Canad. Math. Bull. Vol. **27** (3), 1984

A NOTE ABOUT LINEAR SYSTEMS ON CURVES

BY
ALLEN TANNENBAUM

ABSTRACT. Inverting the Castelnuovo bound in two ways, we show that for given integers $p \geq 0$, $d>1$, $n>1$, we can find a smooth irreducible curve of genus p which contains a linear system of degree d and of maximal dimension relative to the given data p and d, and a smooth irreducible curve of genus p which contains a linear system of dimension n and of minimal degree relative to the data p and n.

1. **Introduction.** A famous result of Castelnuovo states that given integers $d \geq n \geq 2$, the maximal genus of integral non-degenerate curves in $\mathbb{P}^n$ (all our schemes will be defined over $k=\bar{k}$, an algebraically closed field of characteristic 0) is $p_{\max}=M/2(2d-(M+1)(n-1)-2)$ where M is the smallest integer $\geq(d-1)/(n-1)-1$. Moreover this bound is realized by certain smooth curves (called "Castelnuovo curves") and thus is the best possible.

We will assume from now on that all the linear systems which we consider will define a birational map of the given curve onto its image. Then Griffiths–Harris [1], page 253, note that the Castelnuovo inequality $p \leq M/2(2d-(M+1)(n-1)-2)$ can be inverted to give an upper bound on n in terms of p and d, and a lower bound on d in terms of p and n. Indeed, if this is done, one gets:

$$(1) \qquad n \leq \frac{2(l(d-1)-p)}{l(l+1)}+1, \qquad l=\left[\frac{2(p-1)}{d}+1\right];$$

$$(2) \qquad d \geq \frac{(j+1)}{2}(n-1)+\frac{p}{j}+1, \qquad j(j-1)<\frac{2p}{n-1} \leq j(j+1).$$

(Actually Griffiths-Harris instead of our (1) claim that

$$n \leq \frac{2(l(d-1)-p)}{l(l+1)},$$

Received by the editors July 15, 1983 and, in revised form, November 8, 1983.

1980 Mathematics Subject Classification. Primary 14H45; Secondary 14C20.

The revision of this work was supported in part by the U.S. Army Research Office under Contract numbers DAAG29-81-K-0166 and DAAG29-81-K-0136 and U.S. Air Force Grant No. AF0SR81-0238.

https://doi.org/10.4153/CMB-1984-056-3 Published online by Cambridge University Press

where l is as above. This is slightly in error (they need to add 1 to the right side for the proper inequality). The proof of this is exactly the simple manipulation of the Castelnuovo inequality referred to in Griffiths–Harris [1], page 253).

Next set

$$N(p,d)=\left[\frac{2(l(d-1)-p)}{l(l+1)}+1\right],\qquad l=\left[\frac{2(p-1)}{d}+1\right]; \tag{3}$$

$$D(p,n)=\text{smallest integer}\geq\frac{(j+1)}{2}(n-1)+\frac{p}{j}+1, \tag{4}$$

$$j(j-1)<\frac{2p}{n-1}\leq j(j+1).$$

Finally in the standard way, let g_d^n denote a linear system of dimension n and degree d on a smooth irreducible curve C of genus p. Then analogously to the problem of Castelnuovo stated above concerning the existence of curves of maximal genus relative to the data d, n, we pose the following two questions (we let $p\geq 0$ be a given integer):

(i) For a given integer $d>1$, can we find a smooth irreducible curve C of genus p which contains a linear system $g_d^{N(p,d)}$ of maximal dimension relative to d and p?

(ii) For a given integer $n>1$, can we find a smooth irreducible curve C of genus p containing a linear system $g_{D(p,n)}^n$ of minimal degree relative to n and p?

The point of this note is to show that the answer to both of these questions is yes, completing the analogy to the Castelnuovo problem.

2. **Linear systems on curves.** Using the notation of the Introduction, we now formally state our result:

THEOREM 1. *Let $p\geq 0$ be a given integer. Then*

(i) *For a given integer $d>1$, there exists a smooth irreducible curve C of genus p containing a linear system $g_d^{N(p,d)}$.*

(ii) *For a given integer $n>1$, there exists a smooth irreducible curve C of genus p containing a linear system $g_{D(p,n)}^N$.*

The proof is based on three facts, two trivial and one deeper. The two trivial facts are contained in the following lemma:

LEMMA 2. (i) *Let C' be a smooth Castelnuovo curve in $\mathbb{P}^n$ of degree d, and maximal genus $p_{\max}$. Then $n=N(p_{\max},d)$, $d=D(p_{\max},n)$.*

(ii) *If $p>p_{\max}$, then $N(p,d)<N(p_{\max},d)$, and $D(p_{\max},n)<D(p,n)$.*

Proof. Both (i) and (ii) follow from simple manipulations of the inequalities given in the Introduction. For the sake of completeness we will prove in (i) the statement that $n=N(p_{\max},d)$. The other assertions are proved similarly.

So if we take M as above, then we see that $(d-1)=M(n-1)+q$ for some positive integer $q \le n-1$. Since $p_{\max}=M/2(2d-(m+1)(n-1)-2)$, we get that

$$\frac{2(p_{\max}-1)}{d}+1=M+\frac{(M+1)(q-1)}{d}. \tag{5}$$

We claim that $(M+1)(q-1)/d<1$. Indeed, since $M=(d-1-q)/(n-1)$, we have that $M(q-1)=(d-1-q)((q-1)/(n-1))<d-1-q$. Thus $(M(q-1)+q+1)/d<1$, from which the claim follows. But then from (5) and (1) from the Introduction, we see that $M=l$.

We now have the formula that $p_{\max}=l/2(2d-(l+1)(n-1)-2)$. From this it follows that

$$\frac{2(l(d-1)-p_{\max})}{l(l+1)}+1=n$$

as required. Q.E.D.

The third fact we will need is the following result from [2]:

THEOREM 2. *There exist integral non-degenerate curves of degree d in $\mathbb{P}^n$ $(n \ge 2)$ with precisely δ nodes (and no other singular points) for all δ from* 0 *to the Castelnuovo bound.*

Proof. Just combine (3.1) and (3.3) from [2]. Q.E.D.

REMARKS 3. (i) Note in particular that Theorem 2 implies that there exist integral non-degenerate curves of all geometric genera from 0 to the maximum (defined by the Castelnuovo bound) in $\mathbb{P}^n$ for given degree $d \ge n \ge 2$.

(ii) Actually from the proofs of (3.1) and (3.3) from [2], the integral curves with the required number of nodes lie in the same linear system as a Castelnuovo curve (on the surface of degree $n-1$ in $\mathbb{P}^n$ containing the Castelnuovo curve).

We now conclude with:

Proof of Theorem 1. (i) Set $n=N(p,d)$. If $n=0$ or $n=1$, the result is immediate so we may clearly assume that $n=N(p,d)>1$. In this case, let C' be a Castelnuovo curve in $\mathbb{P}^n$ of degree d of maximal genus $p_{\max}$. Then from Lemma 2(i). $N(p_{\max},d)=n=N(p,d)$, and so by Lemma 1(ii), $p \le p_{\max}$. But then by Theorem 2, there exists an integral non-degenerate curve of geometric genus p in $\mathbb{P}^n$ of degree d. Taking a hyperplane section of such a curve we immediately get the required result.

(ii) Again letting $p_{\max}$ be the maximal genus of integral non-degenerate curves in $\mathbb{P}^n$ of degree $d=D(p,n)$, by Lemma 2, $D(p_{\max},n)=d=D(p,n)$ and consequently $p \le p_{\max}$. From this the result follows as in (i) above. Q.E.D.

References

1. P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. John Wiley and Sons, New York (1978).

2. A. Tannenbaum. *Families of algebraic curves with nodes.* Compositio Math. **41** (1980), 107–126.

Center for Mathematical System Theory
Department of Mathematics
University of Florida
Gainesville, Florida 32611
U.S.A.

Department of Mathematics
Ben-Gurion University of the Negev
Beer-Sheva, Israel 84 105