Canad. Math. Bull. Vol. **54** (2), 2011 pp. 338–346
doi:10.4153/CMB-2011-017-4

Szegö's Theorem and Uniform Algebras

Takahiko Nakazi

Abstract. We study Szegö's theorem for a uniform algebra. In particular, we do it for the disc algebra or the bidisc algebra.

1 Introduction

Let A be a uniform algebra on a compact Hausdorff space X. Let τ be a complex homomorphism of A and m the representing measure of τ on X. We have that $L^p(m) = L^p(X, m)$ denotes the usual Lebesgue space for $1 \leq p \leq \infty$. For a nonnegative function W in $L^1(m) = L^1(X, m)$, put

$$S(W) = \inf_{g \in A_\tau} \int_X |1 - g|^2 W\, dm$$

where A_τ is the kernel of τ. $S(W)$ is called a Szegö infimum.

Let D be the open unit disc in $\mathbb{C}$. Suppose A is the disc algebra on X and $X = \partial D$. When $dm = d\theta/2\pi$, it is well known that

$$S(W) = \exp \int_{\partial D} \log W\, d\theta/2\pi.$$

This is the celebrated theorem of G. Szegö [5]. In [3], the author studied a Szegö infimum $S(W)$ when A is the bidisc algebra on $\partial D \times \partial D$, and $dm = d\theta_1 d\theta_2/4\pi^2$.

In this paper, we study a Szegö infimum when A is the disc algebra on X, $X = \bar{D}$ and $dm = r dr d\theta/\pi$. Unfortunately we cannot choose the method used in the bidisc algebra on $\partial D \times \partial D$ [3]. We need a new technique.

For $p = 1, 2$, $H^p(m) = H^p(X, m)$ denotes the abstract Hardy space for A, that is, the closure $[A]_m$ of A in $L^p(X, m)$. For a nonnegative function W in $L^1(X, m)$, $H^p(W) = H^p(X, W dm)$ denotes the closure $[A]_{Wdm}$ of A in $L^p(X, W dm)$. In this paper, we will assume that m is a Jensen measure of τ, but also that Jensen's inequality is valid for any function in $H^p(X, m)$ (see [2]). If h is a function in $H^2(X, m)$ and $[hA]_m = H^2(X, m)$ then h is called a generator in $H^2(X, m)$.

In Sections 2 and 3, we study the Szegö infimum for an arbitrary uniform algebra. In Section 2, we study when the Szegö infimum $S(W)$ is the arithmetic mean of the weight W or the geometric mean of W. In Section 3, we study when $S(W)$ is the mixed mean of the arithmetic mean and the geometric mean of W. In Section 4,

Received by the editors June 13, 2008; revised March 4, 2009.
Published electronically February 10, 2011.
This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education
AMS subject classification: **32A35**, 46J15, 60G25.
Keywords: Szegö's theorem, uniform algebras, disc algebra, weighted Bergman space.

we apply the result in Section 3 when A is the bidisc algebra on $X = \partial D \times \partial D$. In Section 5, we apply the results in Section 2 and 3 when A is the disc algebra on $X = \bar{D}$, and we prove our main results in this paper, that is, Theorems 5.1 and 5.2.

2 Arithmetic and Geometric Means

For a nonnegative function W in $L^1(X, m)$,

$$\int_X W\,dm \quad \text{and} \quad \exp \int_X \log W\,dm$$

are called an arithmetic mean and a geometric mean, respectively. Since m is a Jensen measure of τ, it is easy to see that

$$\int_X W\,dm \geq S(W) \geq \exp \int_X \log W\,dm.$$

Theorem 2.1 *Let W be a nonnegative function in $L^1(X, m)$, then $S(W) = \int_X W\,dm$ if and only if $\int_X fW\,dm = \tau(f) \int_X W\,dm$ $(f \in A)$.*

Proof If $S(W) = \int_X W\,dm$, then for any $g \in A_\tau$,

$$\int_X W\,dm \leq \int_X W\,dm - 2Re \int_X gW\,dm + \int_X |g|^2 W\,dm,$$

and so

$$2Re \int_X gW\,dm \leq \int_X |g|^2 W\,dm.$$

Suppose $\int_X gW\,dm \neq 0$. For $\alpha = |\int_X gW\,dm| / \int_X gW\,dm$, consider αg as g. Then

$$2\Big|\int_X gW\,dm\Big| \leq \int_X |g|^2 W\,dm.$$

Consider $tg \in A_\tau$ for $0 < t < 1$. Then

$$2t\Big|\int_X gW\,dm\Big| \leq t^2 \int_X |g|^2 W\,dm$$

and so $\int_X gW\,dm = 0$ as $t \to 0$. This contradiction implies the "only if" part. The "if" part is clear. ■

Theorem 2.2 *Let W be a nonnegative function in $L^1(X, m)$. If $W = |h|^2$ for some generator h in $H^2(X, m)$, then $S(W) = \exp \int_X \log W\,dm > 0$.*

If $S(W) = \exp \int_X \log W\,dm > 0$, then there exists a function h in $H^2(X, m)$ such that $|h|^2 W = c$ a.e.m for some positive constant c.

Proof If $W = |h|^2$ for some generator h, then $S(|h|^2) = |\int_X hdm|^2$ and

$$S(|h|^2) \geq \exp \int_X \log |h|^2 dm \geq \Big| \int_X hdm \Big|^2.$$

Hence $S(|h|^2) = \exp \int_X \log |h|^2 dm$.

Suppose $S(W) = \exp \int_X \log W dm > 0$. Then τ is continuous on $H^2(W)$, and so there exists a function f in $H^2(W)$ such that

$$\int_X f dm = 1 \text{ and } \inf_{g \in A_\tau} \int_X |1 - g|^2 W dm = \int_X |f|^2 W dm.$$

By Jensen's inequality,

$$\int_X |f|^2 W dm \geq \exp \int_X \log W dm \exp \int_X \log |f|^2 dm \geq \Big| \int_X f dm \Big|^2 \exp \int_X \log W dm.$$

Thus

$$\int_X |f|^2 W dm = \exp \int_X \log |f|^2 W dm,$$

and so $|f|^2 W = c$ $a.e.m$ for some positive constant c. ■

3 Intermediate Mean

Let $W = W_1 W_2$ be in $L^1(X, m)$, where W_j is a nonnegative function in $L^1(X, m)$ for $j = 1, 2$. Then

$$\int_X W_1 dm \int_X W_2 dm \geq \int_X W_1 dm \exp \int_X \log W_2 dm \geq \exp \int_X \log W_1 W_2 dm.$$

It may happen that

$$\int_X W_1 W_2 dm = \int_X W_1 dm \int_X W_2 dm.$$

Theorem 3.1 *Let $W_1 dm / \int_X W_1 dm$ be a representing measure for τ and $W_2 = |h|^2$ for some generator h in $H^2(X, m)$. If $W = W_1 W_2$ is in $L^1(X, m)$, then*

$$S(W) \geq \int_X W_1 dm \exp \int_X \log W_2 dm.$$

If W_1 is in $L^\infty(X, m)$, then the equality is valid.

Proof Since $W_2 = |h|^2$ and $h \in H^2(m)$, by Schwarz's inequality

$$\begin{aligned} S(W) &= \inf_{g \in A_\tau} \int_X |h - hg|^2 W_1 dm \\ &\geq \inf_{g \in A_\tau} \Big| \int_X h W_1 dm - \int_X hg W_1 dm \Big|^2 \Big(\int_X W_1 dm \Big)^{-1} \\ &= \Big| \int_X h W_1 dm \Big|^2 \Big(\int_X W_1 dm \Big)^{-1} = \Big| \int_X hdm \Big|^2 \int_X W_1 dm, \end{aligned}$$

https://doi.org/10.4153/CMB-2011-017-4 Published online by Cambridge University Press

because $W_1dm/\int_X W_1dm$ is a representing measure of τ. If $W_1 \in L^\infty(m)$, the closure of A_τ in $L^2(m)$ belongs to the closure of hA_τ in $L^2(W_1dm)$ and hence

$$S(W) = \inf_{g\in A_\tau} \int_X |h - hg|^2 W_1 dm = \Big| \int_X hdm \Big|^2 \int_X W_1 dm.$$

On the other hand, by Theorem 2.2

$$\Big| \int_X hdm \Big|^2 = S(W_2) = \exp \int_X \log W_2 dm.$$

This implies the theorem. ■

If W is a nonnegative function in $L^1(X, m)$, then $H^2(W)$ denotes the closure of A in $L^2(X, Wdm)$. If $W \equiv 1$, then $H^2(W) = H^2(m) = H^2(X, m)$ and

$$[\sqrt{W}A]_m = \sqrt{W}H^2(W).$$

Lemma 3.2 *Let W be a nonnegative function in $L^1(X, m)$. If $[\sqrt{W}A]_m \ominus [\sqrt{W}A_\tau]_m$ contains a cyclic vector u, then $[\sqrt{W}A]_m = q\sqrt{W_1}H^2(W_1)$, where q is a unimodular function and $W_1dm/\int_X W_1dm$ is a representing measure for τ.*

Proof If $u \in [\sqrt{W}A]_m \ominus [\sqrt{W}A_\tau]_m$, then u is orthogonal to uA_τ, and so $|u|^2$ is orthogonal to A_τ. Put

$$q(x) = \begin{cases} u(x)/|u(x)| & \text{if } u(x) \neq 0, \\ 1 & \text{if } u(x) = 0, \end{cases}$$

and $W_1 = |u|^2$, then $u = q\sqrt{W_1}$ and $W_1dm/\int_X W_1dm$ is a representing measure of τ. If u is a cyclic vector, then $[\sqrt{W}A]_m = [uA]_m = q\sqrt{W_1}H^2(W_1)$. ■

Theorem 3.3 *Let W be a nonnegative function in $L^1(X, m)$ and suppose $[\sqrt{W}A]_m \ominus [\sqrt{W}A_\tau]_m$ has a cyclic vector u. Then $W = W_1W_2$, where $W_1 = |u|^2$ and $W_2 = |h|^2$ for some h in $H^2(W_1)$ such that hA is dense in $H^2(W_1)$.*

(i) $S(W) = |\int_X hW_1dm|^2(\int_X W_1dm)^{-1}$.

(ii) *If W_1^{-1} belongs to $L^\infty(X, m)$, then*

$$S(W) = \Big| \int_X hdm \Big|^2 \int_X W_1dm = \int_X W_1dm \exp \int_X \log W_2dm.$$

Proof By Lemma 3.2, $\sqrt{W} = q\sqrt{W_1}h$, $[hA]_{W_1dm} = H^2(W_1)$, and $W_1dm/\int W_1dm$ is a representing measure for τ. Hence $W = W_1|h|^2$ and $h - (\int_X hW_1dm)(\int_X W_1dm)^{-1}$ belongs to $[A_\tau]_{W_1dm}$. Since hA is dense in $H^2(W_1) = [A]_{W_1dm}$, hA_τ is dense in $[A_\tau]_{W_1dm}$. Hence

$$S(W) = \int_X \Big| \Big(\int_X hW_1dm\Big) \Big(\int_X W_1dm\Big)^{-1} \Big|^2 W_1dm.$$

This implies (i). If W_1^{-1} belongs to $L^\infty(m)$, then $H^2(W_1) \subseteq H^2(m)$, and so h belongs to $H^2(m)$. Hence h is a generator in $H^2(m)$. Thus $\int_X hW_1dm = \int_X hdm \int_X W_1dm$ and $\exp \int_X \log|h|dm = |\int_X hdm|$ by the proof of Theorem 3.1. This implies (ii). ■

https://doi.org/10.4153/CMB-2011-017-4 Published online by Cambridge University Press

4 Bidisc Algebra on $\partial D \times \partial D$

In this section, A denotes the bidisc algebra on $X = \partial D \times \partial D$, $\tau(f) = f(0,0)$ $(f \in A)$ and $dm = d\theta_1 d\theta_2/4\pi^2$. Then m is a Jensen measure of τ. In [3], we gave a necessary and sufficient condition for that $S(W) = \int_X W\,dm$. The condition is equivalent to that in Theorem 2.1. In [3], we also proved that $S(W) = \exp \int_X \log W\,dm$ if and only if $W = |h|^2$ for some generator h in $H^2(X, m)$. For the proof, the "if" part is the same as the one in Theorem 2.2. We cannot use Theorem 2.2 for the "only if" part. In [3], we proved it in a different way.

Let r be a rational number and E_r denote a subset of Z. For $-\infty < r < 0$, suppose W_{1r} is a nonnegative function in $L^1(X, m)$ such that

$$W_{1r} \sim \sum_{t \in E_r} a_t \zeta^{t\alpha},$$

where $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = r\alpha_2$ and $\zeta^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}$. For $0 < r < \infty$, suppose W_{2r} is a nonnegative function in $L^1(X, dm)$ such that

$$\log W_{2r} \sim \sum_{t \in E_r} b_t \zeta^{t\alpha},$$

where $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = r\alpha_2$. Suppose $W_1 = \sum_r W_{1r}$ is a finite sum for $-\infty < r < 0$, and $W_2 = \prod_r W_{2r}$ is a finite product for $0 < r < \infty$, respectively. Then W_j belongs to $L^1(X, m)$ for $j = 1, 2$, $W_1 / \int_X W_1 dm$ is a representing measure of the origin, and it is easy to see that there exists a generator h in $H^2(X, m)$ with $W_2 = |h|^2$ when $E - r = Z$, for $0 < r < \infty$.

In fact, if F and G are in $L^1(X, m)$, and

$$F \sim \sum_{t \in E_r} a_t \zeta^{\alpha t} \ \text{ and } \ G \sim \sum_{t \in E_\ell} b_t \zeta^{\beta t}$$

where $\alpha_1 = r\alpha_2$, $\beta_1 = \ell\beta_2$ and $r \neq \ell$, then

$$\int_X FG dm = \int_X F dm \int_X G dm.$$

This implies that W_2 belongs to $L^1(X, m)$. By one variable theory, for each $0 < r < \infty$ there exists a generator h_r in $H^2(X, m)$ such that

$$h_r = \sum_{t \in E_r \cap Z_+} c_t \zeta^{\beta t} \ \text{ and } \ W_{2r} = |h_r|^2.$$

Then it is easy to see that $W_2 = |\prod_r h_r|^2$ and $\prod_r h_r$ is a generator in $H^2(X, m)$. For if h_r and h_s are generators and $r \neq s$ then there exist sequences h_{rn} and h_{sn} such that

$$\int_X |h_r h_{rn} - 1|^2 dm \to 0 \ \text{ and } \ \int_X |h_s h_{sn} - 1|^2 dm \to 0,$$

where h_{rn} and h_{sn} are polynomials. Then

$$\begin{aligned}
&\Big(\int_X |h_r h_s h_{rn} h_{sn} - 1|^2 dm\Big)^{1/2} \\
&\leqq \Big(\int_X |(h_r h_{rn} - 1) h_s h_{sn}|^2 dm\Big)^{1/2} + \Big(\int_X |h_s h_{sn} - 1|^2 dm\Big)^{1/2} \\
&= \Big(\int_X |h_r h_{rn} - 1|^2 dm\Big)^{1/2} \Big(\int_X |h_s h_{sn}|^2 dm\Big)^{1/2} + \Big(\int_X |h_s h_{sn} - 1|^2 dm\Big)^{1/2}.
\end{aligned}$$

Hence the product $h_r h_s$ is also a generator. Hence if W_1 is in $L^\infty(X, m)$ applying Theorem 3.1 to $W = W_1 W_2$, $S(W) = \int_X W_1 dm \exp \int_X \log W_2 dm$. On the other hand, if $W_2 = W_{2r}$ for some $0 < r < \infty$ without assuming W_1 in $L^\infty(X, m)$, then we can show that $S(W) = \int_X W_1 dm \exp \int_X \log W_2 dm$. In fact, if $g_r = \sum_{t \in E_r \cap Z_+} c_t \zeta^{\beta_t}$ with $c_0 = 0$, then

$$\int_X |1 - g_r|^2 W_1 W_2 dm = \int_X W_1 dm \int_X |1 - g_r|^2 W_2 dm \geq S(W_1 W_2).$$

Now Theorem 3.1 implies that $S(W) = \int_X W_1 dm \exp \int_X \log W_2 dm$.

5 Disc Algebra on $\bar{D}$

In this section, A denotes the disc algebra on $X = \bar{D}$, $\tau(f) = f(0)$ ($f \in A$), and $dm = r dr d\theta/\pi$. Then m is a Jensen measure of τ. In this situation, $S(W)$ has not been studied. $H(D)$ denotes the set of all holomorphic functions on D. In the following theorem, the "if" part is just a corollary of Theorem 2.2. For the "only if" part we can use Theorem 2.2 unlike in the case of the bidisc algebra (see §4 or [3]).

Theorem 5.1 *Let $X = \bar{D}$, $A =$ the disc algebra on $\bar{D}$ and $dm = r dr d\theta/\pi$. Suppose W is a nonnegative function in $L^1(\bar{D}, m)$ and $\log W$ is in $L^1(\bar{D}, m)$. Then $S(W) = \exp \int_X \log W dm$ if and only if $W = |h|^2$ for some generator h in $H^2(\bar{D}, m)$.*

Proof By Theorem 2.2, if $W = |h|^2$ for some generator h in $H^2(m)$, then $S(W) = \exp \int_{\bar{D}} \log W dm > 0$.

If $S(W) = \exp \int_{\bar{D}} \log W dm > 0$, then τ is continuous on $H^2(W)$ and so there exists a function f in $H^2(W)$ such that

$$f(0) = \int_{\bar{D}} f dm = 1 \text{ and } S(W) = \int_{\bar{D}} |f|^2 W dm.$$

Since $\int_{\bar{D}} \log W dm > -\infty$, $H^2(W) \subset H(D)$ by [4] and so f is analytic on D. Since

$$\int_{\bar{D}} |f|^2 W dm \geq \exp \int_{\bar{D}} \log W dm \exp \int_{\bar{D}} \log |f|^2 dm \geq \exp \int_{\bar{D}} \log W dm$$

because $\exp \int_{\bar{D}} \log |f|^2 dm \geq |f(0)|^2 = 1$, $|f|^2 W = c > 0$ *a.e.m.* If $f(a) = 0$ for some $a \in D$, then there exists a positive integer ℓ such that $f = (z - a)^\ell g$, $g \neq 0$ on

https://doi.org/10.4153/CMB-2011-017-4 Published online by Cambridge University Press

$D(a,2\varepsilon)$ for some $\varepsilon > 0$ and $g \in H(D)$. Hence

$$\int_{\bar D} W\,dm = c\int_{\bar D} |f^{-1}|^2 dm = c\int_D \frac{1}{|z-a|^{2\ell}}|g^{-1}|^2 dm$$
$$\geq c\delta \int_{D(a,\varepsilon)} \frac{1}{|z-a|^{2\ell}} dm$$

where $\delta^{-1} = \inf\{|g(z)|^2 : z \in D(a,\varepsilon)\}$. While

$$\int_{D(a,\varepsilon)} \frac{1}{|z-a|^{2\ell}} dm = \int_{D(0,\varepsilon)} \frac{1}{|z|^{2\ell}} \frac{(1-|a|^2)^2}{|1-\bar a z|^4} dm$$
$$\geq \Big(\frac{1-|a|}{1+|a|}\Big)^2 \int_{D(0,\varepsilon)} \frac{1}{|z|^{2\ell}} dm = \infty.$$

This contradiction implies that f has no zeros on D. Hence f^{-1} belongs to $H(D) \cap L^2(m)$. Since it is known that $H(D) \cap L^2(m) = H^2(m)$, f^{-1} belongs to $H^2(m)$. Hence $\int_{\bar D} \log|f|dm = \log|f(0)|$ because $\int_{\bar D} \log|f|dm \geq \log|f(0)|$. Put $h = \sqrt{c}f^{-1}$, then $W = |h|^2$ and

$$S(W) = \exp\int_{\bar D} \log W\,dm = |h(0)|^2$$

and so $h - h(0)$ belongs to $[hA_\tau]_m$. This implies that $h(0)$ belongs to $[hA]_m$, and so h is a generator in $H^2(m)$. ■

If $W = |f|^2$ for some $f \in H^2(\bar D, m)$, then $W = W_1 W_2$, where $W_1 = |Q|^2$ for some inner function in $H^2(\bar D, m)$, $W_1 dm / \int_{\bar D} W_1 dm$ is a representing measure of the origin and $W_2 = |h|^2$ for some generator h in $H^2(\bar D, m)$. This is a deep result of a factorization theorem for a function in the Bergman space [1]. Hence if W_1 is in $L^\infty(\bar D, m)$, then $H^2(\bar D, m) \subseteq H^2(\bar D, W_1 dm)$, and so hA is dense in $H^2(\bar D, W_1 dm)$. Hence

$$S(W) = \Big|\int_{\bar D} hW_1 dm\Big|^2 \Big(\int_{\bar D} W_1 dm\Big)^{-1}.$$

Moreover, if W_1^{-1} is in $L^\infty(\bar D, m)$, then

$$S(W) = \Big|\int_{\bar D} h\,dm\Big|^2 \int_{\bar D} W_1 dm$$
$$= \int_{\bar D} W_1 dm \exp\int_{\bar D} \log W_2 dm.$$

In general, it is easy to see that

$$S(W) \geq \int_0^1 2r\,dr \exp\int_0^{2\pi} \log W(re^{i\theta})d\theta/2\pi$$
$$\geq \exp\int_{\bar D} \log W\,dm.$$

https://doi.org/10.4153/CMB-2011-017-4 Published online by Cambridge University Press

Theorem 5.2 *Let $X = \bar{D}$, $A =$ the disc algebra on $\bar{D}$ and $dm = rdrd\theta/\pi$. Suppose W is a positive function in $L^1(\bar{D}, m)$ and $\log W$ is in $L^1(\bar{D}, m)$. If*

$$S(W) = \int_0^1 2rdr \exp \int_0^{2\pi} \log W(re^{i\theta}) d\theta/2\pi,$$

then $W(re^{i\theta}) = \phi(r)|h(re^{i\theta})|^2$, where

$$h \in H(D), \quad \exp \int_0^{2\pi} \log |h(re^{i\theta})| d\theta/2\pi = |h(0)| > 0$$

and ϕ is a positive function in $L^1([0,1], dr)$. Conversely if $W(re^{i\theta}) = \phi(r)|h(re^{i\theta})|^2$ and h is a generator in $H^2(\bar{D}, \phi dm)$, then

$$S(W) = \int_0^1 2rdr \exp \int_0^{2\pi} \log W(re^{i\theta}) d\theta/2\pi.$$

Proof Since $\log W \in L^1(m)$, by the proof of Theorem 5.1 there exists f in $H(D)$ with $f(0) = 1$ such that

$$\begin{aligned}\inf_{g \in A_\tau} \int_{\bar{D}} |1-g|^2 W dm = \int_{\bar{D}} |f|^2 W dm &= \int_0^1 2rdr \int_0^{2\pi} |f(re^{i\theta})|^2 W(re^{i\theta}) d\theta/2\pi \\ &\geq \int_0^1 2rdr \exp \int_0^{2\pi} \log |f(re^{i\theta})|^2 W(re^{i\theta}) d\theta/2\pi \\ &\geq \int_0^1 2rdr \exp \int_0^{2\pi} \log W(re^{i\theta}) d\theta/2\pi.\end{aligned}$$

If $S(W) = \int_0^1 2rdr \exp \int_0^{2\pi} \log W(re^{i\theta}) d\theta/2\pi$, then

$$\int_0^1 2rdr \int_0^{2\pi} |f(re^{i\theta})|^2 W(re^{i\theta}) d\theta/2\pi = \int_0^1 2rdr \exp \int_0^{2\pi} \log |f(re^{i\theta})|^2 W(re^{i\theta}) d\theta/2\pi.$$

Hence for $a.e. r \in [0,1]$

$$\int_0^{2\pi} |f(re^{i\theta})|^2 W(re^{i\theta}) d\theta/2\pi = \exp \int_0^{2\pi} \log |f(re^{i\theta})|^2 W(re^{i\theta}) d\theta/2\pi$$

and $\int_0^{2\pi} \log |f(re^{i\theta})| d\theta/2\pi = \log |f(0)|$. Therefore $|f(re^{i\theta})|^2 W(re^{i\theta}) = \phi(r)$ for $a.e. \theta \in [0, 2\pi]$. If $f(a) = 0$ for some $a \in D$, then $\phi(r) = 0$ for $r = |a|$. Since $W(re^{i\theta}) > 0$, $f(re^{i\theta}) = 0$ for $r = |a|$. This contradiction implies that $|f(z)| > 0$ for

https://doi.org/10.4153/CMB-2011-017-4 Published online by Cambridge University Press

$z \in D$. Put $h(re^{i\theta}) = 1/f(re^{i\theta})$, then $W(re^{i\theta}) = \phi(r)|h(re^{i\theta})|^2$, where $h \in H(D)$ and $\exp \int_0^{2\pi} \log |h(re^{i\theta})| d\theta/2\pi = \log |h(0)|$.

Conversely, if $W(re^{i\theta}) = \phi(r)|h(re^{i\theta})|^2$ and h is a generator in $H^2(\bar{D}, \phi dm)$, then

$$\begin{aligned}\inf_{g \in A_\tau} \int_{\bar{D}} |1-g|^2 W\, dm = \Big| \int_{\bar{D}} h\phi dm \Big|^2 &= |h(0)|^2 \int_{\bar{D}} \phi dm \\ &= \int_0^1 2r\phi(r) dr \exp \int_0^{2\pi} \log |h(re^{i\theta})|^2 d\theta/2\pi \\ &= \int_0^1 2r dr \exp \int_0^{2\pi} \log W(re^{i\theta}) d\theta/2\pi.\end{aligned}$$

Here we used that $\phi dm / \int_{\bar{D}} \phi dm$ is a representing measure of τ. ■

Suppose $W = |f|^2$ for some f in the Hardy space $H^2(\partial D, d\theta/2\pi)$ and $f = qh$, where q is inner and h is outer. Then

$$\exp \int_{\bar{D}} \log |h|^2 dm \geq S(W) \geq \exp \int_{\bar{D}} \log |q|^2 dm \exp \int_{\bar{D}} \log |h|^2 dm.$$

In fact, $|h|^2 \geq W$ and $S(|h|^2) = \exp \int_{\bar{D}} \log |h|^2 dm$ by Theorem 5.1.

Suppose $W = \chi_E |h|^2$, where χ_E is the characteristic function of $E = \{z \in \bar{D} : r_0 \leq |z| < 1\}$, and h is a generator. Put $W_1 = \chi_E$ then $W_1 dm/(1 - r_0^2)$ is a representing measure of τ. By the proof of Theorem 3.1,

$$S(W) = \Big| \int_{\bar{D}} h W_1 dm \Big|^2 \Big(\int_{\bar{D}} W_1 dm \Big)^{-1} = |h(0)|^2 (1 - r_0^2).$$

Hence

$$S(W) = \int_{r_0}^1 2r dr \exp \int_0^{2\pi} \log W(re^{i\theta}) d\theta/2\pi.$$

Suppose E is a simply connected domain in D whose boundary contains the origin. Then $S(\chi_E) = 0$, since

$$\inf_{g \in A_\tau} \int_E |1-g|^2 dm = \inf_{f \in A} \int_{\bar{D}} |1 - (z-1) f|^2 dm = 0.$$

References

[1] A. Aleman, S. Richter, and C. Sundberg, *Beurling's theorem for the Bergman space.* Acta Math.**177**(1996), no. 2, 275–310. doi=10.1007/BF02392623 doi:10.1007/BF02392623

[2] T. W. Gamelin, *Uniform algebras.* Prentice-Hall, Englewood Cliffs, NJ, 1969.

[3] T. Nakazi, *Szegö's theorem on a bidisc.* Trans. Amer. Math. Soc. **328**(1991), no. 1, 421–432. doi=10.2307/2001888 doi:10.2307/2001888

[4] T. Nakazi and M. Yamada, *Riesz's functions in weighted Hardy and Bergman spaces.* Canad. J. Math. **48**(1996), no. 5, 930–945.

[5] G. Szegö, *Beiträge zur Theorie der Toeplitzschen Formen.* Math. Z. **6**(1920), no. 3–4, 167–202. doi:10.1007/BF01199955

Hokusei Gakuen University, Sapporo 004-8631, Japan
e-mail: z00547@hokusei.ac.jp