Draw CF parallel to AB. Take any point G in AC, make CH = CG, and join GH cutting BC in K.
On GH describe a segment of a circle containing an angle equal to $\frac{1}{2}(\angle ABC + \angle ACB)$, and let the arc of the segment cut BC at R. Through B draw BE parallel to RG, and BF parallel to RH.

Then by similar triangles it will follow that
EC = FC, and $\angle EBF = \angle GRH = \frac{1}{2}(\angle ABC + \angle ACB)$.
Let BE meet at O the arc of the segment described on BC, and containing an angle equal to the supplement of $\frac{1}{2}(\angle ABC + \angle ACB)$.
Then $\angle EOC = \frac{1}{2}(\angle ABC + \angle ACB) = \angle EBF$;
therefore BDCF is a parallelogram, and BD = FC.
Now EC has been proved equal to FC, and O is on the circumference of the known circle;
therefore BD = DE = EC.

Fourth Meeting, February 13*th*, 1885.

A. J. G. Barclay, Esq., M.A., President, in the Chair.

Note on a Plane Strain.

By Professor Tait.

The object of this note is to point out, by a few remarks on a single case, how well worth the attention of younger mathematicians is the *full* study of certain problems, suggested by physics, but limited (so far as that science is concerned) by properties of matter.

In de St Venant's beautiful investigations of the flexure of prisms, there occurs a plane strain involving the displacements

$$\xi = \frac{xy}{D}, \quad \eta = \frac{y^2 - x^2}{2D}.$$

Physically, this is applicable to de St Venant's problem only when x and y are each small compared with D. But it is interesting to consider the results of extending it to all values of the coördinates. This I shall do, but very briefly.

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1. The altered cöordinates of any point are given, in terms of the original cöordinates, by

$$x' = x\left(1 + \frac{y}{D}\right),\ y' = y + \frac{y^2 - x^2}{2D}.$$

Hence

$$\delta x' = \delta x\left(1 + \frac{y}{D}\right) + \delta y\frac{x}{D},$$

$$\delta y' = -\delta x\frac{x}{D} + \delta y\left(1 + \frac{y}{D}\right).$$

From these we see at once that, so far as an indefinitely small area is concerned, the strain is a mere extension in all directions in the ratio

$$\sqrt{\left(1 + \frac{y}{D}\right)^2 + \frac{x^2}{D^2}} : 1,$$

combined with a rotation through an angle whose tangent is

$$-\frac{x}{D + y}.$$

2. Hence elementary squares remain squares; and any two series of lines, dividing the plane into little squares, will continue to do so after the strain.

One simple case is furnished by sets of lines parallel to the axes. Thus $y = b$ becomes the parabola

$$x'^2 = -\frac{2D^3}{(D+b)^2}\left(y' - b - \frac{b^2}{2D}\right), \quad \ldots \quad \ldots \quad (1.)$$

and $x = a$ becomes a parabola

$$x'^2 = \frac{2a^2}{D}\left(y' + \frac{a^2 + D^2}{2D}\right). \quad \ldots \quad \ldots \quad (2.)$$

These groups of parabolas, (1) and (2), must evidently be othogonal, and if the simultaneous small increments of a and b be *equal*, must divide the plane into little squares. But, as it is clear from (2) that the sign of a is immaterial, the two lines

$$x = a,\ x = -a$$

are *both* deformed into the same parabola. Hence it appears that every part of the area becomes *duplex*. This will be examined by another and more suitable method later.

Having thus obtained another set of lines which divide the plane into squares, we may begin again with it and obtain a third set, &c.

3. A line, $y = mx$, passing through the origin, becomes the parabola

$$\left(\frac{m}{m^2-1}y' - \frac{x'}{2}\right)^2 = \frac{2D}{m^2+1}(mx' - y').$$

The orthogonal trajectories of all such parabolas are the curves into which the circles

$$x^2 + y^2 = c^2$$

are deformed. Their equation may be put in the form

$$\sqrt{x'^2 + y''^2} - \frac{c}{2} = \pm \sqrt{\frac{c^2 y''}{D} + \frac{c^2}{4}},$$

where y'' is written instead of $y' + \dfrac{c^2}{2D}$.

These curves have the property that, at every point, *the sum (or difference) of the distance from a given point, and of a multiple of the square root of the distance from a given line, is constant.*

4. But, if we express the new rectangular cöordinates of a point in terms of its original polar cöordinates, we have

$$x' = r\cos\theta + \frac{r^2}{2D}\cos\left(2\theta - \frac{\pi}{2}\right),$$

$$y' = r\sin\theta + \frac{r^2}{2D}\sin\left(2\theta - \frac{\pi}{2}\right).$$

Thus the deformed circles, above spoken of, are seen to be *epicycloids of the cardioid series.* Their orthogonal trajectories are the parabolas just mentioned.

5. Another curious set of questions is, as it were, the reverse of these :—*i.e.*, what were the curves, in the unstrained plate, which became the system

$$x = a,\ y = b,$$

or the other (also orthogonal) system

$$y = mx,\ x^2 + y^2 = c^2?$$

6. But a different transformation is still more explicit in the information it gives. Shift the origin to $(0, -D)$, and we have

$$x' = \frac{xy}{D},\ y' = \frac{y^2 - x^2 + D^2}{2D}.$$

If we put $x = \rho\sin\phi$, $y = \rho\cos\phi$, these give

$$x' = \frac{\rho^2}{2D}\sin 2\phi,\ \ y' - \frac{D}{2} = \frac{\rho^2}{2D}\cos 2\phi.$$

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Hence a circle, of radius ρ, surrounding the new origin, becomes a circle of radius $\frac{\rho^2}{2D}$ surrounding the point $\left(0, -\frac{D}{2}\right)$ half-way between the new and old origins. The ϕ of any point in the circle becomes 2ϕ.

Hence the whole surface is opened up like a fan round the new origin, *every* radius through this origin having its inclination to the axis of y *doubled*. Thus the parts of a diameter, on opposite sides of the centre, are brought to coincide; and an infinitely extended line, through the centre, becomes limited at the centre. Thus what was a single sheet becomes duplex, as was said above.

7. It suffices to have indicated, by a partial examination of some of the curious features of a single case, the stores of novelties which are thus easily reached. See especially, for additional materials of the same kind, the investigation in §§ 706-7 of Thomson and Tait's *Natural Philosophy*.

38 George Square, Edinburgh,
15*th January* 1885.

Boole's and other proofs of Fourier's Double-Integral Theorem.

By Peter Alexander, M.A.

In my former paper on Fourier's double-integral I remarked that Poisson's form of the integral gave the same incorrect result as Fourier's form in an example by which I tested it, and seemed subject to the same limitations.

This, I now find, is not the case. The assertion, though quite true of the form

$$f(x) = \mathop{\mathrm{L}}_{\kappa=0} \frac{1}{\pi}\int_0^\infty dq \int_{-\infty}^{\infty} d\alpha\, \epsilon^{-\kappa q} \cos(qx - q\alpha).f(\alpha)$$

which I then dealt with, and which I understood to be Poisson's formula, having it on the authority of Freeman (*Fourier's Theory of Heat*, p. 351), is not true of the correct form.

Dr Muir having a difficulty in accepting my statement, asked me to reconsider the matter, and kindly referred me to several papers on