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Introduction and Overview

This is the second volume of our two-volume book on factorization algebras as they apply to quantum field theory. In Volume 1, we focused on the theory of factorization algebras while keeping the quantum field theory to a minimum. Indeed, we only ever discussed free theories. In Volume 2, we will focus on the factorization algebras associated with interacting classical and quantum field theories.

In this introduction, we will state in outline the main results that we prove in this volume. The centerpiece is a deformation quantization approach to quantum field theory, analogous to that for quantum mechanics, and the introduction to the first volume provides an extensive motivation for this perspective, which is put on solid footing here. Subsequently we explore symmetries of field theories that fit into this approach, leading to classical and quantum versions of the Noether theorem in the language of factorization algebras.

Remark: Throughout the text, we refer to results from the first volume in the style “see Chapter I.2” to indicate the second chapter of Volume 1. ◊

1.1 The Factorization Algebra of Classical Observables

We will start with the factorization algebra associated with a classical field theory. Suppose we have a classical field theory on a manifold M, given by some action functional, possibly with some gauge symmetry. To this data we will associate a factorization algebra of *classical observables*. The construction goes as follows. First, for every open subset $U \subset M$, consider the space $\mathcal{EL}(U)$ of solutions to the Euler–Lagrange equations on U, modulo gauge. We work in perturbation theory, which means we consider solutions that live in the formal neighborhood of a fixed solution. We also work in the derived sense, which

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means we "impose" the Euler–Lagrange equations by a Koszul complex. In sophisticated terms, we take $\mathcal{EL}(U)$ to be the formal derived stack of solutions to the equations of motion. As U varies, the collection $\mathcal{EL}(U)$ forms a sheaf of formal derived stacks on M.

The factorization algebra Obs^{cl} of classical observables of the field theory assigns to an open U, the dg commutative algebra $\mathscr{O}(\mathcal{EL}(U))$ of functions on this formal derived stack $\mathcal{EL}(U)$. This construction is simply the derived version of functions on solutions to the Euler–Lagrange equations, and hence provides a somewhat sophisticated refinement for classical observables in the typical sense.

It takes a little work to set up a theory of formal derived geometry that can handle formal moduli spaces of solutions to nonlinear partial differential equations like the Euler–Lagrange equations. In the setting of derived geometry, formal derived stacks are equivalent to homotopy Lie algebras (i.e., Lie algebras up to homotopy, often modeled by dg Lie algebras or L_∞ algebras). The theory we develop in Chapter 3 takes this characterization as a definition. We define a formal elliptic moduli problem on a manifold M to be a sheaf of homotopy Lie algebras satisfying certain properties. Of course, for the field theories considered in this book, the formal moduli of solutions to the Euler–Lagrange equations always define a formal elliptic moduli problem. We develop the theory of formal elliptic moduli problems sufficiently to define the dg algebra of functions, as well as other geometric concepts.

1.2 The Factorization Algebra of Quantum Observables

In Chapter 8, we give our main construction. It gives a factorization algebra Obs^q of *quantum* observables for any quantum field theory in the sense of Costello (2011b). A quantum field theory is, by that definition, something that lives over $\mathbb{C}[[\hbar]]$ and reduces modulo $\hbar$ to a classical field theory. The factorization algebra Obs^q is then a factorization algebra over $\mathbb{C}[[\hbar]]$, and modulo $\hbar$ it reduces to a factorization algebra quasi-isomorphic to the algebra Obs^{cl} of classical observables.

The construction of the factorization algebra of quantum observables is a bit technical. The techniques arise from the approach to quantum field theory developed in Costello (2011b). In that book a quantum field theory is defined to be a collection of functionals $I[L]$ on the fields that are *approximately* local. They play a role analogous to the action functional of a classical field theory. These functionals depend on a "length scale" L, and when L is close to zero the functional $I[L]$ is close to being local. The axioms of a quantum field theory are:

(i) As L varies, $I[L]$ and $I[L']$ are related by the operation of "renormalization group flow." Intuitively, if $L' > L$, then $I[L']$ is obtained from $I[L]$ by integrating out certain high-energy fluctuations of the fields.
(ii) Each $I[L]$ satisfies a scale L quantum master equation (the quantum version of a compatibility with gauge symmetry).
(iii) When we reduce modulo $\hbar$ and send $L \to 0$, then $I[L]$ becomes the interaction term in the classical Lagrangian.

The fact that $I[L]$ is never local, just close to being local as $L \to 0$, means that we have to work a bit to define the factorization algebra. The essential idea is simple, however. If $U \subset M$, we define the cochain complex $\text{Obs}^q(U)$ to be the space of first-order deformations $\{I[L] + \epsilon\mathcal{O}[L]\}$ of the collection of functionals $I[L]$ that define the theory. We ask that this first-order deformation satisfies the renormalization group flow property modulo ϵ^2. This condition gives a linear expression for $\mathcal{O}[L]$ in terms of any other $\mathcal{O}[L']$. This idea reflects the familiar intuition from the path integral that observables are first-order deformations of the action functional.

The observables we are interested in do not need to be localized at a point, or indeed given by the integral over the manifold of something localized at a point. Therefore, we should not ask that $\mathcal{O}[L]$ becomes local as $L \to 0$. Instead, we ask that $\mathcal{O}[L]$ becomes supported on U as $L \to 0$.

Moreover, we do not ask that $I[L] + \epsilon\mathcal{O}[L]$ satisfies the scale L quantum master equation (modulo ϵ^2). Instead, its failure to satisfy the quantum master equation defines the differential on the cochain complex of quantum observables.

With a certain amount of work, we show that this definition defines a factorization algebra Obs^q that quantizes the factorization algebra Obs^{cl} of classical observables.

1.3 The Physical Importance of Factorization Algebras

Our key claim is that factorization algebras encode, in a mathematically clean way, the features of a quantum field theory that are important in physics.

This formalism must thus include the most important examples of quantum field theories from physics. Fortunately, the techniques developed in Costello (2011b) give a cohomological technique for constructing quantum field theories, which applies easily to many examples. For instance, the Yang–Mills theory and the ϕ^4 theory on $\mathbb{R}^4$ were both constructed in Costello (2011b). (Note that we work throughout on Riemannian manifolds, not Lorentzian ones.)

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As a consequence, each theory has a factorization algebra on $\mathbb{R}^4$ that encodes its observables.

1.3.1 Correlation Functions from Factorization Algebras

In the physics literature on quantum field theory, the fundamental objects are *correlation functions* of observables. The factorization algebra of a quantum field theory contains enough data to encode the correlation functions. In this sense, its factorization algebra encodes the essential data of a quantum field theory.

Let us explain how this encoding works. Assume that we have a field theory on a compact manifold M. Suppose that we work near an *isolated* solution to the equations of motion, that is, one that admits no small deformations. (Strictly speaking, we require that the cohomology of the tangent complex to the space of solutions to the equations of motion is zero, which is a little stronger, as it means that there are also no gauge symmetries preserving this solution to the equations of motion.) Some examples of theories where we have an isolated solution to the equations of motion are a massive scalar field theory, on any compact manifold, or a massless scalar field theory on the four-torus T^4 where the field has monodromy -1 around some of the cycles. In each case, we can include an interaction, such as the ϕ^4 interaction.

Since the classical observables are functions on the space of solution to the equations of motion, our assumption implies $H^*(\mathrm{Obs}^{cl}(M)) = \mathbb{C}$. A spectral sequence argument then lets us conclude that $H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[[\hbar]]$.

If $U_1, \ldots, U_n \subset M$ are disjoint open subsets of M, the factorization algebra structure gives a map

$$\langle - \rangle : H^*(\mathrm{Obs}^q(U_1)) \otimes \cdots \otimes H^*(\mathrm{Obs}^q(U_n)) \to H^*(\mathrm{Obs}^q(M)) = \mathbb{C}[[\hbar]].$$

If $\mathcal{O}_i \in H^*(\mathrm{Obs}^q(U_i))$ are observables on the open subsets U_i, then $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$ is the correlation function of these observables.

Consider again the ϕ^4 theory on T^4, where the ϕ field has monodromy -1 around one of the four circles. In the formalism of Costello (2011b), it is possible to construct the ϕ^4 theory on $\mathbb{R}^4$ so that the $\mathbb{Z}/2$ action sending ϕ to $-\phi$ is preserved. By descent, the theory – and hence the factorization algebra – exists on T^4 as well. Thus, this theory provides an example where the quantum theory can be constructed and the correlation functions defined.

1.3.2 Factorization Algebras and Renormalization Group Flow

Factorization algebras provide a satisfying geometric understanding of the RG flow, which we discuss in detail in Chapter 9 but sketch now.

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In Costello (2011b), a scaling action of $\mathbb{R}_{>0}$ on the collection of theories on $\mathbb{R}^n$ was given. It provides a rigorous version of the RG flow as defined by Wilson.

There is also a natural action by scaling of the group $\mathbb{R}_{>0}$ (under multiplication) on the collection of translation-invariant factorization algebras on $\mathbb{R}^n$. Let $R_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ denote the diffeomorphism that rescales the coordinates, and let $\mathcal{F}$ be a translation-invariant factorization algebra on $\mathbb{R}^n$. Then the pullback $R_\lambda^* \mathcal{F}$ is a new factorization algebra on $\mathbb{R}^n$.

We show that the map from theories on $\mathbb{R}^n$ to factorization algebras on $\mathbb{R}^n$ intertwines these two $\mathbb{R}_{>0}$ actions. Thus, this simple scaling action on factorization algebras is the RG flow.

In order to define have a quantum field theory with finitely many free parameters, it is generally essential to only consider *renormalizable* quantum field theories. In Costello (2011b), it was shown that for any translation-invariant field theory on $\mathbb{R}^n$, the dependence of the field theory on the scalar parameter $\lambda \in \mathbb{R}_{>0}$ is via powers of λ and of $\log \lambda$. A strictly renormalizable theory is one in which the dependence is only via $\log \lambda$, and the quantizations of the Yang–Mills theory and ϕ^4 theory constructed in Costello (2011b) both have this feature.

We can translate the concept of renormalizability into the language of factorization algebras. For any translation-invariant factorization algebra $\mathcal{F}$ on $\mathbb{R}^n$, there is a family of factorization algebras $\mathcal{F}_\lambda = R_\lambda^* \mathcal{F}$ on $\mathbb{R}^n$. Because this family depends smoothly on λ, a priori it defines a factorization algebra over the base ring $C^\infty(\mathbb{R}_{>0})$ of smooth functions of the variable λ. We say this family is *strictly renormalizable* if it arises by extension of scalars from a factorization algebra over the base ring $\mathbb{C}[\log \lambda]$ of polynomials in $\log \lambda$. The factorization algebras associated with the Yang–Mills theory and the ϕ^4 theory both have this feature.

In this way, we formulate via factorization algebras the concept of renormalizability of a quantum field theory.

1.3.3 Factorization Algebras and the Operator Product Expansion

One disadvantage of the language of factorization algebras is that the factorization algebra structure is often very difficult to describe explicitly. The reason is that for an open set U, the space $\mathrm{Obs}^q(U)$ of quantum observables on U is a very large topological vector space, and it is not obvious how one can give it a topological basis. To extract more explicit computations, we introduce the concept of a *point observable* in Chapter 10. The space of point

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observables is defined to be the limit $\lim_{r\to 0} \mathrm{Obs}^q(D(0,r))$ of the space of quantum observables on a disc of radius r around the origin, as $r \to 0$. Point observables capture what physicists call *local operators*; however, we eschew the term operator as our formalism does not include a Hilbert space on which we can operate.

Given two point observables $\mathcal{O}_1$ and $\mathcal{O}_2$, we can place $\mathcal{O}_1$ at 0 and $\mathcal{O}_2$ at x and then use the factorization product on sufficiently small discs centered at 0 and x to define a product element

$$\mathcal{O}_1(0) \cdot \mathcal{O}_2(x) \in \mathrm{Obs}^q(\mathbb{R}^n).$$

The operator product is defined by expanding the product $\mathcal{O}_1(0) \cdot \mathcal{O}_2(x)$ as a function of x and extracting the “singular part.” It is not guaranteed that such an expansion exists in general, but we prove in Chapter 10 that it does exist to order $\hbar$. This order $\hbar$ operator product expansion can be computed explicitly, and we do so in detail for several theories in Chapter 10. The methods exhibited there provide a source of concrete examples in which mathematicians can rigorously compute quantities of quantum field theory.

That chapter also contains the longest and most detailed example in this book. It has recently become clear (Costello 2013b; Costello et al. 2019) that one can understand quantum groups, such as the Yangian and related algebras, using Feynman diagram computations in quantum field theory. The general idea is that one should take a quantum field theory that has one topological direction, so that the factorization product in this direction gives us a (homotopy) associative algebra. By taking the Koszul dual of this associative algebra, one finds a new algebra that in certain examples is a quantum group. For the Yangian, the relevant Feynman diagram computations are given in Costello et al. (2019). We present in detail an example related to a different infinite-dimensional quantum group, following Costello (2017). We perform one-loop Feynman diagram computations that reproduce the commutation relations in this associative algebra. (We chose this example as the relevant Feynman diagram computations are considerably easier than those that lead to the Yangian algebra in Costello et al. 2019.)

1.4 Poisson Structures and Deformation Quantization

In the deformation quantization approach to quantum mechanics, the associative algebra of quantum operators reduces, modulo $\hbar$, to the commutative algebra of classical operators. But this algebra of classical operators has a little more structure: It is a Poisson algebra. Deformation quantization posits that

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the failure of the algebra of quantum operators to be commutative is given, to first order in $\hbar$, by the Poisson bracket.

Something similar happens in our story. Classical observables are given by the algebra of functions on the derived space of solutions to the Euler–Lagrange equations. The Euler–Lagrange equations are not just any partial differential equations (PDEs), however: They describe the critical locus of an action functional. The derived critical locus of a function on a finite-dimensional manifold carries a shifted Poisson (or P_0) structure, meaning that its dg algebra of functions has a Poisson bracket of degree 1. In the physics literature, this Poisson bracket is sometimes called the BV bracket or antibracket.

This feature suggests that the space of solutions to the Euler–Lagrange equations should also have a P_0 structure, and so the factorization algebra Obs^{cl} of classical observables has the structure of a P_0 algebra. We show that this guess is indeed true, as long as we use a certain homotopical version of P_0 factorization algebras.

Just as in the case of quantum mechanics, we would like the Poisson bracket on classical observables to reflect the first-order deformation into quantum observables. We find that this behavior is the case, although the statement is not as nice as that in the familiar quantum mechanical case.

Let us explain how it works. The factorization algebra of classical observables has compatible structures of dg commutative algebra and shifted Poisson bracket. The factorization algebra of quantum observables has, by contrast, no extra structure: it is simply a factorization algebra valued in cochain complexes. Modulo $\hbar^2$, the factorization algebra of quantum observables lives in an exact sequence:

$$0 \to \hbar\,\mathrm{Obs}^{cl} \to \mathrm{Obs}^{q} \bmod \hbar^2 \to \mathrm{Obs}^{cl} \to 0.$$

The boundary map for this exact sequence is an operator, for every open $U \subset M$,

$$D\colon\ \mathrm{Obs}^{cl}(U) \to \mathrm{Obs}^{cl}(U).$$

This operator is a cochain map of cohomological degree 1. Because Obs^{q} is not a factorization algebra valued in commutative algebras, D is not a derivation for the commutative algebra structure on $\mathrm{Obs}^{cl}(U)$.

We can measure the failure of D to be a derivation by the expression

$$D(ab) - (-1)^{|a|} aDb - (Da)b.$$

We find that this quantity is the same, up to homotopy, as the shifted Poisson bracket on classical observables.

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We should view this identity as being the analog of the fact that the failure of the algebra of observables of quantum mechanics to be commutative is measured, modulo $\hbar^2$, by the Poisson bracket. Here, we find that the failure of the factorization algebra of quantum observables to have a commutative algebra structure compatible with the differential is measured by the shifted Poisson bracket on classical observables.

This analogy has been strengthened to a theorem by Safronov (2018) and Rozenblyum (unpublished). Locally constant factorization algebras on $\mathbb{R}$ are equivalent to homotopy associative algebras. Safronov and Rozenblyum show that locally constant P_0 factorization algebras on $\mathbb{R}$ are equivalent to ordinary, unshifted Poisson algebras. Therefore a deformation quantization of a P_0 factorization algebra on $\mathbb{R}$ into a plain factorization algebra is precisely the same as a deformation of a Poisson algebra into an associative algebra; in this sense, our work recovers the usual notion of deformation quantization.

1.5 The Noether Theorem

The second main theorem we prove in this volume is a factorization-algebraic version of the Noether theorem. The formulation we find of the Noether theorem is significantly more general than the traditional formulation. We will start by reminding the reader of the traditional formulation before explaining our factorization-algebraic generalization.

1.5.1 Symmetries in Classical Mechanics

The simplest version of the Noether theorem applies to classical mechanics.

Suppose we have a classical-mechanical system with a continuous symmetry given by a Lie algebra $\mathfrak{g}$. Let A be the Poisson algebra of operators of the system, which is the algebra of functions on the phase space. Then, the Noether theorem, as traditionally phrased, says that there is a central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ and a map of Lie algebras

$$\widehat{\mathfrak{g}} \to A$$

where A is given the Lie bracket coming from the Poisson bracket. This map sends the central element in $\widehat{\mathfrak{g}}$ to a multiple of the identity in A. Further, the image of $\widehat{\mathfrak{g}}$ in A commutes with the Hamiltonian.

From a modern point of view, this is easily understood. The phase space of the classical mechanical system is a symplectic manifold X, with a function H on it, which is the Hamiltonian. The algebra of operators is the Poisson algebra of functions on X. If a Lie algebra $\mathfrak{g}$ acts as symmetries of the classical system,

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then it acts on X by symplectic vector fields preserving the Hamiltonian function. There is a central extension of $\mathfrak{g}$ that acts on X by Hamiltonian vector fields, assuming that $H^1(X) = 0$.

At the quantum level, the Poisson algebra of functions on X is upgraded to a noncommutative algebra (which we continue to call A), which is its deformation quantization. The quantum version of the Noether theorem says that if we have an action of a Lie algebra $\mathfrak{g}$ acting on the quantum mechanical system, there is a central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ (possibly depending on $\hbar$) and a Lie algebra map $\widehat{\mathfrak{g}} \to A$. This Lie algebra map lifts canonically to a map of associative algebras

$$U\widehat{\mathfrak{g}} \to A,$$

sending the central element to $1 \in A$.

1.5.2 The Noether Theorem in the Language of Factorization Algebras

Let us rewrite the quantum-mechanical Noether theorem in terms of factorization algebras on $\mathbb{R}$. As we saw in Section I.3.2, factorization algebras on $\mathbb{R}$ satisfying a certain local-constancy condition are the same as associative algebras. When translation invariant, these factorization algebras on $\mathbb{R}$ are the same as associative algebras with a derivation. A quantum mechanical system is a quantum field theory on $\mathbb{R}$, and so has as a factorization algebra Obs^q of observables. Under the equivalence between factorization algebras on $\mathbb{R}$ and associative algebra, the factorization algebra Obs^q becomes the associative algebra A of operators, and the derivation becomes the Hamiltonian.

Similarly, we can view $U_c(\mathfrak{g})$ as being a translation-invariant factorization algebra on $\mathbb{R}$, where the translation action is trivial. In Section I.3.6, we give a general construction of a factorization algebra – the *factorization envelope* – associated with a sheaf of dg Lie algebras on a manifold. The associative algebra $U_c(\mathfrak{g})$ is, when interpreted as a factorization algebra on $\mathbb{R}$, the twisted factorization envelope of the sheaf $\Omega^*_{\mathbb{R}} \otimes \mathfrak{g}$ of dg Lie algebras on $\mathbb{R}$. We write this twisted factorization envelope as $U_c(\Omega^*_{\mathbb{R}} \otimes \mathfrak{g})$.

The Noether theorem then tells us that there is a map of translation-invariant factorization algebras

$$U_c(\Omega^*_{\mathbb{R}} \otimes \mathfrak{g}) \to \mathrm{Obs}^q$$

on $\mathbb{R}$. We have simply reformulated the Noether theorem in factorization-algebraic language. This rewriting will become useful shortly, however, when we state a far-reaching generalization of the Noether theorem.

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1.5.3 The Noether Theorem in Quantum Field Theory

Let us now phrase our general theorem. Suppose we have a quantum field theory (QFT) on a manifold X, with factorization algebra Obs^q of observables. The usual formulation of the Noether theorem starts with a field theory with some Lie algebra of symmetries. We will work more generally and ask that there is some sheaf $\mathcal{L}$ of homotopy Lie algebras on X that acts as symmetries of our QFT. (Strictly speaking, we work with sheaves of homotopy Lie algebras of a special type, which we call local L_∞ algebras. A local L_∞ algebra is a sheaf of homotopy Lie algebras whose underlying sheaf is the smooth sections of a graded vector bundle and whose structure maps are given by multidifferential operators.) Our formulation of the Noether theorem then takes the following form.

Theorem 1.5.1 *In this situation, there is a canonical ℏ-dependent (shifted) central extension of $\mathcal{L}$, and a map of factorization algebras,*

$$U_c(\mathcal{L}) \to \mathrm{Obs}^q,$$

from the twisted factorization envelope of $\mathcal{L}$ to the factorization algebra of observables of the quantum field theory.

Let us explain how a special case of this statement recovers the traditional formulation of the Noether theorem, under the assumption (merely to simplify the notation) that the central extension is trivial.

Suppose we have a theory with a Lie algebra $\mathfrak{g}$ of symmetries. One can show that this implies the sheaf $\Omega_X^* \otimes \mathfrak{g}$ of dg Lie algebras also acts on the theory. Indeed, this sheaf is simply a resolution of the constant sheaf with stalk $\mathfrak{g}$.

The factorization envelope $U(\mathcal{L})$ assigns the Chevalley–Eilenberg chain complex $C_*(\mathcal{L}_c(U))$ to an open subset $U \subset X$. This construction implies that there is a map of pre-cosheaves $\mathcal{L}_c[1] \to U(\mathcal{L})$. Applied to $\mathcal{L} = \Omega_X^* \otimes \mathfrak{g}$, we find that a $\mathfrak{g}$-action on our theory gives a cochain map

$$\Omega_c^*(U) \otimes \mathfrak{g}[1] \to \mathrm{Obs}^q(U)$$

for every open. In degree 0, this map $\Omega_c^1 \otimes \mathfrak{g} \to \mathrm{Obs}^q$ can be viewed as an $n-1$-form on X valued in observables. This $n-1$ form is the Noether current. (The other components of this map contain important homotopical information.)

If $X = M \times \mathbb{R}$, where M is compact and connected, we get a map

$$\mathfrak{g} = H^0(M) \otimes H_c^1(\mathbb{R}) \otimes \mathfrak{g} \to H^0(\mathrm{Obs}^q(U)).$$

This map is the Noether charge.

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We have seen that specializing to observables of cohomological degree 0 and the sheaf $\mathcal{L} = \Omega^* \otimes \mathfrak{g}$, we recover the traditional formulation of the Noether theorem in quantum field theory. Our formulation, however, is considerably more general.

1.5.4 The Noether Theorem Applied to Two-Dimensional Chiral Theories

As an example of this general form of the Noether theorem, let us consider the case of two-dimensional chiral theories with symmetry group G.

In this situation, the symmetry Lie algebra is not simply the constant sheaf with values in $\mathfrak{g}$ but the sheaf $\Omega^{0,*}_\Sigma \otimes \mathfrak{g}$, the Dolbeault complex on Σ valued in $\mathfrak{g}$. In other words, it encodes the sheaf of $\mathfrak{g}$-valued holomorphic functions. This sheaf of dg Lie algebra acts on the sheaf of fields $\Omega^{1/2,*}(\Sigma, R)$ in the evident way.

Our formulation of the Noether theorem then tells us that there is some central extension of $\Omega^{0,*}_\Sigma \otimes \mathfrak{g}$ and a map of factorization algebras

$$U_c\left(\Omega^{0,*}_\Sigma \otimes \mathfrak{g}\right) \to \mathrm{Obs}^q$$

from the twisted factorization envelope to the observables of the system of free fermions.

In Section I.5.5, we calculated the twisted factorization algebra of $\Omega^{0,*}_\Sigma \otimes \mathfrak{g}$, and we found that it encodes the Kac–Moody vertex algebra at the level determined by the central extension.

Thus, in this example, our formulation of the Noether theorem recovers something relatively familiar: In any chiral theory with an action of G, we find a copy of the Kac–Moody algebra at an appropriate level.

1.6 Brief Orienting Remarks toward the Literature

Since we began this project in 2008, we have been pleased to see how themes that animated our own work have gotten substantial attention from others:

- Encoding classical field theories, particularly in the BV formalism, using L_∞ algebras (Hohm and Zwiebach 2017; Jurčo et al. 2019b,a).
- The meaning and properties of derived critical loci (Vezzosi 2020; Joyce 2015; Pridham 2019).

- The role of shifted symplectic structures in derived geometry and enlarged notions of deformation quantization (Pantev et al. 2013; Calaque et al. 2017; Ben-Bassat et al. 2015; Brav et al. 2019; Pridham 2017; Melani and Safronov 2018a,b; Safronov 2017; Toën 2014).
- Factorization algebras as a natural tool in field theory, particularly for topological field theories (Scheimbauer 2014; Kapranov et al. 2016; Benini et al. 2019, 2020; Beem et al. 2020).

We are grateful to take part in such a dynamic community, where we benefit from others' insights and critiques and we also have the chance to share our own. This book does not document all that activity, which is only partially represented by the published literature anyhow; we offer only a scattering of the relevant references, typically those that played a direct role in our own work or in our learning, and hence exhibit an unfortunate but hard-to-avoid bias toward close collaborators or interlocutors.

Our work builds, of course, upon the work and insights of generations of mathematicians and physicists who precede us. As time goes on, we discover how many of our insights appear in some guise in the past. In particular, it should be clear how much Albert Schwarz and Maxim Kontsevich shaped our views and our approach by their vision and by their results, and how much we gained from engaging with the work of Alberto Cattaneo, Giovanni Felder, and Andrei Losev.

There is a rich literature on BRST and BV methods in physics that we hope to help open up to mathematicians, but we do not make an attempt here to survey it, a task that is beyond us. We recommend Henneaux and Teitelboim (1992) as a point for jumping into that literature, tracking who cites it and whom it cites. A nice starting point to explore current activity in Lorentzian signature is Rejzner (2016), where these BRST/BV ideas cross-fertilize with the algebraic quantum field theory approach.

1.7 Acknowledgments

The project of writing this book has stretched over more than a decade. In that course of time, we have benefited from conversations with many people, the chance to present this work at several workshops and conferences, and the feedback of various readers. The book and our understanding of this material is much better due to the interest and engagement of so many others. Thank you.

We would like to thank directly the following people, although this list is undoubtedly incomplete: David Ayala, David Ben-Zvi, Dan Berwick-Evans, Damien Calaque, Alberto Cattaneo, Lee Cohn, Ivan Contreras, Vivek Dhand, Chris Douglas, Chris Elliott, John Francis, Davide Gaiotto, Dennis Gaitsgory, Sachin Gautam, Ezra Getzler, Greg Ginot, Ryan Grady, André Henriques, Rune Haugseng, Theo Johnson-Freyd, David Kazhdan, Si Li, Jacob Lurie, Takuo Matsuoka, Aaron Mazel-Gee, Pavel Mnev, David Nadler, Thomas Nikolaus, Fred Paugam, Dmitri Pavlov, Toly Preygel, Eugene Rabinovich, Kasia Rejzner, Kolya Reshetikhin, Nick Rozenblyum, Pavel Safronov, Claudia Scheimbauer, Graeme Segal, Jackie Shadlen, Yuan Shen, Jim Stasheff, Stephan Stolz, Dennis Sullivan, Matt Szczesny, Hiro Tanaka, Peter Teichner, David Treumann, Alessandro Valentino, Brian Williams, Philsang Yoo, and Eric Zaslow for helpful conversations. K.C. is particularly grateful to Graeme Segal for many illuminating conversations about quantum field theory over the years. O.G. would like to thank the participants in the Spring 2014 Berkeley seminar and Fall 2014 MPIM seminar for their interest and extensive feedback. Particular thanks are owed to Brian Williams and Eugene Rabinovich: their work and collaborations using the Noether machinery have substantially enhanced that part of this book.

Our work has taken place at many institutions – Northwestern University; the University of California, Berkeley; the Max Planck Institute for Mathematics; the Perimeter Institute for Theoretical Physics; and the University of Masschusetts, Amherst – which each provided a supportive environment for writing and research. Research at the Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation. The Perimeter Institute is located within the Haldimand Tract, land promised to the Six Nations (Haudenosaunee) peoples. K.C.'s research at the Perimeter Institute is supported by the Krembil Foundation and by the NSERC discovery grant program. We have also benefited from the support of the NSF: K.C. and O.G. were partially supported by NSF grants DMS 0706945 and DMS 1007168, and O.G. was supported by NSF postdoctoral fellowship DMS 1204826 and DMS 1812049. K.C. was also partially supported by a Sloan fellowship and a Simons fellowship in the early days of this long project.

Finally, during the period of this project, we have depended upon the unrelenting support and love of our families. In fact, we have both gotten married and had children over these years! Thank you Josie, Dara, Laszlo, and Hadrian for being models of insatiable curiosity, ready sources of exuberant distraction, and endless founts of joy. Thank you Lauren and Sophie for joining and building our lives together. Amid all the tumult and the moves, you always encourage us and you always share your wit, wisdom, and warmth. And it makes all the difference.