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ON A CANONICAL LIE ALGEBRA $\mathfrak{sl}(2r+2)$-BUNDLE OVER GRASS(n, r)

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In the present note we shall construct a Lie algebra sl $(2r + 2)$-bundle over Grass(n, r), canonically, which may be useful for theory of holomorphic vector bundles probably.

1. We use the following notations:

$U(m)$: unitary group of degree m,
$u(m)$: unitary Lie algebra of degree m,
$U(r+1, n+1) = \{w \,|\, (r+1) \times (n+1)\text{-matrices such that } w{}^t\overline{w} = I\}$,
$u(r+1, n+1) = \{(A, B) \,|\, (r+1) \times (n+1)\text{-matrices such that}$
$A + {}^t\overline{A} = 0\}$,
T^*M: the cotangent bundle of M,
$\overline{T}^*M$: the complex conjugate bundle of T^*M,
$E(M)$: the exterior algebra bundle over M generated by $T^*M \oplus \overline{T}^*M$ for a complex manifold M.

We mean by the upper $(r+1)$-part of an $(n+1) \times (n+1)$-matrix

$$X = \begin{pmatrix} Y \\ Z \end{pmatrix} \begin{matrix} \}r+1 \\ \}n-r \end{matrix}$$

the $(r+1) \times (n+1)$-matrix Y, then $U(r+1, n+1)$ and $u(r+1, n+1)$ are regarded as the upper $(r+1)$-parts of $U(n+1)$ and $u(n+1)$, respectively. There exist the natural bundle structures

$$\begin{array}{l} U(n+1) \\ \big\downarrow \tilde{\pi} \\ U(r+1, n+1) = U(n-r)\backslash U(n+1) \end{array}$$

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$$\begin{array}{l} \downarrow \pi \\ \text{Grass}(n, r) = U(r+1)\backslash U(r+1, n+1) \\ \qquad\qquad\quad = U(r+1) \times U(n-r)\backslash U(n+1) . \end{array}$$

The space $U(r+1, n+1)$ is the canonical $U(r+1)$-bundle over complex Grassmann manifold Grass(n, r).

An exponential map

$$\exp : u(r+1, n+1) \longrightarrow U(r+1, n+1)$$

is defined by

$$(1) \qquad \exp(A, B) = \text{the upper } (r+1)\text{-part of } \exp\begin{pmatrix} A & B \\ -{}^t\bar{B} & 0 \end{pmatrix} .$$

We choose a positive number κ such that the projection π induces a homeomorphism of the submanifold

$$\{\exp(0, B) \mid \|B\| < \kappa\}$$

in $U(r+1, n+1)$ onto an open neighbourhood W of $\pi(I, 0)$ on Grass(n, r). Then we get a system of system of real analytic local cross sections σ_α $(\alpha \in U(n+1))$ of Grass(n, r) into $U(r+1, n+1)$ given by

$$(2) \qquad \sigma_\alpha(\pi(\exp(0, B)\cdot\alpha)) = \exp(0, B)\alpha \qquad (\|B\| < \kappa, \alpha \in U(n+1)) .$$

The local cross section σ_α is defined on the image W_α of W by the action α:

$$\sigma_\alpha : W\alpha \longrightarrow \sigma_\alpha(W\alpha) .$$

There exist real analytic maps

$$\tau_{\beta,\alpha} : W\alpha \cap W\beta \longrightarrow U(r+1) \qquad (\alpha, \beta \in U(n+1))$$

such that

$$(3) \qquad \tau_{\gamma,\beta}\tau_{\beta,\alpha} = \tau_{\gamma,\alpha} , \qquad (\alpha, \beta, \gamma \in U(n+1)) .$$

$$(4) \qquad \sigma_\beta = \tau_{\beta,\alpha}\sigma_\alpha$$

2. Denoting by $w = (w_{i\ell})$ the system of coordinates on $U(r+1, n+1)$, then

$$(5) \qquad \theta = w d {}^t\bar{w}$$

is a $U(n+1)$-invariant connection form on $U(r+1, n+1)$, because

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$$\theta + {}^t\bar{\theta} = wd^t\overline{w} = d(w^t\overline{w}) = dI = 0$$

and

$$(w\alpha)d^t(\overline{w}\overline{\alpha}) = w\alpha^t\overline{\alpha}d^t\overline{w} = wd^t\overline{w} \qquad (\alpha \in U(n+1)) .$$

We mean by ω the curvature form of θ, i.e.

(6) $$\omega = d\theta + \theta \wedge \theta .$$

LEMMA 1. *Putting* $w = (\overbrace{w^{(1)}}^{r+1}, \overbrace{w^{(2)}}^{n-r})$, *we have*

(7) $$\theta_{(I,0)} = -dw^{(1)} = d^t\overline{w}^{(1)} ,$$

(8) $$\omega_{(I,0)} = dw^{(2)} \wedge d^t\overline{w}^{(2)} .$$

Proof. From the definitions it follows;

$$dw^{(1)}d^t\overline{w}^{(1)} = d(w^{(1)t}\overline{w}^{(1)} + w^{(2)t}\overline{w}^{(2)})_{(I,0)} = 0 ,$$

$$\theta_{(I,0)} = (I,0)\begin{pmatrix} d^t\overline{w}^{(1)} \\ d^t\overline{w}^{(2)} \end{pmatrix} = d^t\overline{w}^{(1)} = -dw^{(1)} ,$$

$$\begin{aligned} \omega_{(I,0)} &= (d\theta)_{(I,0)} + \theta_{(I,0)} \wedge \theta_{(I,0)} \\ &= dw^{(1)} \wedge d^t\overline{w}^{(1)} + dw^{(2)} \wedge d^t\overline{w}^{(2)} - dw^{(1)} \wedge d^t\overline{w}^{(2)} \\ &= dw^{(2)} \wedge d^t\overline{w}^{(2)} . \end{aligned}$$

LEMMA 2.

(9) $$\tau^*_{\beta,\alpha}(\omega) = \tau_{\beta,\alpha}\omega\tau^{-1}_{\beta,\alpha} \qquad (\alpha, \beta \in U(n+1)) .$$

Proof. Since $\tau^*_{\beta,\alpha}(w) = \tau_{\beta,\alpha}w$, it follows:

$$\begin{aligned} \tau^*_{\beta,\alpha}(\theta) &= \tau^*_{\beta,\alpha}(w)d^t\overline{(\tau^*_{\beta\alpha}(w))} = \tau_{\beta,\alpha}wd({}^t\overline{w}\tau^{-1}_{\beta,\alpha}) \\ &= \tau_{\beta,\alpha}wd^t\overline{w}\tau^{-1}_{\beta,\alpha} - \tau_{\beta,\alpha}w^t\overline{w}\tau^{-1}_{\beta,\alpha}d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} \\ &= \tau_{\beta,\alpha}\theta\tau^{-1}_{\beta,\alpha} - d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} , \end{aligned}$$

and

$$\begin{aligned} \tau^*_{\beta,\alpha}(\omega) &= \tau^*_{\beta,\alpha}(d\theta) + \tau^*_{\beta,\alpha}(\theta) \wedge \tau^*_{\beta,\alpha}(\theta) \\ &= d\tau_{\beta,\alpha} \wedge \theta\tau^{-1}_{\beta,\alpha} + \tau_{\beta,\alpha}d\theta\tau^{-1}_{\beta,\alpha} \\ &\quad + \tau_{\beta,\alpha}\theta \wedge \tau^{-1}_{\beta,\alpha}d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} - d\tau_{\beta,\alpha} \wedge \tau^{-1}_{\beta,\alpha}d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} \\ &\quad + \tau_{\beta,\alpha}\theta \wedge \theta\tau^{-1}_{\beta,\alpha} - \tau_{\beta,\alpha}\theta\tau^{-1}_{\beta,\alpha} \wedge d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} \\ &\quad - d\tau_{\beta,\alpha} \wedge \theta\tau^{-1}_{\beta,\alpha} + d\tau_{\beta,\alpha}\tau^{-1}_{\beta,\alpha} \wedge d\tau_{\beta,\alpha}\tau^{-1}_{\beta.\alpha} \\ &= \tau_{\beta,\alpha}(d\theta + \theta \wedge \theta)\tau^{-1}_{\beta,\alpha} . \end{aligned}$$

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PROPOSITION 1. *Let $\omega^{(\alpha)}$ be the restriction of the curvature form on the local cross section $\sigma_\alpha(W\alpha)$ of* Grass(n, r) *in $U(r+1, n+1)$. Then it follows*:

$$\tau^*_{\beta,\alpha}(\omega^{(\beta)}) = \tau_{\beta,\alpha}\omega^{(\alpha)}\tau^{-1}_{\beta,\alpha} \qquad (\alpha, \beta \in U(n+1)) . \tag{10}$$

Proof. From Lemma 2 it follows

$$\tau^*_{\beta,\alpha}(\omega^{(\beta)}) = \tau^*_{\beta,\alpha}(\omega\,|_{\sigma_\beta(W\beta)}) = \tau^*_{\beta,\alpha}(\omega)\,|_{\sigma_\alpha(W\alpha)}$$
$$= (\tau_{\beta,\alpha}\omega\tau^{-1}_{\beta,\alpha})\,|_{\sigma_\alpha(W\alpha)} = \tau_{\beta,\alpha}\omega\,|_{\sigma_\alpha(W\alpha)}\tau^{-1}_{\beta,\alpha} = \tau_{\beta,\alpha}\omega^{(\alpha)}\tau^{-1}_{\beta,\alpha} .$$

3. Let E be the exterior algebra generated by dw_{ip}, $d\overline{w}_{ip}$ $(0 \leq i \leq r;\ r+1 \leq p \leq n)$. We mean by $e(\xi)\eta$ and $i(\xi)\eta$ respectively the order and inner product $\xi \wedge \eta$ and the inner product of ξ with η with respect to the metric

$$2\sum_{i=0}^{r}\sum_{p=r+1}^{n}(dw_{ip}, d\overline{w}_{ip}) .$$

Since w_{ip} $(0 \leq i \leq r;\ r+1 \leq p \leq n)$ are independent complex variables, the inner and outer products satisfy

$$e(dw_{jp})i(dw_{jp}) + i(dw_{jp})e(dw_{jp}) = \mathrm{id} ,$$
$$e(d\overline{w}_{jp})i(d\overline{w}_{jp}) + i(d\overline{w}_{jp})e(d\overline{w}_{jp}) = \mathrm{id} .$$

Except these two cases $e(dw_{jp}), i(dw_{jp}), e(d\overline{w}_{jp}), i(d\overline{w}_{jp})$ $(0 \leq j \leq r;\ r+1 \leq p \leq n)$ are auti-commutative each other.

LEMMA 3.

$$\Big[\sum_{p=r+1}^{n} e(dw_{jp})e(d\overline{w}_{ip}),\ \sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{\ell p})\Big]$$
$$= -\delta_{k\ell}\sum_{p=r+1}^{n} i(d\overline{w}_{\ell p})e(d\overline{w}_{ip}) + \delta_{i\ell}\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{kp}) , \tag{11}$$

$$\Big[\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{ip}),\ \sum_{p=r+1}^{n} e(dw_{\ell p})e(d\overline{w}_{kp})\Big]$$
$$= \delta_{\ell i}\sum_{p=r+1}^{n} e(w_{jp})e(d\overline{w}_{kp}) , \tag{12}$$

$$\Big[\sum_{p=r+1}^{n} i(d\overline{w}_{jp})e(d\overline{w}_{ip}),\ \sum_{p=r+1}^{n} e(dw_{ip})e(d\overline{w}_{kp})\Big]$$
$$= -\delta_{jk}\sum_{p=r+1}^{n} e(dw_{\ell p})e(d\overline{w}_{ip}) , \tag{13}$$

(14)
$$\left[\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{ip}),\ \sum_{p=r+1}^{n} i(dw_{kp})i(\overline{w}_{ip})\right] = -\delta_{jk}\sum_{p=r+1}^{n} e(dw_{ip})i(d\overline{w}_{\ell p}) ,$$

(15)
$$\left[\sum_{p=r+1}^{n} i(d\overline{w}_{jp})e(d\overline{w}_{ip}),\ \sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{\ell p})\right] = \delta_{\ell i}\sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{jp}) .$$

Proof. From the above remark it follows

$$\begin{aligned}
&\sum_{p,q=r+1}^{n}\{e(dw_{jp})e(d\overline{w}_{ip})i(dw_{kq})i(d\overline{w}_{\ell q}) - i(dw_{kq})i(d\overline{w}_{\ell q})e(dw_{jp})e(d\overline{w}_{ip})\}\\
&= \sum_{p=r+1}^{n}\{-e(dw_{jp})i(dw_{kp})e(d\overline{w}_{ip})i(d\overline{w}_{\ell p}) + i(dw_{kp})e(dw_{jp})i(d\overline{w}_{\ell p})e(d\overline{w}_{ip})\}\\
&= -\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{kp})\{e(d\overline{w}_{ip})i(d\overline{w}_{\ell p}) + i(d\overline{w}_{\ell p})e(d\overline{w}_{ip})\}\\
&\quad + \delta_{kj}\sum_{p=r+1}^{n} i(d\overline{w}_{\ell p})e(d\overline{w}_{ip})\\
&= \delta_{kj}\sum_{p=r+1}^{n} i(d\overline{w}_{\ell p})e(d\overline{w}_{ip}) - \delta_{i\ell}\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{kp}) ,
\end{aligned}$$

$$\begin{aligned}
&\sum_{p,q=r+1}^{n}\{e(dw_{jp})i(dw_{ip})e(dw_{\ell p})e(d\overline{w}_{kq}) - e(dw_{\ell q})e(d\overline{w}_{kq})e(dw_{jp})i(dw_{ip})\}\\
&= \sum_{p=r+1}^{n} e(dw_{jp})\{i(dw_{ip})e(dw_{\ell p}) + e(dw_{\ell p})i(dw_{ip})\}e(d\overline{w}_{kp})\\
&= \delta_{i\ell}\sum_{p=r+1}^{n} e(dw_{jp})e(d\overline{w}_{kp}) ,
\end{aligned}$$

$$\begin{aligned}
&\sum_{p,q=r+1}^{n}\{i(d\overline{w}_{jp})e(d\overline{w}_{ip})e(dw_{\ell q})e(d\overline{w}_{kq}) - e(dw_{\ell q})e(d\overline{w}_{kq})i(d\overline{w}_{jp})e(d\overline{w}_{ip})\}\\
&= -\sum_{p=r+1}^{n} e(dw_{\ell p})\{i(d\overline{w}_{jp})e(d\overline{w}_{kp}) + e(d\overline{w}_{kp})i(d\overline{w}_{jp})\}e(d\overline{w}_{ip})\\
&= -\delta_{kj}\sum_{p=r+1}^{n} e(dw_{\ell p})e(d\overline{w}_{ip}) ,
\end{aligned}$$

$$\begin{aligned}
&\sum_{p,q=r+1}^{n}\{e(dw_{jp})i(dw_{ip})i(dw_{kq})i(d\overline{w}_{\ell q}) - i(dw_{kq})i(d\overline{w}_{\ell q})e(dw_{jp})i(dw_{jp})\}\\
&= -\sum_{p=r+1}^{n}\{e(dw_{jp})i(dw_{kp}) + i(dw_{kp})e(dw_{jp})\}i(dw_{ip})i(d\overline{w}_{\ell p})\\
&= -\delta_{jk}\sum_{p=r+1}^{n} i(dw_{ip})i(d\overline{w}_{\ell p}) ,
\end{aligned}$$

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$$\sum_{p,q=r+1}^{n}\{i(d\overline{w}_{jp})e(d\overline{w}_{ip})i(dw_{kq})i(d\overline{w}_{\ell q}) - i(dw_{kq})i(d\overline{w}_{\ell q})i(d\overline{w}_{jp})e(d\overline{w}_{ip})\}$$
$$= \sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{ip})\{e(d\overline{w}_{ip})i(d\overline{w}_{\ell p}) + i(d\overline{w}_{\ell p})e(d\overline{w}_{ip})\}$$
$$= \delta_{i\ell}\sum_{p=r+1}^{n} i(dw_{kp})i(d\overline{w}_{jp}) .$$

THEOREM 1. *Let L and Λ be the $(r+1)\times(r+1)$-matrices whose (i, j)-th entries are given by*

$$L_{ij} = \sqrt{-1}\sum_{p=r+1}^{n} e(dw_{ip})e(d\overline{w}_{jp})$$

and

$$\Lambda_{ij} = -\sqrt{-1}\sum_{p=r+1}^{n} i(dw_{ip})i(d\overline{w}_{jp}) .$$

Then there exists a representation ρ of Lie algebra sl$(2r+2)$ *such that*

$$\rho\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \mathrm{tr}\,(AL) = \sum_{i,j=0}^{r} a_{ij}L_{ji} \tag{16}$$

$$\rho\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \mathrm{tr}\,({}^tB\Lambda) = \sum_{i,j=0}^{r} b_{ij}\Lambda_{ij} . \tag{17}$$

Proof. Let ε_{ij} be the $(r+1)\times(r+1)$-matrix whose only non-zero entry is the (i, j)-th entry 1. We denote by ρ the linear mapping given by

$$\rho\begin{pmatrix} 0 & 0 \\ \varepsilon_{ij} & 0 \end{pmatrix} = L_{ij} ,$$
$$\rho\begin{pmatrix} 0 & \varepsilon_{ij} \\ 0 & 0 \end{pmatrix} = \Lambda_{ij} ,$$
$$\rho\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{ij} \end{pmatrix} = -\sum_{p=r+1}^{n} i(d\overline{w}_{jp})e(d\overline{w}_{ip}) ,$$
$$\rho\begin{pmatrix} \varepsilon_{ij} & 0 \\ 0 & 0 \end{pmatrix} = -\sum_{p=r+1}^{n} e(dw_{jp})i(dw_{ip}) .$$

Then by virtue of (11), (12), (13), (14), (15) it follows

$$\left[\rho\begin{pmatrix} 0 & 0 \\ \varepsilon_{ij} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_{k} \\ 0 & 0 \end{pmatrix}\right] = [L_{ji}, \Lambda_{k\ell}] = \delta_{kj}\rho\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{i\ell} \end{pmatrix} - \delta_{i\ell}\rho\begin{pmatrix} \varepsilon_{kj} & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \rho\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_{ij}\varepsilon_{k\ell} \end{pmatrix} - \rho\begin{pmatrix} \varepsilon_{k\ell}\varepsilon_{ij} & 0 \\ 0 & 0 \end{pmatrix} = \rho\left[\begin{pmatrix} 0 & 0 \\ \varepsilon_{ij} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon_{k\ell} \\ 0 & 0 \end{pmatrix}\right] ,$$

In the previous paper [ ] we have proved the essentially same result.

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$$\left[\rho\begin{pmatrix}\varepsilon_{ij} & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ \varepsilon_{k\ell} & 0\end{pmatrix}\right] = \rho\begin{pmatrix}-\varepsilon_{k\ell}\varepsilon_{ij} & 0\\ 0 & 0\end{pmatrix} = \rho\left[\begin{pmatrix}\varepsilon_{ij} & \\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ \varepsilon_{k\ell} & 0\end{pmatrix}\right],$$

$$\left[\rho\begin{pmatrix}0 & 0\\ 0 & \varepsilon_{ij}\end{pmatrix}, \rho\begin{pmatrix}0 & 0\\ \varepsilon_{k\ell} & 0\end{pmatrix}\right] = \rho\begin{pmatrix}-\varepsilon_{ij}\varepsilon_{k\ell} & 0\\ 0 & 0\end{pmatrix} = \rho\left[\begin{pmatrix}0 & 0\\ 0 & \varepsilon_{ij}\end{pmatrix}, \begin{pmatrix}0 & 0\\ \varepsilon_{k\ell} & 0\end{pmatrix}\right],$$

$$\left[\rho\begin{pmatrix}\varepsilon_{ij} & 0\\ 0 & 0\end{pmatrix}, \rho\begin{pmatrix}0 & \varepsilon_{k\ell}\\ 0 & 0\end{pmatrix}\right] = \rho\begin{pmatrix}0 & \varepsilon_{ij}\varepsilon_{k\ell}\\ 0 & 0\end{pmatrix} = \rho\left[\begin{pmatrix}\varepsilon_{ij} & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & \varepsilon_{k\ell}\\ 0 & 0\end{pmatrix}\right],$$

$$\left[\rho\begin{pmatrix}0 & 0\\ 0 & \varepsilon_{ij}\end{pmatrix}, \rho\begin{pmatrix}0 & \varepsilon_{k\ell}\\ 0 & 0\end{pmatrix}\right] = \rho\begin{pmatrix}0 & -\varepsilon_{k\ell}\varepsilon_{ij}\\ 0 & 0\end{pmatrix} = \rho\left[\begin{pmatrix}0 & 0\\ 0 & \varepsilon_{ij}\end{pmatrix}, \begin{pmatrix}0 & \varepsilon_{k\ell}\\ 0 & 0\end{pmatrix}\right],$$

This means that ρ gives a Lie algebra homomorphism of $(2r + 2)$. For any $(r + 1) \times (r + 1)$ matrices A and B

$$\rho\begin{pmatrix}0 & 0\\ A & 0\end{pmatrix} = \sum_{i,j=0}^{r} a_{ij}\rho\begin{pmatrix}0 & 0\\ \varepsilon_{ij} & 0\end{pmatrix} = \sum_{i,j=0}^{r} a_{ij}L_{ji} = \operatorname{tr}(AL),$$

$$\rho\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix} = \sum_{i,j=0}^{r} b_{ij}\rho\begin{pmatrix}0 & \varepsilon_{ij}\\ 0 & 0\end{pmatrix} = \sum_{i,j=0}^{r} b_{ij}\Lambda_{ij} = \operatorname{tr}(B\Lambda).$$

4. We mean by $E(\sigma_\alpha(W\alpha))$ the exterior algebra bundle generated by $T^*\sigma_\alpha(W\alpha) \oplus \bar{T}^*\sigma_\alpha(\alpha)$. We mean by $\omega^{(\alpha)}$ the restriction of the curvature form ω on the local cross section $\sigma_\alpha(W\alpha)$ of Grass(n, r), and we define linear operators acting on $E(\sigma_\alpha(W\alpha))$ as follows:

$$L_{ij}^{(\alpha)} = \sqrt{-1}\, e(\omega_{ij}^{(\alpha)}) \tag{18}$$

$$\Lambda_{ij}^{(\alpha)} = -\sqrt{-1}\, i_\alpha(\omega_{ij}^{(\alpha)}), \tag{19}$$

where $\omega_{ij}^{(\alpha)} = (\omega_{ij}^{(\alpha)})$ and $i_\alpha(\ )$ means the inner product with respect to the metric corresponding to $\operatorname{tr}\omega^{(\alpha)} = \sum_{i=0}^{r} \omega_{ii}^{(\alpha)}$.

LEMMA 5. *We denote*

$$\tau_{\beta,\alpha}^*(L_{ij}^{(\beta)}) = \sqrt{-1}\, e(\tau_{\beta,\alpha}^*(\omega_{ij}^{(\beta)})), \tag{20}$$

$$\tau_{\beta,\alpha}^*(\Lambda_{ij}^{(\beta)}) = \sqrt{-1}\, i_\alpha(\tau_{\beta,\alpha}^*(\omega_{ij}^{(\beta)})). \tag{20}$$

Then we have

$$\tau_{\beta,\alpha}^*(L^{(\beta)}) = \tau_{\beta,\alpha}L^{(\alpha)}\tau_{\beta,\alpha}^{-1}, \tag{22}$$

$$\tau_{\beta,\alpha}^*(\Lambda^{(\beta)}) = {}^t\tau_{\beta,\alpha}^{-1}\Lambda^{(\alpha)}\,{}^t\tau_{\beta,\alpha} \qquad (\alpha, \beta \in U(n + 1)). \tag{23}$$

Proof. Since $e(dw_{i\ell})$ and $i(dw_{i\ell})$ depend on $dw_{i\ell}$ $(0 \le i \le r;\ 0 \le \ell \le n)$ respectively covariantly and contravariantly, hence we have

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$$e(\tau_{\beta,\alpha}\omega\tau_{\beta,\alpha}^{-1}) = e(\tau_{\beta,\alpha}\omega^t\bar{\tau}_{\beta,\alpha}) = \tau_{\beta,\alpha}e(\omega)^t\bar{\tau}_{\alpha,\beta} = \tau_{\beta,\alpha}e(\omega)\tau_{\beta,\alpha}^{-1} ,$$
$$i(\tau_{\beta,\alpha}\omega\tau_{\beta,\alpha}^{-1}) = i(\tau_{\beta,\alpha}\omega^t\bar{\tau}_{\beta,\alpha}) = {}^t\tau_{\beta,\alpha}^{-1}i(\omega)^t\tau_{\beta,\alpha} .$$

Therefore by virtue of the definitions of L and Λ, we have (26) and (27).

PROPOSITION 2. *There exists a representation $\rho^{(\alpha)}$ of* sl $(2r+2)$ *as linear operators on $E(\sigma_\alpha(W\alpha))$ such that*

$$\rho^{(\alpha)}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \operatorname{tr}(AL^{(\alpha)}) , \tag{24}$$

$$\rho^{(\alpha)}\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \operatorname{tr}({}^tB\Lambda^{(\alpha)}) , \tag{25}$$

$$\tau_{\beta,\alpha}^*(\rho^{(\beta)}(X)\xi_\beta) = \rho^{(\alpha)}\left(\begin{pmatrix} \tau_{\beta,\alpha} & 0 \\ 0 & \tau_{\beta,\alpha} \end{pmatrix}^{-1} \times \begin{pmatrix} \tau_{\beta,\alpha} & 0 \\ 0 & \tau_{\beta,\alpha} \end{pmatrix}\right)\tau_{\beta,\alpha}^*(\xi_\beta) , \tag{26}$$
$$(\alpha, \beta \in U(n+1)\,;\, X \in \mathrm{sl}\,(2r+2)) .$$

Proof. Since $\omega_{(I,0)} = (\sum_{p=r+1}^n dw_{ip} \wedge d\bar{w}_{jp})$ and $dw_{ip}, d\bar{w}_{ip}$ $(0 \le i \le r;$ $r+1 \le p \le n)$ form a linear base of the fibre $T^*_{(I,0)}\sigma_e(W) \oplus \bar{T}^*_{(I,0)}\sigma_e(W)$, the representation ρ of sl $(2r+2)$ in Theorem 1 is the representation of sl $(2r+2)$ acting on the fibre of $E(\sigma_\alpha(W))$ at $(I,0)$ which satisfies (22) and (23). Since Grass (n,r) is homogeneous for $U(n+1)$ and ω is $U(n+1)$-invariant, translating the fibre by α in $U(n+1)$, we get a representation $\rho^{(\alpha)}$ of sl $(2r+2)$ acting on the fibre $E(\sigma_\alpha(W\alpha))$ at $\tilde{\pi}\alpha$ satisfying (22) and and (23), where $\tilde{\pi}\alpha$ means the upper $(r+1)$-part of α. For each point Z_β on $\sigma_\beta(W\beta)$ there exists an element α in $(U(n+1)$ such that $\tau_{\beta,\alpha}(\tilde{\pi}\alpha) = Z_\beta$. Hence it is sufficient to prove that, if there exists a representation $\rho^{(\alpha)}$ of sl $(2r+2)$ acting on the fibre of $E(\sigma_\alpha(W))$ at Z_α satisfying (22) and (23), then for a point $Z_\beta = \tau_{\beta,\alpha}Z_\alpha$ on $\sigma_\beta(W\beta)$ there exists a representation $\rho^{(\beta)}$ of sl $(2r+2)$ acting on the fibre $E(\sigma_\beta(W\beta))$ at Z_β satisfying (22), (23), (24). By virtue of (20) and (21) we have

$$\tau_{\beta,\alpha}^*(L^{(\beta)}) = \tau_{\beta,\alpha}L^{(\alpha)}\tau_{\beta,\alpha}^{-1}$$

and

$$\tau_{\beta,\alpha}^*(\Lambda^{(\beta)}) = {}^t\tau_{\beta,\alpha}^{-1}\Lambda^{(\alpha)t}\tau_{\beta,\alpha} ,$$

hence it follows:

$$\rho^{(\alpha)}\left(\begin{pmatrix} \tau_{\beta,\alpha} & 0 \\ 0 & \tau_{\beta,\alpha} \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}\begin{pmatrix} \tau_{\beta,\alpha} & 0 \\ 0 & \tau_{\beta,\alpha} \end{pmatrix}\right)$$
$$= \operatorname{tr}(\tau_{\beta,\alpha}^{-1}A\tau_{\beta,\alpha}L^{(\alpha)}) = \operatorname{tr}(A\tau_{\beta,\alpha}L^{(\alpha)}\tau_{\beta,\alpha}^{-1}) = \tau_{\beta,\alpha}^* \operatorname{tr}(AL^{(\beta)}) ,$$

$$\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1}\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}\right)$$
$$= \operatorname{tr}\,({}^t(\tau_{\beta,\alpha}^{-1}\beta\tau_{\beta,\alpha})A^{(\alpha)}) = \operatorname{tr}\,({}^tB\,{}^t\tau_{\beta,\alpha}^{-1}A^{(\alpha)}\,\tau_{\beta,\alpha}) = \tau_{\beta,\alpha}^{*}\operatorname{tr}\,(AL^{(\beta)})\ .$$

Since $\mathrm{sl}\,(2r+2)$ is generated by the elements

$$\left\{\begin{pmatrix}0 & 0\\ A & 0\end{pmatrix}, \begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\middle| A, B \text{ are } (r+1)\times(r+1) \text{ matrices}\right\}$$

we get a representation $\rho^{(\beta)}$ of $\mathrm{sl}\,(2r+2)$ acting on the fibre $E(\sigma_\beta(W\beta))$ at Z_β such that

$$\rho^{(\beta)}(X)\xi_\beta = \tau_{\beta,\alpha}^{*}\left(\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}\right)\tau_{\beta,\alpha}^{*}\xi_\beta\right)\ .$$

This representation $\rho^{0\beta)}$ satisfies, (22), (23) and (24). Morever the representation depends only on Z_β, which does not depend on the choise of α and Z_α, because by virtue of (8) and (03)

$$\tau_{\beta\gamma}\tau_{\gamma\alpha} = \tau_{\beta\alpha} \qquad (\alpha, \beta, \gamma \in U(n+1))$$

and

$$\tau_{\beta\alpha}\left\{\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}\right)\tau_{\beta,\alpha}^{*}\xi_\beta\right\}$$
$$= \tau_{\beta,\alpha}^{*}\tau_{\alpha,\gamma}^{*}\left\{\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta(\alpha}\end{pmatrix}^{-1}\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\gamma} & 0\\ 0 & \pi_{\beta,\gamma}\end{pmatrix}\right)\tau_{\beta,\alpha}^{*}(\tau_{\beta,\gamma}^{*}\xi_\beta)\right\}$$
$$= \tau_{\gamma,\beta}^{*}\left\{\rho^{(\gamma)}\begin{pmatrix}\tau_{\beta,\gamma} & 0\\ 0 & \tau_{\beta,\gamma}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\gamma} & 0\\ 0 & \tau_{\beta,\gamma}\end{pmatrix}\tau_{\beta,\gamma}^{*}\xi_\beta\right\}\ .$$

1.5. We can now construct a canonical $\mathrm{sl}\,(2r+2)$-Lie algebra bundle $L(n, r)$ which acts naturally on $E\,(\mathrm{Grass}\,(n, r))$.

We define an equivalence relation $\sim$ in

$$\bigcup_{\alpha\in U(n+1)} \sigma_\alpha(W\alpha) \times \mathrm{sl}\,(2r+2)$$

such that $(Z_\alpha, X_\alpha) \sim (Z_\beta, X_\beta)$ if and only if

$$\pi Z_\alpha = \pi Z_\beta$$

and

$$X_\beta = \begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1} X \begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix} \qquad (\alpha, \beta \in U(n+1))\ .$$

Since $\tau_{\gamma,\alpha}\tau_{\beta,\alpha}=\tau_{\gamma,\alpha}$ $(\alpha,\beta,\gamma\in U(n+1))$, the equivalence classes form a real analytic vector bundle

$$L(n,r)=\bigcup_{\alpha\in U(n+1)}\sigma_\alpha(W\alpha)\times \mathrm{sl}\,(2r+2)/\sim$$

over Grass (n,r). Let us show $L(n,r)$ acts naturally on E (Grass (n,r)). The bundle E (Grass (n,r)) may be expressed

$$E\ (\mathrm{Grass}\ (n,r))=\bigcup_{\alpha\in U(n+1)}E(\sigma_\alpha(W\alpha))/\sim\ .$$

where $\xi_\alpha\sim\xi_\beta$ if and only if $\xi_\alpha=\tau_{\beta,\alpha}\xi_\beta$. Lie algebra sl $(2r+2)$ acts on $E(\sigma_\alpha W\alpha))$ as follows:

$$(X,\xi_\alpha)\longrightarrow\rho^{(\alpha)}(X)\xi_\alpha\ .$$

By virtue of (30) in Proposition 2 it follows

$$\tau^*_{\beta,\alpha}(\rho^{(\beta)}(X)\xi_\beta)=\rho^{(\alpha)}\left(\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}^{-1}X\begin{pmatrix}\tau_{\beta,\alpha} & 0\\ 0 & \tau_{\beta,\alpha}\end{pmatrix}\right)\tau^*_{\beta,\alpha}(\xi_\beta)$$
$$(\alpha,\beta\in U(n+1),\xi_\beta\in E(\sigma_\beta(W\beta))\ .$$

This means that $L(n,r)$ acts on E (Grass (n,r)).

We now concluded that:

THEOREM 2. *Let* $U(n-r)\backslash U(n+1)\xrightarrow{\pi}$ Grass (n,r) *be the canonical* $U(r+1)$*-bundle over* Grass (n,r), *and let* ω *be the curvature form of the* $U(n+1)$*-invariant connections*

$$\theta=wd^t\overline{w}\ ,$$

where $U(n-r)\backslash U(n+1)$ *is regarded the space of* $(r+1)\times(n+1)$*-matrices* w *satisfying* $w\,^t\overline{w}=I$. *Let* E (Grass (n,r)) *be the exterior algebra bundle over* Grass (n,r) *generated by*

$$T^*\ \mathrm{Grass}\ (n,r)\oplus\overline{T}^*\ \mathrm{Grass}\ (n,r)\ .$$

Then there exists a sl $(2r+2)$-Lie algebra bundle $L(n,r)$ over Grass (n,r) with the following properties:

i) The action of $U(n+1)$ on Grass (n,r) is lifted on $L(n,r)$.

ii) The $U(r+1)$-bundle $U(n-r)\backslash U(n+1)$ acts on $L(n,r)$ as follows

$$(\tau,\rho(X))\longrightarrow\rho\left(\begin{pmatrix}\tau & 0\\ 0 & \tau\end{pmatrix}^{-1}X\begin{pmatrix}\tau & 0\\ 0 & \tau\end{pmatrix}\right)$$

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for local sections τ and ρ into $U(n-r)\backslash U(n+1)$ and $L(n,r)$, respectively.

iii) $L(n,r)$ acts on $E(\mathrm{Grass}(n,r))$ as follows:

Let E be the exterior algebra generated over the dual space of the horizontal space at a point Z on $U(n-r)\backslash U(n+1)$ with respect to the connection $wd^t\bar{w}$, and let L_{ij} and Λ_{ij} $(0 \leq i, j \leq r)$ be the linear operators actings on E given by

$$L_{ij} = \sqrt{-1}\, e(\tilde{\omega}_{ij}) \ , \qquad \Lambda_{ij} = -\sqrt{-1}\, i(\tilde{\omega}_{ij}) \ ,$$

where $\tilde{\omega} = (\tilde{\omega}_{ij})$ is the restriction of the curvature form ω on the horizontal space and the inner product $i(\ )$ corresponds to $\mathrm{tr}\,\tilde{\omega}$. Then, identified E with the fibre of $E(\mathrm{Grass}(n,r))$ at the base point, of Z, the action of $L(n,r)$ on E is given by

$$\rho\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} = \mathrm{tr}\,(AL) = \sum_{i,j=0}^{r} a_{ij}L_{ji} \ ,$$

$$\rho\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} = \mathrm{tr}\,({}^tB\Lambda) = \sum_{i,j=0}^{r} b_{ij}\Lambda_{ij} \ .$$

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