 < b_{n,k} \leq b_{n,k+1} \, \forall n, k \in \mathbb{N}$. The space $K(B) = \{(\alpha_n) \subset \mathbb{K} : \lim_n |\alpha_n| b_{n,k} = 0 \text{ for all } k \in \mathbb{N}\}$ with the base of norms (p_k): $p_k((\alpha_n)) = k \max_n |\alpha_n| b_{n,k}, k \in \mathbb{N}$, is a Köthe space. The sequence (e_n) of coordinate vectors forms a basis in $K(B)$; the coordinate basis is 1-orthogonal with respect to the base (p_k) [1, Proposition 2.2].

Put $B_{\mathbb{K}} = \{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$. Let A be a subset of an lcs E. The set $\operatorname{co} A = \{\sum_{i=1}^{n} \alpha_i a_i : n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in B_{\mathbb{K}}, a_1, \dots, a_n \in A\}$ is the *absolutely convex hull* of A; its closure in E is denoted by $\overline{\operatorname{co}} A$.

A subset B of an lcs E is *absolutely convex* if $\operatorname{co} B = B$.

A subset B of an lcs E is *compactoid* if for each neighbourhood U of 0 in E there exists a finite subset A of E such that $B \subset U + \operatorname{co} A$.

By a *Fréchet–Montel space* we mean a Fréchet space in which any bounded subset is compactoid.

Let E and F be locally convex spaces. A linear map $T \colon E \to F$ is *compact* if there exists a neighbourhood U of 0 in E such that $T(U)$ is compactoid in F.

For any seminorm p on an lcs E the map $\overline{p} \colon E_p \to [0, \infty), x + \ker p \to p(x)$ is a norm on $E_p = (E/\ker p)$. Let $\varphi_p \colon E \to E_p, x \to x + \ker p$.

An lcs E is *nuclear* if for every continuous seminorm p on E there exists a contin-

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uous seminorm q on E with $q \geq p$ such that the map

$$\varphi_{pq} \colon (E_q, \overline{q}) \to (E_p, \overline{p}), x + \ker q \to x + \ker p$$

is compact.

Let E be a Fréchet space with a basis (x_n) which is 1-orthogonal with respect to a base of norms (p_k) in $\mathcal{P}(E)$. Then E is nuclear if and only if $\forall k \in \mathbb{N}$, $\exists m > k$: $\lim_n [p_k(x_n)/p_m(x_n)] = 0$ [1, Propositions 2.4 and 3.5].

Results

A sequence (x_n) in a Fréchet space X is a *pseudo-basis* of X, if for any element x of X there is a sequence $(\alpha_n) \subset \mathbb{K}$ such that the series $\sum_{n=1}^{\infty} \alpha_n x_n$ is convergent in X to x.

In [8] we have proved that there exist nuclear Fréchet spaces without a basis. For pseudo-bases we have the following.

Proposition 1 *Any Fréchet space E of countable type has a pseudo-basis.*

Proof Let (p_k) be a base in $\mathcal{P}(E)$ and $U_k = \{x \in E : p_k(x) \leq 1\}, k \in \mathbb{N}$. Let $\beta \in \mathbb{K}$ with $0 < |\beta| < 1$. Choose a linearly independent and linearly dense sequence (z_i) in E. Put $Z_n = \operatorname{lin}\{z_i : 1 \leq i \leq n\}, n \in \mathbb{N}$. Let (N_k) be a partition of $\mathbb{N}$ into infinite subsets. For $n \in N_k, k \in \mathbb{N}$, let $x_{n,1}, \dots, x_{n,n}$ be a basis in Z_n which is $|\beta|$-orthogonal with respect to p_k (see [10, proof of Lemma 1.1]). We will show that the sequence $(x_n) = (x_{1,1}, x_{2,1}, x_{2,2}, x_{3,1}, x_{3,2}, x_{3,3}, \dots)$ is a pseudo-basis in E.

Let $k \in \mathbb{N}, x \in U_k$ and $m \in \mathbb{N}$. Then for some $n \in N_k$ with $n \geq m$ there is $y \in Z_n \cap (x + U_{k+1})$. Thus $\exists \beta_1, \dots, \beta_n \in \mathbb{K} : y = \sum_{i=1}^{n} \beta_i x_{n,i}$ and

$$|\beta| \max_{1 \leq i \leq n} p_k(\beta_i x_{n,i}) \leq p_k(y) \leq \max\{p_k(y - x), p_k(x)\} \leq 1.$$

Hence $\beta_1 x_{n,1}, \dots, \beta_n x_{n,n} \in \beta^{-1} U_k$.

We have proved that $\forall k \in \mathbb{N}\, \forall x \in U_k\, \forall m \in \mathbb{N}\, \exists s \geq m\, \exists \alpha_m, \dots, \alpha_s \in \mathbb{K}$:

$$(x - \sum_{i=m}^{s} \alpha_i x_i) \in U_{k+1} \text{ and } \{\alpha_m x_m, \dots, \alpha_s x_s\} \subset \beta^{-1} U_k.$$

It follows that the sequence (x_n) is a pseudo-basis in E. ∎

Remark 2 It is easy to see that any dense sequence (x_n) in a Fréchet space E is a pseudo-basis of E. Unfortunately, any non-zero Fréchet space over a non-separable field is non-separable.

Using the existence of pseudo-bases in any Fréchet space of countable type we get the following.

Theorem 3 *Any Fréchet space E of countable type is isomorphic to a quotient of some Köthe space.*

Proof Assume that E is not of finite type. Then for some $p \in \mathcal{P}(E)$ the quotient space $(E/\ker p)$ is infinite-dimensional. Let G be an algebraic complement of $\ker p$ in E. Since G is an infinite-dimensional metrizable lcs of countable type, it contains a linearly independent and linearly dense sequence (g_n). Let (s_k) be a linearly dense sequence in $\ker p$ and let (N_k) be a partition of $\mathbb{N}$ into infinite subsets. We can choose a sequence $(\alpha_n) \subset (\mathbb{K} \setminus \{0\})$ with $\lim_{n\in N_k} \alpha_n g_n = 0, k \in \mathbb{N}$. Put $z_n = \alpha_n g_n + s_k$ for $n \in N_k, k \in \mathbb{N}$. The sequence (z_n) is linearly independent and linearly dense in E, and $\operatorname{lin}(z_n) \cap \ker p = \{0\}$.

By Proposition 1 and its proof, the space E has a pseudo-basis (e_n) such that $(e_n) \subset (\operatorname{lin}(z_n) \setminus \{0\})$. Let (p_k) be a base in $\mathcal{P}(E)$ with $p_1 \geq p$. Put $a_{n,k} = p_k(e_n)$ for $n, k \in \mathbb{N}$. Clearly, $0 < a_{n,k} \leq a_{n,k+1}$ for all $n, k \in \mathbb{N}$. Let $A = (a_{n,k})$ and let X be the Köthe space $K(A)$.

For any $\alpha = (\alpha_n) \in X$ the series $\sum_{n=1}^{\infty} \alpha_n e_n$ is convergent in E. Moreover, $p_k(\sum_{n=1}^{\infty} \alpha_n e_n) \leq \max_n |\alpha_n| a_{n,k} \leq q_k(\alpha)$ for $k \in \mathbb{N}, \alpha \in X$, where (q_k) is the standard base of norms in $\mathcal{P}(X)$. Thus the linear operator $T\colon X \to E, T\alpha = \sum_{n=1}^{\infty} \alpha_n e_n$, is well defined and continuous. We show that $T(X) = E$. Let $e \in E$. Then there exists $(\alpha_n) \subset \mathbb{K}$ such that $\sum_{n=1}^{\infty} \alpha_n e_n = e$. Clearly, $\lim_n |\alpha_n| a_{n,k} = \lim_n |\alpha_n| p_k(x_n) = 0$, $k \in \mathbb{N}$. Thus $\alpha = (\alpha_n) \in X$ and $T\alpha = e$. It follows that E is isomorphic to the quotient $(X/\ker T)$ of X.

If E is of finite type, then it is isomorphic to a quotient of $\mathbb{K}^{\mathbb{N}} \times c_0$ and, by the first part of the proof, to a quotient of some Köthe space. ∎

In [12] we have proved that there exists a Köthe space V (unique up to isomorphism) such that any Köthe space is isomorphic to a complemented closed subspace of V. Thus, by Theorem 3, we get

Corollary 4 *Any Fréchet space of countable type is isomorphic to a quotient of the Köthe space V.*

Now we prove the following.

Theorem 5 *Any Köthe space X is isomorphic to a quotient of some Köthe–Montel space.*

Proof Let (x_n) be a basis in X. This basis is 1-orthogonal with respect to a base of norms (p_k) in $\mathcal{P}(X)$. Without loss of generality we can assume that $p_1(x_n) \geq 1, n \in \mathbb{N}$. Put $d_{m,k} = p_k(x_m)$ for $m, k \in \mathbb{N}$. Let $(N_i), (S_m)$ be two partitions of $\mathbb{N}$ such that the set $N_i \cap S_m$ is non-empty for all $i, m \in \mathbb{N}$.

For $n \in N_i \cap S_m, i, m \in \mathbb{N}$ and $k \in \mathbb{N}$ we put $b_{n,k} = k^i d_{m,k}$ if $k \leq i$ and $b_{n,k} = k^{in} d_{m,k}$ if $k > i$. Clearly, $0 < b_{n,k} \leq b_{n,k+1}$ for all $n, k \in \mathbb{N}$. Put $B = (b_{n,k})$. The Köthe space $K(B)$ is a Fréchet–Montel space (see [10, Corollary 1.10, Example 1.9 and its proof]). We will prove that X is isomorphic to a quotient of $K(B)$. Put $Y = K(B)$.

Let $(f_n) \subset Y'$ be the sequence of coefficient functionals associated with the coordinate basis (e_n) in Y. For any $\alpha = (\alpha_n) \in Y$ we have $\lim_n f_n(\alpha) = 0$, since

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$\lim_n |\alpha_n| b_{n,1} = 0$. Put $g_m(\alpha) = \sum_{n \in S_m} f_n(\alpha)$ for $m \in \mathbb{N}$ and $\alpha \in Y$. By the Banach–Steinhaus theorem, the linear functionals g_m, $m \in \mathbb{N}$, are continuous on Y. For all $k, m \in \mathbb{N}$ and $\alpha \in Y$ we have

$$p_k(g_m(\alpha)x_m) = |g_m(\alpha)| d_{m,k} \le \sup_{n \in S_m} |f_n(\alpha)| d_{m,k} \le \sup_{n \in S_m} |\alpha_n| b_{n,k}$$

and $\lim_n |\alpha_n| b_{n,k} = 0$, so $\lim_m g_m(\alpha)x_m = 0$ in X, for any $\alpha \in Y$. Put $T\colon Y \to X$, $T\alpha = \sum_{m=1}^{\infty} g_m(\alpha)x_m$. For $k, m \in \mathbb{N}$ and $\alpha \in Y$ we get

$$p_k(T\alpha) \le \max_m \max_{n \in S_m} |f_n(\alpha)| d_{m,k} \le \max_m \max_{n \in S_m} q_k(\alpha)(d_{m,k} b_{n,k}^{-1}) \le q_k(\alpha),$$

where (q_k) is the standard base of norms in $\mathcal{P}(Y)$. Thus the linear operator T is continuous. We show that $T(Y) = X$. Let $x \in X$. Then $\exists (\alpha_m) \subset \mathbb{K} : x = \sum_{m=1}^{\infty} \alpha_m x_m$ and $\forall k \in \mathbb{N}, \lim_m |\alpha_m| d_{m,k} = 0$. Therefore there exists an increasing sequence $(m_k) \subset \mathbb{N}$ with $m_1 = 1$ such that $|\alpha_m| d_{m,k} \le k^{-k-1} p_1(x)$ for $m_k \le m < m_{k+1}, k \in \mathbb{N}$. Let $t_m \in N_k \cap S_m$ for $m_k \le m < m_{k+1}, k \in \mathbb{N}$. Let $l \in \mathbb{N}$. Then for $k \ge l$ and $m_k \le m < m_{k+1}$ we have

$$|\alpha_m| b_{t_m,l} \le |\alpha_m| b_{t_m,k} = |\alpha_m| d_{m,k} k^k \le k^{-1} p_1(x).$$

Hence $\forall l \in \mathbb{N}, \lim_m |\alpha_m| b_{t_m,l} = 0$. Thus the series $\sum_{m=1}^{\infty} \alpha_m e_{t_m}$ is convergent in Y to some element y. Clearly, $Ty = x$; so $T(Y) = X$. It follows that X is isomorphic to the quotient $(Y/\ker T)$ of Y. ■

By Corollary 4 and Theorem 5 we obtain

Corollary 6 *Any Fréchet space of countable type is isomorphic to a quotient of some Köthe–Montel space W.*

For nuclear Fréchet spaces we shall prove the following.

Theorem 7 *Any nuclear Fréchet space E is isomorphic to a quotient of some nuclear Köthe space.*

Proof Assume that E is not of finite type. Let $\beta \in \mathbb{K}$ with $0 < |\beta| < 1$. Then E possesses a base (p_k) in $\mathcal{P}(E)$ such that:

(1) $\dim(E/\ker p_1) = \infty$;
(2) $\forall k \in \mathbb{N}$, $p_k \le |\beta|^2 p_{k+1}$;
(3) for any $k \in \mathbb{N}$ the canonical map $\varphi_{k,k+1}\colon (E_{k+1}, \overline{p_{k+1}}) \to (E_k, \overline{p_k})$ is compact.

Let (z_n) be a linearly independent and linearly dense sequence in E such that $\operatorname{lin}(z_n) \cap \ker p_1 = \{0\}$ (see the proof of Theorem 3). Put $Z = \operatorname{lin}(z_n)$ and $U_m = \{x \in E : p_m(x) \le 1\}$ for $m \in \mathbb{N}$. Let $k \in \mathbb{N}$.

Let (v_n) be a $|\beta|$-orthogonal basis in $(E_{k+1}, \overline{p_{k+1}})$ with $|\beta| < \overline{p_{k+1}}(v_n) \le 1, n \in \mathbb{N}$, such that $\operatorname{lin}(v_n) = \operatorname{lin}(\varphi_{k+1}(z_n))$ (see [5], Theorem 3.16 (i) and its proof). Put $u_n = (\varphi_{k+1}|Z)^{-1}(v_n), n \in \mathbb{N}$. Then $(u_n) \subset Z \cap U_{k+1}$.

We will show that $U_{k+2} \subset \overline{\text{co}}(u_n)$. Let $x \in U_{k+2}$. Assume that $m \in \mathbb{N}, \alpha_1, \ldots, \alpha_m \in \mathbb{K}$ and $(x - \sum_{i=1}^m \alpha_i u_i) \in U_{k+2}$. Then

$$p_{k+1}\Big(\sum_{i=1}^m \alpha_i u_i\Big) \leq \max\Big\{ p_{k+1}\Big(\sum_{i=1}^m \alpha_i u_i - x\Big), p_{k+1}(x)\Big\} \leq |\beta|^2$$

and

$$p_{k+1}\Big(\sum_{i=1}^m \alpha_i u_i\Big) = \overline{p_{k+1}}\Big(\sum_{i=1}^m \alpha_i v_i\Big) \geq |\beta| \max_{1\leq i\leq m} \overline{p_{k+1}}(\alpha_i v_i) \geq |\beta|^2 \max_{1\leq i\leq m} |\alpha_i|.$$

Hence $\max_{1\leq i\leq m} |\alpha_i| \leq 1$. We have proved that $\sum_{i=1}^m \alpha_i u_i \in \text{co}(u_n)$ provided $(x - \sum_{i=1}^m \alpha_i u_i) \in U_{k+2}$. Thus $x \in \overline{\text{co}}(u_n)$, since (u_n) is linearly dense in E. Hence $U_{k+2} \subset \overline{\text{co}}(u_n)$.

Put $W = Z \cap U_{k+1}$. The set $\varphi_k(W)$ is absolutely convex and compactoid in $(E_k, \overline{p_k})$. Therefore there exists a sequence $(y_i) \subset (\beta^{-1}\varphi_k(W) \setminus \{0\})$ with $\lim_i \overline{p_k}(y_i) = 0$ such that $\varphi_k(W) \subset \overline{\text{co}}(y_i)$ (see [6, Proposition 8.2]).

Let $d_i \in \beta^{-1}W$ with $\varphi_k(d_i) = y_i, i \in \mathbb{N}$. Clearly, $0 < p_k(d_i) \leq |\beta|, i \in \mathbb{N}$, and $\lim_i p_k(d_i) = 0$. Since $(u_n) \subset Z \cap U_{k+1}$, we have

$$\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\ \exists \alpha_1, \ldots, \alpha_m \in B_{\mathbb{K}} : 0 < \overline{p_k}(\varphi_k(u_n) - \sum_{i=1}^m \alpha_i y_i) < n^{-1}.$$

Put $b_n = u_n - \sum_{i=1}^m \alpha_i d_i, n \in \mathbb{N}$. Then $0 < p_k(b_n) < n^{-1}, n \in \mathbb{N}$.

Let $x_{2n-1}^k = d_n, x_{2n}^k = b_n$ for $n \in \mathbb{N}$. Clearly, $(x_n^k) \subset Z \cap (U_k \setminus \{0\})$, $\lim_n p_k(x_n^k) = 0$ and $(u_n) \subset \text{co}(x_n^k)$; hence $U_{k+2} \subset \overline{\text{co}}(u_n) \subset \overline{\text{co}}(x_n^k)$.

Let (S_k) be a partition of $\mathbb{N}$ into infinite subsets and let (x_n) be a sequence in E such that $(x_n)_{n\in S_k} = (x_1^k, x_2^k, \ldots)$ for any $k \in \mathbb{N}$. Let $d_{n,k} = p_k(x_n)$ for $n, k \in \mathbb{N}$. Clearly, $0 < d_{n,k} \leq d_{n,k+1}$ for $n, k \in \mathbb{N}$. Moreover, $0 < d_{n,m} \leq 1$ for $n \in S_m, m \in \mathbb{N}$, and $\lim_{n\in S_m} d_{n,m} = 0, m \in \mathbb{N}$.

Put $b_{n,k} = d_{n,k} d_{n,m}^{-k/m} |\beta|^{-km}$ for $n \in S_m, m \in \mathbb{N}$, and $k \in \mathbb{N}$. Clearly, $0 < b_{n,k} \leq |\beta| b_{n,k+1}$ for all $n, k \in \mathbb{N}$. Let $k \in \mathbb{N}$. For $n \in S_m, m \in \mathbb{N}$, we have $b_{n,k} b_{n,k+1}^{-1} \leq d_{n,m}^{1/m} |\beta|^m$. Let $\epsilon > 0$. Then $\exists l \in \mathbb{N}\ \forall m > l, |\beta|^m \leq \epsilon$ and $\exists t \in \mathbb{N}\ \forall 1 \leq m \leq l\ \forall n \in (S_m \setminus \{1, \ldots, t\}), d_{n,m} \leq \epsilon^m$. Hence $\forall n > t, b_{n,k} b_{n,k+1}^{-1} \leq \epsilon$. Thus $\lim_n b_{n,k} b_{n,k+1}^{-1} = 0, k \in \mathbb{N}$; so the Köthe space $K(B)$, associated with the matrix $B = (b_{n,k})$, is nuclear.

We shall show that E is isomorphic to a quotient of $K(B)$. Put $Y = K(B)$ and $q_k(\alpha) = \max_n |\alpha_n| b_{n,k}$ for $\alpha = (\alpha_n) \in Y$ and $k \in \mathbb{N}$. Clearly, (q_k) is a base in $\mathcal{P}(Y)$. Let $\alpha = (\alpha_n) \in Y$ and $k \in \mathbb{N}$. For $n \in S_m, m \in \mathbb{N}$ we have

$$p_k(\alpha_n x_n) = |\alpha_n| d_{n,k} \leq q_k(\alpha) b_{n,k}^{-1} d_{n,k} = q_k(\alpha)(d_{n,m}^{1/m} |\beta|^m)^k.$$

Thus $\lim_n p_k(\alpha_n x_n) = 0$ and $\max_n p_k(\alpha_n x_n) \leq q_k(\alpha)$ for all $\alpha = (\alpha_n) \in Y$ and $k \in \mathbb{N}$. It follows that the linear map

$$T\colon Y \to E, T\alpha = \sum_{n=1}^\infty \alpha_n x_n$$

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is well defined and continuous. Put $V_m = \{\alpha \in Y : q_m(\alpha) \le 1\}, m \in \mathbb{N}$. Let (e_n) be the coordinate basis in Y. Let $m \in \mathbb{N}$. Since $q_m(\beta^{m^2} e_n) = |\beta|^{m^2} b_{n,m} = 1$ for $n \in S_m$, we have $T(V_m) \supset \{\beta^{m^2} x_n : n \in S_m\}$; so $\overline{T(V_m)} \supset \beta^{m^2} \overline{\text{co}}\{x_n^m : n \in \mathbb{N}\} \supset \beta^{m^2} U_{m+2}$. Thus the map T is almost open. By the open mapping theorem [4, Theorem 2.72] we infer that $T(Y) = E$ and E is isomorphic to the quotient $(Y/\ker T)$ of Y.

If E is of finite type and $K(B)$ is a nuclear Köthe space, then E is isomorphic to a quotient of $\mathbb{K}^{\mathbb{N}} \times K(B)$ and, by the first part of the proof, to a quotient of some nuclear Köthe space. ∎

Finally, we shall show that there is no nuclear Fréchet space X such that any nuclear Köthe space is isomorphic to a quotient of X.

For arbitrary subsets A, B in a linear space E and a linear subspace L of E we denote $d(A, B, L) = \inf\{|\beta| : \beta \in \mathbb{K} \text{ and } A \subset \beta B + L\}$ (we put $\inf \varnothing = \infty$). Let $d_n(A, B) = \inf\{d(A, B, L) : L < E \text{ and } \dim L < n\}, n \in \mathbb{N}$.

It is easy to check the following.

Remark 8 Let E and F be linear spaces. If $A, B \subset E$ and T is a linear map from E onto F, then $d_n(A, B) \ge d_n(T(A), T(B))$ for $n \in \mathbb{N}$. If $A' \subset A \subset E$ and $B \subset B' \subset E$, then $d_n(A, B) \ge d_n(A', B')$ for $n \in \mathbb{N}$.

By the second part of the proof of [11, Lemma 2], we get

Lemma 9 *Let (f_n) be the sequence of coefficient functionals associated with a basis (x_n) in an lcs E. Let $(a_k), (b_k) \subset (0, \infty)$. Put $A = \{x \in E : \max_k |f_k(x)| a_k^{-1} \le 1\}$ and $B = \{x \in E : \max_k |f_k(x)| b_k^{-1} \le 1\}$. Then for any $n \in \mathbb{N}$ and $\alpha \in \mathbb{K}$ with $|\alpha| < 1$ we have $d_n(A, B) \ge |\alpha| a_n b_n^{-1}$.*

If $a = (a_n) \subset (0, \infty)$ is a non-decreasing sequence with $\lim a_n = \infty$, then the following Köthe space is nuclear: $A_\infty(a) = K(B)$ with $B = (b_{k,n}), b_{k,n} = k^{a_n}$ (see [1]); $A_\infty(a)$ is a *power series space of infinite type*.

Now we can prove our last theorem.

Theorem 10 *For any nuclear Köthe space X there exists a non-decreasing sequence $(a_n) \subset (0, \infty)$ with $\lim_n a_n = \infty$ such the space $A_\infty(a)$ is not isomorphic to any quotient of X.*

Proof Let $\beta \in \mathbb{K}$ with $0 < |\beta| < 1$. Let (x_n) be a basis of X which is 1-orthogonal with respect to a base of norms (p_k) in $\mathcal{P}(X)$ with $\lim_n [p_k(x_n) p_{k+1}^{-1}(x_n)] = 0, k \in \mathbb{N}$. Put $U_k = \{x \in X : p_k(x) \le 1\}$ for $k \in \mathbb{N}$. It is easy to see that

$$\forall i \in \mathbb{N} \forall m \in \mathbb{N}\ \exists n \in \mathbb{N} : U_{i+1} \subset \beta^m U_i + \text{lin}\{x_1, \dots, x_n\}.$$

Hence $\lim_n d_n(U_{i+1}, U_i) = 0, i \in \mathbb{N}$. Thus there exists an increasing sequence $(v_n) \subset \mathbb{N}$ such that for any $n \in \mathbb{N}$ we have

$$\max_{1 \le k \le n} d_{v_n}(U_{k+1}, U_k) < |\beta| n^{-n}.$$

Put $a_m = \min\{n \in \mathbb{N} : \nu_n \geq m\}, m \in \mathbb{N}$, and $a = (a_n)$. Clearly, $0 < a_m \leq a_{m+1}$ for $m \in \mathbb{N}$, and $\lim_m a_m = \infty$.

Assume that the space $A_\infty(a)$ is isomorphic to a quotient of X. Then there exists a linear continuous and open mapping T from X onto $A_\infty(a)$. Thus for some $k, s \in \mathbb{N}$ we have

$$V_1 \supset T(U_k) \supset T(U_{k+1}) \supset V_s,$$

where $V_i = \{\alpha = (\alpha_n) \in A_\infty(a) : \max_n |\alpha_n| i^{a_n} \leq 1\}, i \in \mathbb{N}$.

Using Remark 8, we get

$$d_m(U_{k+1}, U_k) \geq d_m(T(U_{k+1}), T(U_k)) \geq d_m(V_s, V_1), m \in \mathbb{N}.$$

Let $n \in \mathbb{N}$ with $a_{\nu_n} \geq \max\{k, s\}$. Put $m = \nu_n$; then $a_n = n \geq \max\{k, s\}$. By Lemma 9 we have

$$d_m(V_s, V_1) \geq |\beta| s^{-a_m} \geq |\beta| n^{-n} > d_m(U_{k+1}, U_k);$$

a contradiction. ■

Similarly to the proof of Theorem 10 one can show the following

Remark 11 For any nuclear Köthe space $K(A)$ with $A = (a_{n,k})$ there exists a non-decreasing sequence $(t_n) \subset \mathbb{N}$ with $\lim_n t_n = \infty$ such that for $B = (b_{n,k})$ with $b_{n,k} = a_{t_n,k}$, $n, k \in \mathbb{N}$, the nuclear Köthe space $K(B)$ is not isomorphic to a quotient of $K(A)$.

By Theorems 7 and 10, we obtain

Corollary 12 *There is no nuclear Fréchet space X such that any nuclear Köthe space is isomorphic to a quotient of X.*

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