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F-SIGNATURE UNDER BIRATIONAL MORPHISMS

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Abstract

We study F-signature under proper birational morphisms $\pi : Y \to X$, showing that F-signature strictly increases for small morphisms or if $K_Y \leqslant \pi^* K_X$. In certain cases, we can even show that the F-signature of Y is at least twice as that of X. We also provide examples of F-signature dropping and Hilbert–Kunz multiplicity increasing under birational maps without these hypotheses.

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1. Introduction

Kunz showed that a local ring $(R, \mathfrak{m}, \boldsymbol{k} = \boldsymbol{k}^p)$ of positive characteristic is regular if and only if $F^e_* R$ is a free R-module [**Kun69**]. The F-signature is a measure of singularities that simply states the percentage of $F^e_* R$ that is free (measured in terms of a rank of a maximal free summand). F-signature was implicitly introduced by Smith and Van Den Bergh [**SVdB97**] and formally defined by Huneke and Leuschke in [**HL02**], although it was not shown to exist until [**Tuc12**].

In this paper, we study the behavior of F-signature under birational morphisms. Our main result is as follows.

MAIN THEOREM (Theorems 4.5 and 3.2). *Let X be a strongly F-regular variety of dimension n over an algebraically closed field $\boldsymbol{k}$ of characteristic $p > 0$.*

Suppose $\pi : Y \to X$ *is a proper birational morphism from a normal variety* Y *and fix a point* $y \in \mathrm{Exc}(\pi)$ *with* $\pi(y) = x$. *Suppose additionally that either:*

(a) π *is small, that is,* π *is an isomorphism outside a set of codimension at least two in* Y, *or;*

(b) *The canonical divisor* K_X *is* $\mathbb{Q}$*-Cartier and for every exceptional divisor* E *containing* y, *we have that* $\mathrm{coeff}_E(K_Y - \pi^* K_X) \leqslant 0$. *For instance, if all discrepancies are nonpositive,*

then we have

$$s(\mathcal{O}_{X,x}) < s(\mathcal{O}_{Y,y}).$$

Furthermore, if X *is not Gorenstein at* x *and* $\pi : Y \to X$ *is a small morphism obtained as the blowup of either* $\mathcal{O}_X(K_X)$ *or* $\mathcal{O}_X(-K_X)$, *then*

$$2 \cdot s(\mathcal{O}_{X,x}) \leqslant s(\mathcal{O}_{Y,y}).$$

The first part of our main theorem is a characteristic $p > 0$ analog of a result on normalized volume by Liu and Xu [LX17, Corollary 2.11]. We thank both Liu and Xu for inspiring discussions about the relation between F-signature and normalized volume; also see [LLX18, Theorem 6.14] and [Liu18].

We finally note that the condition that the blowup of $\mathcal{O}_X(K_X)$ (respectively of $\mathcal{O}_X(-K_X)$) is small can be interpreted as requiring that the graded ring $S = \mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \mathcal{O}_X(2K_X) \oplus \cdots$ is generated in degree 1 (respectively that $\mathcal{O}_X \oplus \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-2K_X) \oplus \cdots$ is generated in degree 1). Note that smallness of $\operatorname{Proj} S \to X$ is equivalent to the finite generation of S by [KM98, Lemma 6.2]. On the other hand, if S is generated in degree 1, then S is in fact the Rees algebra of $\mathcal{O}_X(K_X)$ (respectively $\mathcal{O}_X(-K_X)$). This condition is satisfied in surprisingly many rings, including determinantal rings [BV88, Corollary 7.10, Theorem 8.8].

For comparison, in [CRST18], Carvajal-Rojas and the last two authors of this paper studied the behavior of F-signature under finite morphisms (showing that it went strictly up in a controllable way) and used their results to show that the étale fundamental group of the punctured spectrum of a strongly F-regular singularity was finite. This was a characteristic $p > 0$ analog of [Xu14] and was later shown to imply Xu's result by [BGO17]. Note that Xu's proof also used ideas related to volume.

In Section 5, we provide examples showing that the F-signature can decrease outside of the hypotheses of the main theorem. We also show that the Hilbert–Kunz multiplicity can increase in that setting as well.

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2. Preliminaries

All schemes and morphisms of schemes considered in this paper will be separated and all rings and schemes will be Noetherian. Rings and schemes of prime characteristic $p > 0$ will be assumed to be F-finite (meaning that the Frobenius map is a finite map).

We are dealing with F-signature in this paper and so we recall its definition. First, we give some notation. If R is a ring of characteristic $p > 0$ and M is an R-module, we use $F_*^e M$ to denote M viewed as an R-module under the action of e-iterated Frobenius. For any R-module M, we use $\mathrm{frk}(M)$ to denote the *free rank of* M, or in other words, the maximal rank of a free R-module appearing in a direct sum decomposition of M, $M = R^{\oplus \mathrm{frk}(M)} \oplus N$. On the other hand, if R is a domain, we use $\mathrm{rk}(M)$ to denote the (generic) rank of M, that is, $\mathrm{rk}(M) = \dim_{K(R)}(M \otimes_R K(R))$, where $K(R)$ denotes the fraction field of R.

Inspired by the fact that for an F-finite local ring $(R, \mathfrak{m})$, $F_*^e R$ is a free R-module if and only if R is regular, we make the following definition which measures how free $F_*^e R$ is, asymptotically.

DEFINITION 2.1 (F-signature, [HL02]). Suppose that R is an F-finite domain. The *F-signature of* R is defined to be

$$s(R) = \lim_{e\to\infty} \frac{\mathrm{frk}(F_*^e R)}{\mathrm{rk}(F_*^e R)}.$$

This limit exists by [Tuc12] and [DPY16]; also see [PT18]. Furthermore, by [DPY16, Theorem B], $s(R) = \min_{\mathfrak{m}\subseteq R}\{s(R_{\mathfrak{m}})\}$, where $\mathfrak{m}$ runs over maximal ideals of R. Hence, for any Noetherian integral F-finite scheme X, we can define

$$s(X) = \min_{x\in X} s(\mathcal{O}_{X,x}).$$

It is clear that $0 \leqslant s(R) \leqslant 1$, and it is a fact that $s(R) = 1$ if and only if R is regular by [HL02] and [DPY16]. Furthermore, $s(R) > 0$ if and only if R is strongly F-regular by [AL03] and [DPY16]. For our purposes, it will be important to recall that strongly F-regular rings are Cohen–Macaulay and normal.

One common tool used to study F-signature is *Frobenius degeneracy ideals*. In particular, if $(R, \mathfrak{m})$ is an F-finite local ring, for each $e > 0$, following [AE05], we define

$$I_e = \{a \in R \mid \phi(F_*^e(aR)) \subseteq \mathfrak{m}, \text{ for all } \phi \in \mathrm{Hom}_R(F_*^e R, R)\}.$$

It is not difficult to see [AE05, Yao06] that

$$s(R) = \lim_{e\to\infty} \frac{\lambda_R(R/I_e)}{p^{e\dim(R)}},$$

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where $\lambda_R(\bullet)$ denotes the length of the module $\bullet$. We refer the reader to [HL02, Pol18, PT18, Tuc12] for additional properties of F-signature.

Since we are going to study the behavior of F-signature under birational maps, we need to understand how maps like $F^e_*\mathcal{O}_X \to \mathcal{O}_X$ (for example, picking out a summand) extend to birational maps. Suppose X is an F-finite normal and integral scheme. We first note that $\phi : F^e_*\mathcal{O}_X \to \mathcal{O}_X$ induces a map $F^e_*K(X) \to K(X)$ (simply by tensoring with the fraction field of X). If $\pi : Y \to X$ is a birational map, we obtain an induced map $\widetilde{\phi} : F^e_*\mathcal{O}_Y \to K(Y)$ (since the fraction fields of X and Y are isomorphic). It is natural to ask whether

$$\widetilde{\phi}(F^e_*\mathcal{O}_Y) \subseteq \mathcal{O}_Y$$

in which case we say that ϕ *extends to a map on* Y, $\widetilde{\phi} : F^e_*\mathcal{O}_Y \to \mathcal{O}_Y$. On the other hand, each $\phi : F^e_*\mathcal{O}_X \to \mathcal{O}_X$ induces a $\mathbb{Q}$-divisor $\Delta_\phi \geqslant 0$ such that $(1-p^e)(K_X + \Delta_\phi) \sim 0$; see [BS13, Section 4].

For a proper birational map $\pi : Y \to X$ with Y normal, we may pick canonical divisors K_Y and K_X that agree wherever π is an isomorphism. When working on charts or at stalks of Y and X, we continue to use these fixed canonical divisors K_Y and K_X.

LEMMA 2.2. *Suppose that X is an F-finite normal scheme and that $\pi : Y \to X$ is a finite-type birational map from a normal scheme Y with fixed K_Y and K_X as above. A map $\phi : F^e_*\mathcal{O}_X \to \mathcal{O}_X$ extends to a map $\widetilde{\phi} : F^e_*\mathcal{O}_Y \to \mathcal{O}_Y$ as above if and only if $K_Y - \pi^*(K_X + \Delta_\phi) \leqslant 0$.*

Furthermore, all $\phi : F^e_*\mathcal{O}_X \to \mathcal{O}_X$ extend to Y if either of the following two conditions are satisfied:

(a) *π is small; in other words, there is a set $W \subseteq X$ of codimension $\geqslant 2$ such that $\pi^{-1}(W)$ also has codimension $\geqslant 2$ in Y and $\pi : Y \setminus \pi^{-1}(W) \to X \setminus W$ is an isomorphism.*

(b) *K_X is $\mathbb{Q}$-Cartier and $K_Y - \pi^*K_X \leqslant 0$.*

Proof. The first statement is [BS13, Lemma 7.2.1] (note that $\Delta_{\widetilde{\phi}} \geqslant 0$ if and only if ϕ extends to a map on Y). For (a), note that $K_Y - \pi^*(K_X + \Delta_\phi) = -\pi^{-1}_*\Delta_\phi \leqslant 0$. Condition (b) is immediate. □

3. Finitely generated canonical and anticanonical algebras

Before handling the case of more general blowups, we consider the case of a small proper birational map obtained by blowing up either the canonical or anticanonical local algebra under the special assumption that *those algebras are standard graded.*

LEMMA 3.1 (See [San15, Proposition 3.10]). *Suppose that* $(R, \mathfrak{m}, \boldsymbol{k})$ *is an* F*-finite strongly* F*-regular local ring which is not Gorenstein. Then we can write*

$$F_*^e R = R^{\oplus a_e} \oplus \omega_R^{\oplus b_e} \oplus M_e$$

where M_e *has no free* R *or* ω_R*-summands. Furthermore,* $\lim_{e\to\infty} (b_e/\mathrm{rk}(F_*^e R)) = s(R)$ *and in particular* $\lim_{e\to\infty} (a_e/b_e) = 1$.

Proof. Consider a split surjection $F_*^e R \to \omega_R^{\oplus b_e}$. Then the induced map

$$\mathrm{Hom}_R(\omega_R^{\oplus b_e}, \omega_R) \to \mathrm{Hom}_R(F_*^e R, \omega_R)$$

remains split. Moreover, $\mathrm{Hom}_R(F_*^e R, \omega_R) \cong F_*^e \mathrm{Hom}_R(R, \omega_R) \cong F_*^e \omega_R$ and $\mathrm{Hom}_R(\omega_R^{\oplus b_e}, \omega_R) \cong R^{\oplus b_e}$. Therefore, b_e is no more than $\mathrm{frk}(F_*^e \omega_R)$. Conversely, if we set $c_e = \mathrm{frk}(F_*^e \omega_R)$ and consider a split surjective map $F_*^e \omega_R \to R^{\oplus c_e}$, then the induced map $\mathrm{Hom}_R(R^{\oplus c_e}, \omega_R) \to \mathrm{Hom}_R(F_*^e \omega_R, \omega_R)$ remains split. Moreover, $\mathrm{Hom}_R(R^{\oplus c_e}, \omega_R) \cong R^{\oplus c_e}$ and $\mathrm{Hom}_R(F_*^e \omega_R, \omega_R) \cong F_*^e \mathrm{Hom}_R(\omega_R, \omega_R) \cong F_*^e R$. Therefore, $c_e = \mathrm{frk}(F_*^e \omega_R)$ is no more than b_e and so the two numbers coincide. In particular, $b_e = \mathrm{frk}(F_*^e \omega_R)$ and in conclusion

$$\lim_{e\to\infty} \frac{b_e}{\mathrm{rk}(F_*^e R)} = \lim_{e\to\infty} \frac{\mathrm{frk}(F_*^e \omega_R)}{\mathrm{rk}(F_*^e R)} = s(\omega_R) = s(R)\,\mathrm{rk}(\omega_R) = s(R).$$

The equality of $s(\omega_R)$ and $s(R)\,\mathrm{rk}(\omega_R)$ is the content of [Tuc12, Theorem 4.11]. $\square$

THEOREM 3.2. *Suppose that an* F*-finite local ring* $(R, \mathfrak{m})$ *is not Gorenstein and that either*

(a) $S = \bigoplus_n R(nK_R)$ *is generated as a graded ring in degree* 1 *or*

(b) $S = \bigoplus_n R(-nK_R)$ *is generated as a graded ring in degree* 1.

Set $Y = \mathrm{Proj}\, S$ *with* $\pi : Y \to \mathrm{Spec}\, R$ *being the induced map. Then we have* $s(Y) \geqslant 2s(R)$.

Proof. The statement is trivial if R is not strongly F-regular since then $s(R) = 0$. Hence, we may assume that R is strongly F-regular. In the case that $S = \bigoplus_n R(nK_R)$, we have that the small morphism $\pi : Y \to \mathrm{Spec}\, R$ is the blowup of $R(K_R)$ and, hence, K_Y is Cartier. In the case that $S = \bigoplus_n R(-nK_R)$, we have

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that Y is the blowup of $R(-K_R)$ and so $-K_Y$ is Cartier, but then the inverse K_Y is Cartier too.

Consider the split surjection

$$F^e_* R \to R^{\oplus a_e} \oplus \omega_R^{\oplus b_e}$$

guaranteed by Lemma 3.1. We pull back via π^* and reflexify and we obtain a split surjection

$$F^e_* \mathcal{O}_Y \to \mathcal{O}_Y^{\oplus a_e} \oplus \mathcal{O}_Y(K_Y)^{\oplus b_e}.$$

However, $\mathcal{O}_Y(K_Y)$ is locally free and, hence, the result follows since b_e grows at the same rate as a_e, again by Lemma 3.1. □

4. Behavior under more general blowups

We now come to the proof of the more general case. Throughout this section, we work with varieties over an algebraically closed field $\boldsymbol{k}$ of positive characteristic p. We begin with several lemmas.

LEMMA 4.1 (See [**LM09**, Lemma 3.9]). *Let X be a projective variety, $x \in X$ a closed point of dimension n, and A an ample Cartier divisor on X. For all $1 \gg \epsilon > 0$, there exists $\delta > 0$ such that*

$$h^i(X, \mathcal{O}_X(kA) \otimes \mathfrak{m}_x^{\lceil \epsilon k \rceil}) = 0 \quad \textit{for } i > 0$$

and

$$h^0(X, \mathcal{O}_X(kA)) - h^0(X, \mathcal{O}_X(kA) \otimes \mathfrak{m}_x^{\lceil \epsilon k \rceil}) \geqslant \delta \cdot k^n$$

for all $k \gg 1$.

Proof. Let $\mu \colon X' \to X$ be the blowup of X along $\mathfrak{m}_x$, with $\mathfrak{m}_x \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-E)$. Since $-E$ is μ-ample, for a sufficiently large integer $m > 1$, we have that $m\mu^*A - E$ is ample on X'. Shrinking ϵ if necessary, we may assume $m\epsilon < 1$ and thus $\lceil \epsilon k \rceil m \leqslant k$ for $k \gg 0$. Since

$$k\mu^*A - \lceil \epsilon k \rceil E = (k - \lceil \epsilon k \rceil m)\mu^*A - \lceil \epsilon k \rceil (m\mu^*A - E),$$

by Fujita vanishing [**Laz04**, Ch. 1.4.D, Theorem 1.4.35 and Remark 1.4.36] (using that μ^*A is nef), we have that

$$H^i(X', \mathcal{O}_{X'}(k\mu^*A - \lceil \epsilon k \rceil E) = 0 \quad \text{for } i > 0.$$

We recall that

$$\mu_*(\mathcal{O}_{X'}(-\lceil \epsilon k \rceil E)) = \mathfrak{m}_x^{\lceil \epsilon k \rceil}, \quad R^j \mu_*(\mathcal{O}_{X'}(-\lceil \epsilon k \rceil E)) = 0 \quad \text{for } j > 0$$

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provided $k \gg 1$ as shown [Laz04, Lemma 5.4.24]. It follows that

$$H^i(X, \mathcal{O}_X(kA) \otimes \mathfrak{m}_x^{\lceil \epsilon k \rceil}) = H^i(X', \mathcal{O}_{X'}(k\mu^*A - \lceil \epsilon k \rceil E)) = 0 \quad \text{for } i > 0$$

when $k \gg 1$ by the vanishing above. In particular, this holds for $i = 1$, and, hence, from the short exact sequence

$$0 \to \mathcal{O}_X(kA) \otimes \mathfrak{m}_x^{\lceil \epsilon k \rceil} \to \mathcal{O}_X(kA) \to \mathcal{O}_{X,x}/\mathfrak{m}_x^{\lceil \epsilon k \rceil} \to 0,$$

we have

$$h^0(X, \mathcal{O}_X(kA)) - h^0(X, \mathcal{O}_X(kA) \otimes \mathfrak{m}_x^{\lceil \epsilon k \rceil}) = \dim_k(\mathcal{O}_{X,x}/\mathfrak{m}_x^{\lceil \epsilon k \rceil}) = P_{X,x}(\lceil \epsilon k \rceil),$$

where $P_{X,x}$ is the Hilbert–Samuel polynomial of $\mathcal{O}_{X,x}$. Thus, choosing $0 < \delta < (\epsilon^n/n!)e(\mathcal{O}_{X,x})$ gives

$$h^0(X, \mathcal{O}_X(kA)) - h^0(X, \mathcal{O}_X(kA) \otimes \mathfrak{m}_x^{\lceil \epsilon k \rceil}) = P_{X,x}(\lceil \epsilon k \rceil) > \frac{\delta}{\epsilon^n}(\lceil \epsilon k \rceil)^n \geqslant \delta k^n$$

for $k \gg 1$. □

LEMMA 4.2 (See [LX17, Lemma 2.9]). *Let Y be a normal projective variety of dimension n over a field $\boldsymbol{k}$ of prime characteristic $p > 0$ and L a nef and big Cartier divisor on Y. Let $y \in Y$ be a closed point of an irreducible curve C satisfying $(L \cdot C) = 0$. Then there exists $\epsilon > 0$ so that*

$$h^1(Y, \mathcal{O}_Y(kL) \otimes \mathfrak{m}_y^k) \geqslant \epsilon k^n \quad \textit{for } k \gg 1.$$

The following proof was provided to us by Takumi Murayama. We will provide an alternative (and somewhat longer) proof below.

Proof. Let $\psi : \hat{Y} \to Y$ be the normalized blowup of Y along $\mathfrak{m}_y$ and let $\mathfrak{m}_y \mathcal{O}_{\hat{Y}} = \mathcal{O}_{\hat{Y}}(-E)$. Let $\hat{C}$ be the strict transform of C in $\hat{Y}$, in which case $\hat{C} \cdot E > 0$. Let A be a very ample Cartier divisor on $\hat{Y}$ so that the $\mathbb{Q}$-Cartier divisor $\psi^*L - E + \delta A$ is not ample for $1 \gg \delta > 0$ since

$$(\psi^*L - E + \delta A) \cdot \hat{C} = -E \cdot \hat{C} + \delta A \cdot \hat{C},$$

which is negative for all $1 \gg \delta > 0$.

Fix $1 \gg \delta > 0$. Since $\psi^*L - E + \delta A$ is not ample and $\psi^*L - E = \psi^*L - E + \delta A - \delta A$, there exists some $i > 0$ and $\epsilon > 0$ such that

$$h^i(\hat{Y}, \mathcal{O}_{\hat{Y}}(m(\psi^*L - E))) \geqslant \epsilon m^n$$

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for all $m \gg 0$ by [Mur18, Theorem B]. By [Laz04, Lemma 5.4.24], for $k \gg 0$, we have

$$h^i(\hat{Y}, \mathcal{O}_{\hat{Y}}(k(\psi^* L - E)) = h^i(Y, \mathcal{O}_Y(kL) \otimes \mathfrak{m}_y^k)).$$

But if $i \geqslant 2$, then the exact sequence of cohomology derived from twisting the short exact

$$0 \to \mathfrak{m}_y^k \to \mathcal{O}_Y \to \mathcal{O}_Y/\mathfrak{m}_y^k \to 0$$

by kL shows

$$h^i(Y, \mathcal{O}_Y(kL) \otimes \mathfrak{m}_y^k)) = h^i(Y, \mathcal{O}_Y(kL))$$

for all $i \geqslant 2$. By [Laz04, Theorem 1.4.40],

$$h^i(Y, \mathcal{O}_Y(kL)) = O(k^{n-i}).$$

Therefore, it is only possible for $h^i(Y, \mathcal{O}_Y(kL) \otimes \mathfrak{m}_y^k)) \geqslant \epsilon k^n$ for all $m \gg 0$ when $i = 1$, which completes the proof of the lemma. □

Note that the above proof of Lemma 4.2 provides an alternative proof to [LX17, Lemma 2.9]. One would need to replace the reference of [Mur18, Theorem B] with [dFKL07, Theorem A]. Nevertheless, we present a second proof of Lemma 4.2 which closely resembles the proof of [LX17, Lemma 2.9]. We suspect that the alternative proof will be of independent interest.

LEMMA 4.3. *Suppose V is a normal projective variety and D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with nonnegative Iitaka dimension that is not nef. Let $Z \subseteq V$ be an irreducible curve such that $D \cdot Z < 0$. Let $g\colon W' \to V$ be a regular alteration dominating the blowup of I_Z such that $g^{-1}(Z)$ has simple normal crossings. Then $\tau(W', m\|g^*D\|)$ vanishes along $g^{-1}(Z)$ for all integers $m \gg 1$. In particular, if D is big, every irreducible component of $g^{-1}(Z)$ is contained in the non-nef locus of $g^*(D)$.*

Proof. Replacing D with a positive multiple, we may assume that D is a Cartier divisor. Let $\mu\colon V' \to V$ be the normalized blowup of I_Z, with $I_Z\mathcal{O}_{V'} = \mathcal{O}_{V'}(-E)$, and $f'\colon W' \to V'$ the induced map factoring g so that the divisor $E' = (f')^*E$ has simple normal crossing support. Let $g\colon W' \xrightarrow{\nu} W \xrightarrow{f} V$ be the Stein factorization of g (in other words, $W := \mathbf{Spec}\ g_*\mathcal{O}_{W'}$) so that we have a commutative diagram

$$\begin{tikzcd} W' \arrow[r, "f'"] \arrow[d, "\nu"'] \arrow[dr, "g"] & V' \arrow[d, "\mu"] \\ W \arrow[r, "f"'] & V \end{tikzcd}$$

where f is finite, ν is birational, and W is normal. Since f is finite, $f^{-1}(\{Z\})$ is a union of finitely many irreducible curves $Z_1, \ldots, Z_r$ that dominate Z. Note that if $C \subseteq W'$ is an irreducible curve that dominates Z, we have, by the projection formula, that $(g^*D) \cdot C = (\deg g|_C)(D \cdot Z) < 0$.

Given $m \geqslant 1$, consider the asymptotic test ideal $\tau(W', m\|g^*D\|) \subseteq \mathcal{O}_{W'}$. If H is a very ample divisor on W' and $A = K_{W'} + (\dim W' + 1)H$, then

$$\mathcal{O}_{W'}(m(g^*D) + A) \otimes \tau(m\|g^*D\|)$$

is globally generated for all $m \geqslant 1$ by [Mus13, Theorem A]. Therefore, if $C \subseteq W'$ is an irreducible curve that is not contained in the zero locus of $\tau(W', m\|g^*D\|)$, then

$$(m(g^*D) + A) \cdot C \geqslant 0. \tag{4.1}$$

Thus, if C dominates Z and hence $(g^*D) \cdot C < 0$ similarly to the above, we must have that C is contained in the zero locus of $\tau(W', m\|g^*D\|)$ for all $m > -(A \cdot C)/((g^*D) \cdot C)$. Note that this condition on m comes from negating (4.1) and solving for m.

Consider a component E'_i of E' that dominates Z. A general complete intersection curve C on E'_i then dominates Z, and, thus, $\tau(W', m_i\|g^*D\|)$ must vanish along C for some $m_i \gg 1$. As we vary the complete intersection that defines C, the condition on m_i does not change. Thus, in fact, $\tau(W', m_i\|g^*D\|)$ must vanish along all of E'_i.

Supposing now that E'_i is a component of E' that maps to a point of Z, we again wish to show that $\tau(W', m_i\|g^*D\|)$ must vanish along all of E'_i for $m_i \gg 1$. We have that E'_i necessarily also maps to a point of $Z_s \subseteq f^{-1}(Z) \subseteq W$ for some s. Note that some component of E' must dominate Z_s (since ν is surjective and $\nu^{-1}(Z_s) \subseteq \mathrm{Supp}(E')$) and $\nu^{-1}(Z_s)$ is connected as W is normal. In light of the previous paragraph, it suffices to show $\tau(W', m_i\|g^*D\|)$ must vanish along all of E'_i for $m_i \gg 1$. We may assume that E'_i intersects another component E'_j of E' along which $\tau(W', m_j\|g^*D\|)$ is known to vanish for some $m_j \gg 0$.

Take a general complete intersection curve $C \subseteq E'_i$ that meets E'_j in at least one point P, which we may assume to be a smooth point of C. We know that

$$\mathcal{O}_{W'}(lm_j(g^*D) + A) \otimes \tau(W', lm_j\|g^*D\|)$$

is globally generated for any $l \geqslant 1$. Thus, whenever $\tau(W', lm_j\|g^*D\|)$ does not vanish along C, we can find an effective divisor $F \sim_{\mathbb{Z}} (lm_j(g^*D) + A)$ not containing C that vanishes along $\tau(W', lm_j\|g^*D\|)$. Let us consider what happens when we restrict F to C. Note that since E'_i maps to a point of Z, so too does $C \subseteq E'_i$, whence $(g^*D) \cdot C = 0$. Furthermore,

$$\tau(W', m_j\|g^*D\|) \subseteq \mathcal{O}_{W'}(-E'_j)$$

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by assumption; so we have that

$$\tau(W', lm_j\|g^*D\|) \subseteq \tau(W', m_j\|g^*D\|)^l \subseteq \mathcal{O}_{W'}(-lE'_j)$$

for all $l \geqslant 1$ by subadditivity [HY03, Theorem 4.5]. Thus, F must vanish at least to order l at P so that

$$A \cdot C = (0 + A) \cdot C = F \cdot C \geqslant l.$$

But A does not depend on l; so this is impossible, and so $\tau(W', lm_j\|g^*D\|)$ vanishes along C. Fix $l > A \cdot C$ and set $m_i = lm_j$. It follows that $\tau(W', m_i\|g^*D\|)$ must vanish along C and hence also E'_i, as desired.

Thus, taking m' sufficiently large and divisible, we conclude from above that

$$\tau(W', m'\|g^*D\|) \subseteq \mathcal{O}_{W'}(-E'_{\mathrm{red}})$$

so that $\tau(W', m\|g^*D\|)$ vanishes along $g^{-1}(Z) = E'_{\mathrm{red}}$ for all integers $m \gg 1$. In particular, if D is big, every irreducible component E'_i of $g^{-1}(Z)$ is contained in the non-nef locus of $g^*(D)$ by [Mus13, Theorem 6.2]. $\square$

LEMMA 4.4 (See [dFKL07, Proposition 1.1] and [Mur18, Proposition 4.5]). *Suppose that V is a normal projective variety and $Z \subseteq V$ is an irreducible curve. Let L and E be Cartier divisors, with L big and E effective. Assume that $L \cdot Z = 0$, that E does not contain Z, and that $E \cdot Z > 0$.*

If $0 < \gamma_1 < \gamma_2$ are real numbers such that $L - \gamma_2 E$ remains big, then there exists $\epsilon > 0$ and a positive integer c such that $\mathfrak{b}(|kL - mE|) \subseteq I_Z^{\lfloor \epsilon k \rfloor - c}$ for all integers m and k such that $\gamma_1 k \leqslant m \leqslant \gamma_2 k$.

Proof. Without loss of generality, we assume that the base field $\boldsymbol{k} = \overline{\boldsymbol{k}}$ is uncountable. Using [dJ96, Theorem 4.1], we may take a regular alteration $g\colon W' \to V$, dominating the blowup of I_Z such that $g^{-1}(Z)$ has simple normal crossings. Let $\mu\colon V' \to V$ be the normalized blowup of I_Z, with $I_Z\mathcal{O}_{V'} = \mathcal{O}_{V'}(-G)$ and $f'\colon W' \to V'$ the induced map factoring g so that $g = \mu \circ f'$. For any $t \in \mathbb{Q} \cap [\gamma_1, \gamma_2]$, $L - tE$ is big and $(L - tE) \cdot Z = -t(E \cdot Z) < 0$. Applying Lemma 4.3, it follows that every irreducible component $g^{-1}(Z)$ is contained in the non-nef locus of $g^*(L - tE)$.

Thus, if $(f')^*G = G' = \sum a_i G'_i$ so that $(g^{-1}(Z))_{\mathrm{red}} = G'_{\mathrm{red}}$, it follows from [Mus13, Theorem 6.2] that

$$\operatorname{ord}_{G'_i}(\|g^*L - tg^*E\|) = \inf_{\substack{l \geqslant 1 \\ tl \in \mathbb{Z}}} \frac{1}{l} \operatorname{ord}_{G'_i}(\mathfrak{b}(|l(g^*L - tg^*E)|)) > 0$$

for all $t \in \mathbb{Q} \cap [\gamma_1, \gamma_2]$ and any i. Since the asymptotic order of vanishing $\mathrm{ord}_{G'_i}(\| - \|)$ is continuous on the open cone of big divisors in $N^1(X)_{\mathbb{R}}$ by [Mus13, Theorem 6.1], there exists an $\epsilon' > 0$ so that

$$\mathrm{ord}_{G'_i}(\|g^*L - tg^*E\|) > \epsilon'$$

for all $t \in [\gamma_1, \gamma_2]$ and any i. In particular, we have that

$$\mathrm{ord}_{G'_i}(\mathfrak{b}(|kg^*L - mg^*E|)) \geqslant k\epsilon'$$

for all integers m, k satisfying $\gamma_1 k \leqslant m \leqslant \gamma_2 k$. In this case, setting $a = \max_i a_i$ (the largest coefficient of $(f')^*G$) and $\epsilon = \epsilon'/a$ gives

$$\mathfrak{b}(|kg^*L - mg^*E|) \subseteq \mathcal{O}_{W'}(-\lfloor \epsilon k \rfloor G').$$

Since $(f')^*|k\mu^*L - m\mu^*E| \subseteq |(kg^*L - mg^*E)|$, we have

$$\mathfrak{b}(|k\mu^*L - m\mu^*E|) \cdot \mathcal{O}_{W'} \subseteq \mathfrak{b}(|kg^*L - mg^*E)|) \subseteq \mathcal{O}_{W'}(-\lfloor \epsilon k \rfloor G'),$$

and pushing forward along f' gives

$$\mathfrak{b}(|k\mu^*L - m\mu^*E|) \cdot (f')_*\mathcal{O}_{W'} \subseteq (f')_*\mathcal{O}_{W'}(-\lfloor \epsilon k \rfloor G') = \mathcal{O}_{V'}(-\lfloor \epsilon k \rfloor G) \cdot (f')_*\mathcal{O}_{W'}.$$

Thus, using that $\mathcal{O}_{V'}$ is normal and $\mathcal{O}_{V'} \subseteq (f')_*\mathcal{O}_{W'}$ is a finite and hence integral extension, we see

$$\mathfrak{b}(|k\mu^*L - m\mu^*E|) \subseteq \overline{(\mathcal{O}_{V'}(-\lfloor \epsilon k \rfloor G) \cdot (f')_*\mathcal{O}_{W'})} \cap \mathcal{O}_{V'} = \mathcal{O}_{V'}(-\lfloor \epsilon k \rfloor G)$$

from [HS06, Propositions 1.5.2 and 1.6.1]. On the other hand, we have that $H^0(V, \mathcal{O}_V(kL - mE)) = H^0(V', \mathcal{O}_{V'}(k\mu^*L - m\mu^*E))$ again by normality, and in particular

$$\mathfrak{b}(|kL - mE|) \cdot \mathcal{O}_{V'} = \mathfrak{b}(|k\mu^*L - m\mu^*E|).$$

Pushing forward along $\mu : V' \to V$ then gives $\mathfrak{b}(|kL - mE|) \subseteq \overline{I_Z^{\lfloor \epsilon k \rfloor}}$. Using [HS06, Proposition 5.3.4], there exists a positive integer c so that $\overline{I_Z^{\ell}} \subseteq I_Z^{\ell - c}$ for all integers $\ell \geqslant c$, and the result now follows. $\square$

Second proof of Lemma 4.2. We may assume that $\boldsymbol{k} = \overline{\boldsymbol{k}}$ is an uncountable field of prime characteristic. Let $\psi : \hat{Y} \to Y$ be the normalized blowup of Y along $\mathfrak{m}_y$, with $\mathfrak{m}_y \cdot \mathcal{O}_{\hat{Y}} = \mathcal{O}_{\hat{Y}}(-E)$. Take $\hat{C}$ to be the strict transform of C in $\hat{Y}$, noting that $\hat{C}$ is not contained in E and $\hat{C} \cdot E > 0$. We have that ψ^*L is big with $\psi^*L \cdot \hat{C} = L \cdot C = 0$. Moreover, for some sufficiently large integer $\ell > 0$, we have that $\ell\psi^*L - E$ is also big. Set $\gamma_2 = 1/\ell$ and choose $0 < \gamma_1 < \gamma_2$.

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Consider the long exact sequence

$$\cdots \to H^1(Y, \mathcal{O}_Y(kL) \otimes \mathfrak{m}_y^k) \to H^1(Y, \mathcal{O}_Y(kL) \otimes \overline{\mathfrak{m}_y^k}) \\ \to H^1(Y, \mathcal{O}_Y(kL) \otimes \overline{\mathfrak{m}_y^k}/\mathfrak{m}_y^k) = 0$$

where the vanishing holds since $(\overline{\mathfrak{m}_y^k})/\mathfrak{m}_y^k$ is a skyscraper sheaf with support contained in $\{y\}$. Using this sequence and the fact that $R^j\psi_*\mathcal{O}_{\hat{Y}}(-kE) = 0$ for $j > 0$ and sufficiently large k (as $-E$ is ψ-ample), we have

$$h^1(Y, \mathcal{O}_Y(kL) \otimes \mathfrak{m}_y^k) \geqslant h^1(Y, \psi_*\mathcal{O}_{\hat{Y}}(k\psi^*L - kE)) = h^1(\hat{Y}, \mathcal{O}_{\hat{Y}}(k\psi^*L - kE)) \tag{4.2}$$

for all $k \gg 1$. Consider now the differences

$$\Delta_k(m) := h^1(\hat{Y}, \mathcal{O}_{\hat{Y}}(k\psi^*L - (m+1)E)) - h^1(\hat{Y}, \mathcal{O}_{\hat{Y}}(k\psi^*L - mE))$$

for $k \geqslant m > 0$. If $m \gg 1$, and using that $\mathcal{O}_{\hat{Y}}(\psi^*L)|_E = \mathcal{O}_E$ and $\mathcal{O}_{\hat{Y}}(-E)|_E \sim \mathcal{O}_E(1)$ is ample, we have that

$$h^1(E, \mathcal{O}_{\hat{Y}}((k\psi^*L - mE)|_E)) = h^1(E, \mathcal{O}_E(m)) = 0$$

using Serre vanishing. Thus, if $m \geqslant \gamma_1 k$ and $k \gg 1$, it follows that $\Delta_k(m) \geqslant 0$. On the other hand, if additionally $\gamma_2 k \geqslant m \geqslant \gamma_1 k$, we have from Lemma 4.4 that there is some $\epsilon' > 0$ and a positive integer c so that

$$\mathfrak{b}(|k\psi^*L - mE|) \subseteq I_{\hat{C}}^{\lfloor \epsilon' k \rfloor - c}.$$

Thus, if $\hat{y} \in \hat{C} \cap \operatorname{Supp}(E)$ is a closed point, we have an inclusion

$$\operatorname{Im}\big(H^0(\hat{Y}, \mathcal{O}_{\hat{Y}}(k\psi^*L - mE)) \to H^0(E, \mathcal{O}_{\hat{Y}}(k\psi^*L - mE)|_E)\big) \\ \subseteq H^0(E, \mathcal{O}_E(m) \otimes \mathfrak{m}_{\hat{y}}^{\lfloor \epsilon' k \rfloor - c}). \tag{4.3}$$

Choosing $0 < \epsilon'' < \epsilon'/\gamma_2$, we have that for $k \gg 1$,

$$\lfloor \epsilon' k \rfloor - c > \epsilon' k - c - 1 \geqslant \epsilon'' \gamma_2 k + 1 \geqslant \epsilon'' m + 1 > \lceil \epsilon'' m \rceil. \tag{4.4}$$

Shrinking ϵ'' further if necessary, by Lemma 4.1, there exists $\delta > 0$ such that

$$\begin{aligned} \Delta_k(m) &= h^0(E, \mathcal{O}_E(m)) \\ &\quad - \operatorname{rk}\big(H^0(\hat{Y}, \mathcal{O}_{\hat{Y}}(k\psi^*L - mE)) \to H^0(E, \mathcal{O}_{\hat{Y}}(k\psi^*L - mE)|_E)\big) \\ &\geqslant h^0(E, \mathcal{O}_E(m)) - h^0(E, \mathcal{O}_E(m) \otimes \mathfrak{m}_{\hat{y}}^{\lfloor \epsilon' k \rfloor - c}) \quad \text{by (4.3)} \\ &\geqslant h^0(E, \mathcal{O}_E(m)) - h^0(E, \mathcal{O}_E(m) \otimes \mathfrak{m}_{\hat{y}}^{\lceil \epsilon'' m \rceil}) \quad \text{by (4.4)} \\ &\geqslant \delta m^{n-1} \end{aligned}$$

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for all $\gamma_1 k \leqslant m \leqslant \gamma_2 k$ and $k \gg 1$. Thus, we compute

$$\begin{aligned} h^1(\hat{Y}, \mathcal{O}_{\hat{Y}}(k\psi^* L - kE)) &= \left(\sum_{m=\lceil \gamma_1 k \rceil}^{k-1} \Delta_k(m) \right) + h^1(\hat{Y}, \mathcal{O}_{\hat{Y}}(k\psi^* L - \lceil \gamma_1 k \rceil E)) \\ &\geqslant \sum_{m=\lceil \gamma_1 k \rceil}^{\lceil \gamma_2 k \rceil - 1} \Delta_k(m) \quad \text{(since the dropped } \Delta_k(m) \geqslant 0) \\ &\geqslant \sum_{m=\lceil \gamma_1 k \rceil}^{\lceil \gamma_2 k \rceil - 1} \delta m^{n-1} \\ &\geqslant \delta(\lceil \gamma_1 k \rceil)^{n-1}(\lceil \gamma_2 k \rceil - \lceil \gamma_1 k \rceil) \\ &\geqslant \delta(\lceil \gamma_1 k \rceil)^{n-1}(\gamma_2 k - 1 - \gamma_1 k) \\ &\geqslant \delta \gamma_1^{n-1}(\gamma_2 - \gamma_1)k^n - \delta \gamma_1^{n-1} k^{n-1} \end{aligned}$$

for all $k \gg 1$. Thus, choosing $\epsilon < \delta \gamma_1^{n-1}(\gamma_2 - \gamma_1)$ implies that

$$\epsilon k^n < \delta \gamma_1^{n-1}(\gamma_2 - \gamma_1)k^n - \delta \gamma_1^{n-1} k^{n-1} \leqslant h^1(\hat{Y}, \mathcal{O}_{\hat{Y}}(k\psi^* L - kE))$$

for $k \gg 1$. Therefore, by (4.2),

$$h^1(Y, \mathcal{O}_Y(kL) \otimes \mathfrak{m}_y^k) \geqslant h^1(\hat{Y}, \mathcal{O}_{\hat{Y}}(k\psi^* L - kE)) \geqslant \epsilon k^n$$

for all $k \gg 1$ as desired. □

Now we come to the main theorem of the section.

THEOREM 4.5. *Let X be a strongly F-regular variety of dimension n over an algebraically closed field $\boldsymbol{k}$ of characteristic $p > 0$. Suppose $\pi : Y \to X$ is a proper birational morphism from a normal variety Y and fix a point $y \in \operatorname{Exc}(\pi)$ with $\pi(y) = x$. Suppose additionally that either:*

(a) *π is small, that is, π is an isomorphism outside of a set of codimension at least two in Y, or;*

(b) *the canonical divisor K_X is $\mathbb{Q}$-Cartier and for every exceptional divisor E containing y, we have that $\operatorname{coeff}_E(K_Y - \pi^* K_X) \leqslant 0$. For instance, this holds if all the discrepancies are nonpositive.*

Then we have $s(\mathcal{O}_{X,x}) < s(\mathcal{O}_{Y,y})$.

Proof. If $\boldsymbol{k}$ is not uncountable then we base change by the field obtained by adjoining uncountably many indeterminants to $\boldsymbol{k}$ and then taking its algebraic closure. Any closed points on the original varieties will correspond to points

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on the base-changed varieties, and their signatures will not change by [Yao06, Theorem 5.4]. Thus, we may assume that $\boldsymbol{k}$ is uncountable and algebraically closed.

Set $R = \mathcal{O}_{X,x}$ and $S = \mathcal{O}_{Y,y}$ so that we have a local inclusion $R \subseteq S$. By the assumption that π is either small or has nonpositive discrepancy at y, it follows that p^{-e}-linear map on R extends naturally to a p^{-e}-linear map on S; see Lemma 2.2. Consider the Frobenius degeneracy ideals I_e^S of S used to define the F-signature (Section 2) so that $s(\mathcal{O}_{Y,y}) = \lim_{e\to\infty} (1/p^{ne})\ell(S/I_e^S)$, and similarly for the Frobenius degeneracy ideals I_e^R of R. Set $J_e = I_e^S \cap R$. Observe that if $\mathfrak{m}_R$ can be generated by d elements, we have

$$\mathfrak{m}_R^{dp^e} \subseteq \mathfrak{m}_R^{[p^e]} \subseteq J_e \subseteq I_e^R.$$

Indeed, the first inclusion is standard by looking at one monomial in the generators at a time. The second inclusion follows from the fact that $\mathfrak{m}_R^{[p^e]} \subseteq \mathfrak{m}_S^{[p^e]} \subseteq I_e^S$. For the last inclusion, suppose that $r \in R \setminus I_e^R$. Then we know there exists a p^{-e}-linear map ϕ on R so that $\phi(r) = 1$. But then ϕ extends to S, and we still have $\phi(r) = 1$, so that $r \notin I_e^S$. Note also that $J_e^{[p]} \subseteq J_{e+1}$ so that $\lim_{e\to\infty} (1/p^{ne})\ell(R/J_e)$ exists and is at least as large as $s(R) = s(\mathcal{O}_{X,x})$; see [PT18, Theorem B].

Let us take suitable projective closures of X, Y such that π extends to a birational morphism between normal projective varieties. Note that condition (a) or (b) from the statement of the theorem will not necessarily hold on the entire compactifications; however, we will not need this. Let M' be an ample line bundle on X. By Lemma 4.1, for all $1 \gg \epsilon' > 0$, $i > 0$, and $k \gg 1$, we have

$$H^i(X, (M')^{\otimes k} \otimes \mathfrak{m}_R^{\lceil \epsilon' k\rceil}) = 0.$$

Taking $\ell \gg 1$ so that $1/\ell < \epsilon'$, it follows that $H^i(X, (M')^{\otimes \ell dp^e} \otimes \mathfrak{m}_R^{dp^e}) = 0$ for $i > 0$ and $e \gg 1$. Setting $M = M'^{\otimes \ell d}$ and using that $J_e/\mathfrak{m}_R^{dp^e}$ is supported only at $x \in X$, it follows that

$$\begin{aligned} &H^1(X, M^{\otimes p^e} \otimes \mathfrak{m}_R^{dp^e}) \twoheadrightarrow H^1(X, M^{\otimes p^e} \otimes J_e), \\ &H^i(X, M^{\otimes p^e} \otimes \mathfrak{m}_R^{dp^e}) \xrightarrow{\cong} H^i(X, M^{\otimes p^e} \otimes J_e) \quad \text{for } i \geqslant 2. \end{aligned}$$

Hence, $H^i(X, M^{\otimes p^e} \otimes J_e) = 0$ for $i > 0$ and $e \gg 1$. Thus, we have

$$\lim_{e\to\infty} \frac{1}{p^{en}} h^0(X, M^{\otimes p^e} \otimes J_e) = \frac{1}{n!}\operatorname{vol}_X(M) - \lim_{e\to\infty} \frac{1}{p^{ne}} \ell(R/J_e). \tag{4.5}$$

On the other hand, since X is strongly F-regular at x, so too is Y at y, and it follows from the proof of the positivity of the F-signature that there is some e_0

with $I_e^S \subseteq \mathfrak{m}_S^{p^{e-e_0}}$ for all $e \gg 1$ ([**BST12**, Theorem 3.21], *see* [**PT18**, Section 5 and the second proof of Theorem 5.1]). We have the following relations:

$$H^1(Y, \pi^* M^{\otimes p^e} \otimes I_e^S) \twoheadrightarrow H^1(Y, \pi^* M^{\otimes p^e} \otimes \mathfrak{m}_S^{p^{e-e_0}}),$$
$$H^i(Y, \pi^* M^{\otimes p^e} \otimes I_e^S) \xrightarrow{\cong} H^i(Y, \pi^* M^{\otimes p^e}) \quad \text{for } i \geqslant 2.$$

Since $h^i(Y, \pi^* M^{\otimes p^e}) = O(p^{e(n-1)})$ for $i > 0$ as $\pi^* M$ is nef [**Laz04**, Theorem 1.4.40], we have that

$$\begin{aligned}
&\limsup_{e\to\infty} \frac{1}{p^{en}} h^0(Y, \pi^* M^{\otimes p^e} \otimes I_e^S) \\
&\quad = \frac{1}{n!} \operatorname{vol}_Y(\pi^* M) - s(\mathcal{O}_{Y,y}) + \limsup_{e\to\infty} \frac{1}{p^{en}} h^1(Y, \pi^* M^{\otimes p^e} \otimes I_e^S) \\
&\quad \geqslant \frac{1}{n!} \operatorname{vol}_X(M) - s(\mathcal{O}_{Y,y}) + \limsup_{e\to\infty} \frac{1}{p^{en}} h^1(Y, \pi^* M^{\otimes p^e} \otimes \mathfrak{m}_S^{p^{e-e_0}}).
\end{aligned}$$

By Lemma 4.2 applied with $L = \pi^* M^{\otimes p^{e_0}}$, there exists an $\epsilon > 0$ so that

$$h^1(Y, \pi^* M^{\otimes p^e} \otimes \mathfrak{m}_S^{p^{e-e_0}}) = h^1(Y, L^{\otimes (p^{e-e_0})} \otimes \mathfrak{m}_S^{p^{e-e_0}}) \geqslant \epsilon p^{(e-e_0)n}$$

for all $e \gg 1$, so that

$$\limsup_{e\to\infty} \frac{1}{p^{en}} h^0(Y, \pi^* M^{\otimes p^e} \otimes I_e^S) \geqslant \frac{1}{n!} \operatorname{vol}_X(M) - s(\mathcal{O}_{Y,y}) + \frac{\epsilon}{p^{ne_0}}. \tag{4.6}$$

Observe that $\pi_* I_e^S \subseteq \pi_* \mathcal{O}_Y = \mathcal{O}_X$, and so $J_e = \pi_* I_e^S$ which implies that

$$\limsup_{e\to\infty} \frac{1}{p^{en}} h^0(Y, \pi^* M^{\otimes p^e} \otimes I_e^S) = \lim_{e\to\infty} \frac{1}{p^{en}} h^0(X, M^{\otimes p^e} \otimes J_e).$$

Thus, combining equation (4.6) and equation (4.5), we have

$$\frac{1}{n!} \operatorname{vol}_X(M) - \lim_{e\to\infty} \frac{1}{p^{ne}} \ell(R/J_e) \geqslant \frac{1}{n!} \operatorname{vol}_X(M) - s(\mathcal{O}_{Y,y}) + \frac{\epsilon}{p^{ne_0}}$$

whence it follows

$$s(\mathcal{O}_{Y,y}) \geqslant \lim_{e\to\infty} \frac{1}{p^{ne}} \ell(R/J_e) + \frac{\epsilon}{p^{ne_0}} > s(\mathcal{O}_{X,x}).$$

This completes the proof. □

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5. Examples of prime characteristic invariants and blowups of isolated singularities

In this section, we observe that without the hypothesis (a) or (b), the conclusion of Theorem 4.5 may not hold even if $\pi : Y \to X$ is the blowup of an isolated singularity. We provide several examples demonstrating various negative behaviors. We fix the following notation for all of our examples: X will be an affine scheme of a strongly F-regular hypersurface. Specifically,

$$X = \operatorname{Spec}(R), \qquad R = \boldsymbol{k}[x_1, \ldots, x_n]/(f),$$

$\boldsymbol{k}$ will be an algebraically closed field of prime characteristic $p > 0$, and X will have an isolated singularity at the origin $(x_1, \ldots, x_n)$. We denote by $\pi : Y \to X$ the blowup of X at the origin. Then π is proper and birational, and has n standard affine charts:

$$Y_i = \operatorname{Spec}\left(R\left[\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}\right]\right) \cong \operatorname{Spec}\left(\frac{k[x_1/x_i, \ldots, x_i, \ldots, x_n/x_i]}{(f : x_i^\infty)}\right)$$

where $(f : x_i^\infty) = \bigcup_{\ell \in \mathbb{N}} \{g \in k[x_1/x_i, \ldots, x_i, \ldots x_n/x_i] \mid x_i^\ell g \in (f)\}$.

Our strategy of showing that F-signature can strictly decrease under the blowup of an isolated singularity avoids any technical computations or explicit formulas of F-signature. Instead, we show that a strongly F-regular isolated singularity can be blown up to create a variety which has nonstrongly F-regular points. We first discuss a method of determining whether an isolated hypersurface singularity is strongly F-regular.

LEMMA 5.1. *Let $\boldsymbol{k}$ be an F-finite field of prime characteristic*

$$p > 0, \qquad S = \boldsymbol{k}[x_1, \ldots, x_n], \qquad \textit{and} \qquad f \in S$$

an element such that $S/(f)$ is a domain with isolated singularity at the maximal ideal $(x_1, \ldots, x_n)$. Then $S/(f)$ is strongly F-regular if and only if there exists $e \in \mathbb{N}$ such that $x_1 f^{p^e-1} \notin (x_1^{p^e}, \ldots, x_n^{p^e})$.

Proof. The property of being strongly F-regular is a local condition. Let $\mathfrak{m} = (x_1, \ldots, x_n)$. Then R is strongly F-regular if and only if $R_\mathfrak{m}$ is a strongly F-regular local ring. By [AE05], the set

$$\mathcal{P} = \bigcap_{e \in \mathbb{N}} \left\{c \in R_\mathfrak{m} \mid R_\mathfrak{m} \xrightarrow{\cdot c^{1/p^e}} R_\mathfrak{m}^{1/p^e} \text{ does not split}\right\}$$

is an ideal of $R_\mathfrak{m}$ satisfying the following:

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(a) $R_{\mathfrak{m}}$ is F-pure if and only if $\mathcal{P} \neq R_{\mathfrak{m}}$;

(b) if $R_{\mathfrak{m}}$ is F-pure, then $\mathcal{P}$ is a prime ideal;

(c) if $R_{\mathfrak{m}}$ is not strongly F-regular, then the closed set $V(\mathcal{P})$ of $\operatorname{Spec}(R_{\mathfrak{m}})$ defines the nonstrongly F-regular locus of $R_{\mathfrak{m}}$.

Thus, the assumption that R has an isolated singularity implies that $\mathcal{P}$ is 0 if R is strongly F-regular, the unique maximal ideal of $R_{\mathfrak{m}}$ if R is F-pure but not strongly F-regular, or all of $R_{\mathfrak{m}}$ if R is not F-pure. Therefore, $R_{\mathfrak{m}}$ is strongly F-regular if and only if $x_1 \notin \mathcal{P}$. It readily follows by the techniques of [Fed83] that $x_1 \notin \mathcal{P}$ if and only if there exists $e \in \mathbb{N}$ such that $x_1 f^{p^e-1} \notin (x_1^{p^e}, \ldots, x_n^{p^e})S_{\mathfrak{m}}$ (see [Gla96, Theorem 2.3]). Since $(x_1^{p^e}, \ldots, x_n^{p^e})S$ is primary to $\mathfrak{m}$, we have

$$x_1 f^{p^e-1} \notin (x_1^{p^e}, \ldots, x_n^{p^e})S_{\mathfrak{m}}$$

if and only if $x_1 f^{p^e-1} \notin (x_1^{p^e}, \ldots, x_n^{p^e})S$. □

EXAMPLE 5.2. Let

$$R = \frac{\boldsymbol{k}[x_1, x_2, x_3, x_4]}{(x_1^2 + x_2^4 + x_3^5 + x_4^4)}$$

and assume that $\boldsymbol{k}$ is an algebraically closed field of characteristic 7. Then for any $i \in \{2, 3, 4\}$, we have $x_i(x_1^2 + x_2^4 + x_3^5 + x_4^4)^6 \notin (x_1^7, x_2^7, x_3^7, x_4^7)$ and, therefore, R is strongly F-regular by Lemma 5.1. The chart Y_1 is nonsingular. The charts Y_2 and Y_4 are isomorphic and have coordinate rings isomorphic to the hypersurface

$$S = \frac{\boldsymbol{k}[a, b, c, d]}{(a^2 + b^2 + c^5 b^3 + d^4 b^2)}.$$

The hypersurface S is not normal at the point (a, b, c, d), in particular is not strongly F-regular but is F-pure since $(a^2 + b^2 + c^5 b^3 + d^4 b^2)^6 \notin (a^7, b^7, c^7, d^7)$. The remaining chart has a coordinate ring isomorphic to

$$\frac{\boldsymbol{k}[a, b, c, d]}{(a^2 + b^4 c^2 + c^3 + d^4 c^2)},$$

a ring which is neither normal nor F-pure.

Observe that $R_{\mathfrak{m}}$ is a local ring of multiplicity 2. In particular, $\mathrm{e}_{\mathrm{HK}}(R_{\mathfrak{m}}) + s(R_{\mathfrak{m}}) = 2$; see the proof of [Tuc12, Proposition 4.22] for a justification. The same holds for the three singular charts of the blowup. In particular, not only does the F-signature strictly decrease to 0 on points in the exceptional locus of $\pi : Y \to X$, but the Hilbert–Kunz multiplicity of these points has strictly increased to 2.

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We leave it to the reader to verify that if $\widetilde{Y} \to Y$ is the normalization of Y, that is, $\widetilde{Y} \to X$ is the normalized blowup of X at the origin, then $\widetilde{Y}$ is nonsingular and, in particular, the conclusion of Theorem 4.5 is valid for the proper birational morphism $\widetilde{Y} \to X$. This is not an indication that the conclusion of Theorem 4.5 is valid for normalized blowups of isolated strongly F-regular singularities by the following examples.

EXAMPLE 5.3. Let

$$R = \frac{\boldsymbol{k}[x_1, x_2, x_3, x_4]}{(x_1^2 + x_2^3 + x_3^6 + x_4^6)}$$

where $\boldsymbol{k}$ is an algebraically closed field of characteristic 7. Then R is strongly F-regular, but the affine chart Y_4 of the blowup has a coordinate ring isomorphic to

$$\frac{\boldsymbol{k}[a, b, c, d]}{(a^2 + b^3 d + c^6 d^4 + d^4)}$$

which is normal, F-pure, but is not strongly F-regular.

Our final example illustrates that the normalized blowup of an isolated F-regular singularity can produce a normal variety with non-F-pure points. The example is obtained by changing the characteristic of the base field from Example 5.3.

EXAMPLE 5.4. Let

$$R = \frac{\boldsymbol{k}[x_1, x_2, x_3, x_4]}{(x_1^2 + x_2^3 + x_3^6 + x_4^6)}$$

where $\boldsymbol{k}$ is an algebraically closed field of characteristic 11. Then R is strongly F-regular but the affine chart Y_4 of the blowup has a coordinate ring isomorphic to

$$\frac{\boldsymbol{k}[a, b, c, d]}{(a^2 + b^3 d + c^6 d^4 + d^4)}$$

which is normal but not F-pure.

6. Further questions

We conclude the paper by stating two open questions. First, we hope that Hilbert–Kunz multiplicity can also be controlled under certain blowups.

QUESTION 6.1. Can we control the Hilbert–Kunz multiplicity of a local ring $(R, \mathfrak{m})$ under (special) blowups $\pi : Y \to X = \operatorname{Spec} R$?

Second, we would like to generalize the results of Section 3 to the case when the ring $\bigoplus_{i\geqslant 0} R(iK_X)$ or $\bigoplus_{i\geqslant 0} R(-iK_X)$ is finitely generated, instead of being generated in degree 1. Note that we expect that for any strongly F-regular ring and any Weil divisor D, the ring $\bigoplus_{i\geqslant 0} R(iD)$ is finitely generated, and this would hold, for instance, if the minimal model program is known to hold in characteristic $p > 0$ and, hence, we know it if $\dim R = 3$ and $p \geqslant 5$; see [**Bir16**, Theorem 1.3] and also [**SS10**, Theorem 4.3] applied locally.

QUESTION 6.2. If R is a strongly F-regular local ring and $S = \bigoplus_{i\geqslant 0} R(iK_X)$ (respectively, $S = \bigoplus_{i\geqslant 0} R(-iK_X)$) is finitely generated, can we control the F-signature of Proj S?

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