BULL. AUSTRAL. MATH. SOC. 54C35, 54A10, 46E25
VOL. 49 (1994) [427–432]

UNIFORM DENSITY AND m-DENSITY FOR SUBRINGS OF $C(X)$

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This paper deals with the equivalence between u-density and m-density for the subrings of $C(X)$. It was proved by Kurzweil that such equivalence holds for those subrings that are closed under bounded inversion. Here an example is given in $C(\mathbb{N})$ of a u-dense subring that is not m-dense. It is deduced that the two types of density coincide only in the trivial case where these topologies are the same, that is, if and only if X is a pseudocompact space.

For a completely regular space X, $C(X)$ and $C^*(X)$ denote, respectively, the algebra of all real-valued continuous, and continuous and bounded, functions over X. We are interested in the following problem: Is every u-dense subring of $C(X)$ m-dense too?

Recall that the u-topology is defined on $C(X)$ by taking as neighbourhood base of $f \in C(X)$ the sets of the form

$$\{g \in C(X) : |f(x) - g(x)| < \varepsilon \text{ for all } x \in X\}$$

where ε is a positive real number, and that the m-topology is defined by taking the sets of the form

$$\{g \in C(X) : |f(x) - g(x)| < u(x) \text{ for all } x \in X\}$$

where u is a positive unit of $C(X)$.

Obviously the m-topology is finer than the u-topology, and it is well-known that the two coincide if and only if X is a pseudocompact space (Hewitt [5]), namely when $C^*(X) = C(X)$. Although, in general, these topologies are different, many families in $C(X)$ that are u-dense are m-dense too. For instance, it was essentially proved by Kurzweil in [6] that u-density and m-density are equivalent for the subrings of $C(X)$ that are closed under bounded inversion.

Received 21 June 1993
The authors want to thank Professor J.A. Jaramillo of University Complutense in Madrid for his helpful and thoughtful suggestions.

In this note we shall prove that an analogue of Kurzweil's result is not possible for arbitrary subrings of $C(X)$. We first construct an example of a u-dense subring of $C(\mathbb{N})$ that is not m-dense. From this example and taking into account that every non-pseudocompact space contains a C-embedded copy of $\mathbb{N}$ (Gillman-Jerison [4]), we deduce that there is equivalence between u-density and m-density for the subrings of $C(X)$ if and only if X is a pseudocompact space.

We start by setting out a sufficient and necessary condition for the u-dense subrings of $C(X)$ to be m-dense. Throughout this paper the terminology and the notation will be as in Gillman-Jerison [4].

PROPOSITION 1. *Let $\mathfrak{F}$ be a subring of $C(X)$. Then $\mathfrak{F}$ is m-dense if and only if it fulfills the following conditions:*

(i) *$\mathfrak{F}$ is u-dense.*
(ii) *For each $f \in C(X)$ with $f(x) > 0$ for every $x \in X$, there exists $g \in \mathfrak{F}$ such that $0 < g(x) \leqslant f(x)$ for every $x \in X$.*

PROOF: It is enough to prove the sufficient condition because the other follows at once.

Let $h \in C(X)$ and let $u \in C(X)$ be a positive unit. From hypothesis (ii), there exists $g_1 \in \mathfrak{F}$ with $0 < g_1(x) \leqslant u(x)$ for every $x \in X$. Since $\mathfrak{F}$ is u-dense, we can take a function $g_2 \in \mathfrak{F}$ such that

$$|h(x)/g_1(x) - g_2(x)| < 1 \text{ for every } x \in X.$$

Thus, we have that $|h(x) - g_1(x)g_2(x)| < g_1(x) \leqslant u(x)$ for every $x \in X$, which completes the proof. ∎

Although Proposition 1 is a straightforward result on m-density, it will be very useful throughout the paper and we shall now give some of its consequences. Recall that $\mathfrak{F} \subset C(X)$ is *closed under bounded inversion* if for each $f \in \mathfrak{F}$ with $f(x) \geqslant 1$ for every $x \in X$, the function $1/f$ belongs to $\mathfrak{F}$.

COROLLARY 2. (Kurzweil [6]) *Let $\mathfrak{F}$ be a subring of $C(X)$ closed under bounded inversion. Then $\mathfrak{F}$ is u-dense if and only if $\mathfrak{F}$ is m-dense.*

PROOF: By applying Proposition 1, it is enough to see that every u-dense subring $\mathfrak{F}$ of $C(X)$ that is closed under bounded inversion fulfills the above condition (ii).

Let $f \in C(X)$ with $f(x) > 0$ for all $x \in X$. From the u-density of $\mathfrak{F}$ we can choose $g \in \mathfrak{F}$ such that $|2 + 1/f(x) - g(x)| < 1$ for every $x \in X$. It is easy to verify that the function $1/g$ belongs to $\mathfrak{F}$, and $0 < 1/g(x) \leqslant f(x)$ for every $x \in X$, as we required. ∎

Now we need to recall the following results taken from [3].

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THEOREM 3. [3] *A linear subspace over* $\mathbb{Q}$, $\mathfrak{F} \subset C(X)$, *is* u-*dense in* $C(X)$ *if and only if for each countable cover of* X, $\{C_n\}_{n\in\mathbb{Z}}$, *by cozero-sets such that* $C_n \cap C_m = \emptyset$ *if* $|n-m| > 1$, *there is a function* $h \in \mathfrak{F}$ *with* $|h(x) - n| < 2$ *when* $x \in C_n$ $(n \in \mathbb{Z})$.

THEOREM 4. [3] *Let* $\{C_n\}_{n=0}^{\infty}$ *be a countable cover of* X *by cozero-sets such that* $C_n \cap C_m = \emptyset$ *if* $|n-m| > 1$. *If* $\mathfrak{F}$ *is a subring of* $C(X)$ *that completely separates every pair of disjoint zero-sets in* X, *then there exists a partition of unity* $\{g_n\}_{n=0}^{\infty}$ *by functions in* $\mathfrak{F}$ *with* $\operatorname{coz}(g_n) \subset C_n$ *for each* n.

Theorem 3 together with condition (ii) of Proposition 1 provides us with a necessary and sufficient condition of m-density for the subrings of $C(X)$ which are linear subspaces over $\mathbb{Q}$, the so-called *divisible subrings*. These divisible subrings were mainly studied by Anderson in [1], where he obtained Corollary 5 below. Now we have a short way to derive this Corollary from Proposition 1.

COROLLARY 5. (Anderson [1]) *Let* $\mathfrak{F}$ *be a divisible subring of* $C(X)$ *which satisfies:*

(i) $\mathfrak{F}$ *completely separates every pair of disjoint zero-sets in* X.
(ii) *For every sequence* $\{f_n\}_{n=0}^{\infty}$ *of nonnegative functions in* $\mathfrak{F}$ *such that the family of their cozero-sets,* $\{\operatorname{coz}(f_n)\}_{n=0}^{\infty}$, *is a star-finite cover of* X *(that is, each member of the cover meets at most finitely many of the other members), the function* $\sum_{n=0}^{\infty} f_n$ *belongs to* $\mathfrak{F}$.

Under these conditions $\mathfrak{F}$ *is* m-*dense in* $C(X)$.

PROOF: The u-density of $\mathfrak{F}$ was proved by us in [3] as a consequence of the preceding Theorems 3 and 4. Thus it is enough to verify that $\mathfrak{F}$ also fulfills condition (ii) of Proposition 1.

Let $f \in C(X)$ with $f(x) > 0$ for all $x \in X$. By applying Theorem 4 to the cover of X by cozero-sets defined by

$$C_0 = \{x \in X : f(x) > 1/2\}$$
$$C_n = \{x \in X : 1/2^{n+1} < f(x) < 1/2^{n-1}\} \quad \text{when } n \geqslant 1,$$

we obtain a partition of unity $\{g_n\}_{n=0}^{\infty}$ by functions in $\mathfrak{F}$ with $\operatorname{coz}(g_n) \subset C_n$ for all n.

Now from (ii), the function $g = \sum_{n=0}^{\infty} (1/2^{n+3}) g_n$ belongs to $\mathfrak{F}$, and it satisfies $0 < g(x) \leqslant f(x)$ for all $x \in X$. Indeed, for $x \in X$ there exists an n such that $x \in C_n$ and $x \notin C_m$ whenever $|n-m| > 1$. Suppose $n > 0$ (with an analogous argument for $n = 0$), then

$$g(x) = \left(1/2^{n+4}\right) \cdot (4g_{n-1} + 2g_n + g_{n+1}) = \left(1/2^{n+4}\right) \cdot (3g_{n-1} + g_n + 1)$$

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and we have finally that $0 < 1/2^{n+4} < g(x) < 5/2^{n+4} < 1/2^{n+1} < f(x)$. ∎

The following example will be the key to establishing our main result.

EXAMPLE 6. Let $\mathfrak{F}$ be the following subset of $C(\mathbb{N})$

$$\mathfrak{F} = \{(q \cdot z_n)_{n\in\mathbb{N}} : q \in \mathbb{Q} \text{ and } z_n \in \mathbb{Z} \text{ for every } n \in \mathbb{N}\}.$$

1. *$\mathfrak{F}$ is a linear subspace over $\mathbb{Q}$.* Obviously $\mathfrak{F}$ is closed under rational multiplication. On the other hand, let $(q \cdot z_n)_{n\in\mathbb{N}}$ and $(q' \cdot z'_n)_{n\in\mathbb{N}}$ be two sequences in $\mathfrak{F}$. Clearly the set $\{q \cdot z_n + q' \cdot z'_n : n \in \mathbb{N}\}$ is contained in the additive subgroup of $\mathbb{R}$, $q\mathbb{Z} + q'\mathbb{Z}$. Since q/q' is a rational number then $q\mathbb{Z} + q'\mathbb{Z}$ is closed in $\mathbb{R}$ and therefore it must be of the form $p\mathbb{Z}$ for some rational number p. Thus, $(q \cdot z_n)_{n\in\mathbb{N}} + (q' \cdot z'_n)_{n\in\mathbb{N}}$ belongs to $\mathfrak{F}$. (Recall that every additive subgroup of $\mathbb{R}$ is either dense or of the form $\alpha\mathbb{Z}$ for some $\alpha \in \mathbb{R}$. Moreover, the subgroup $\alpha\mathbb{Z} + \beta\mathbb{Z}$ is closed if and only if α/β belongs to $\mathbb{Q}$.)

2. *$\mathfrak{F}$ is a subring.* This is self-evident.

3. *$\mathfrak{F}$ is u-dense in $C(\mathbb{N})$.* For this we can apply Theorem 3. Let $\{C_n\}_{n\in\mathbb{Z}}$ be a countable cover of $\mathbb{N}$ by cozero-sets (in this case, this means arbitrary subsets) such that $C_n \cap C_m = \emptyset$ if $|n - m| > 1$. If we define

$$h(x) = \max\{n \in \mathbb{Z} : x \in C_n\},$$

then h is the desired function because $h \in \mathfrak{F}$ and it is easy to verify that $|h(x) - n| \leqslant 1 < 2$ when $x \in C_n$.

4. *$\mathfrak{F}$ is not m-dense.* It is enough to see that $\mathfrak{F}$ does not satisfy condition (ii) of Proposition 1. Indeed, there is no function $(q \cdot z_n)_{n\in\mathbb{N}}$ in $\mathfrak{F}$ with

$$0 < q \cdot z_n \leqslant 1/n \quad \text{for every } n \in \mathbb{N}.$$

Otherwise, the sequence of positive numbers $(q \cdot z_n)_{n\in\mathbb{N}}$ contained in the subgroup $q\mathbb{Z}$ would have to converge to 0, but this is impossible because clearly $q\mathbb{Z}$ has no accumulation points.

Finally, we shall show that there is equivalence between u-density and m-density for the subrings of $C(X)$ only in the trivial case.

THEOREM 7. *For a completely regular space X, the following conditions are equivalent:*

(a) *X is pseudocompact.*

(b) *Every u-dense subring of $C(X)$ is m-dense.*

PROOF: *(a) implies (b).* This is clear because in this case the two topologies are identical.

(b) implies (a). Suppose X is not pseudocompact. Then X has a C-embedded copy of $\mathbb{N}$, that is, a discrete countable subspace of X such that every continuous function on it can be (continuously) extended to X. We shall denote this copy by $\mathbb{N}$ and take $\mathfrak{F}$ to be the u-dense and not m-dense subring of $C(\mathbb{N})$ constructed in the above Example 6.

Let $\widetilde{\mathfrak{F}} = \{f \in C(X) : f_{|\mathbb{N}} \in \mathfrak{F}\}$. Clearly $\widetilde{\mathfrak{F}}$ inherits every algebraic property of $\mathfrak{F}$ and so $\widetilde{\mathfrak{F}}$ is a subring of $C(X)$.

Moreover $\widetilde{\mathfrak{F}}$ is uniformly dense in $C(X)$. Indeed, let $h \in C(X)$ and $\varepsilon > 0$. Since $\mathfrak{F}$ is u-dense in $C(\mathbb{N})$ then there is $g \in \mathfrak{F}$ such that $|g - h_{|\mathbb{N}}| < \varepsilon$. If we take $\widetilde{g}$ to be any extension to X of the function g, then the function

$$f = [(h + \varepsilon) \wedge \widetilde{g}] \vee (h - \varepsilon)$$

(where $\vee$ and $\wedge$ denote, respectively, supremum and infimum) belongs to $\widetilde{\mathfrak{F}}$ because $f_{|\mathbb{N}} = g$, and it is clear that $|h - f| < \varepsilon$.

Finally, note that $\widetilde{\mathfrak{F}}$ is not m-dense in $C(X)$ because if $\widetilde{u} \geqslant 1$ is an extension to X of the continuous function $u(n) = n$ for every $n \in \mathbb{N}$, then there is no $f \in \widetilde{\mathfrak{F}}$ such that

$$0 < f(x) \leqslant 1/\widetilde{u}(x)$$

since, as we already know, there is no sequence in $\mathfrak{F}$ of positive numbers that tends to zero. ∎

REMARKS. Note that the same proof is valid if, in the above theorem, instead of subring, we consider one of the following algebraic structures: divisible subring, linear subspace over $\mathbb{Q}$, subgroup, or sublattice. The reason is that the family $\mathfrak{F}$ in Example 6 has each of these properties. Therefore, we can also establish the non-equivalence between u-density and m-density in those cases.

But what is the case for linear subspaces over $\mathbb{R}$ or for subalgebras? Note that here we cannot use the same arguments as before since $\mathfrak{F}$ has none of these structures. We proved in [2], with different techniques, that the analogous result holds for linear subspaces over $\mathbb{R}$. Nevertheless we do not know whether the same is true for the subalgebras of $C(X)$.

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