Glasgow Math. J. **50** (2008) 499–507.
doi:10.1017/S0017089508004369. Printed in the United Kingdom

INVARIANT SUBMANIFOLDS OF CONTACT (κ, μ)-MANIFOLDS

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(Received 15 September 2007; revised 19 February 2008; accepted 22 February 2008)

Dedicated to the memory of Late Professor Jerzy J. Konderak on the second anniversary of his departure.

Abstract. Invariant submanifolds of contact (κ, μ)-manifolds are studied. Our main result is that any invariant submanifold of a non-Sasakian contact (κ, μ)-manifold is always totally geodesic and, conversely, every totally geodesic submanifold of a non-Sasakian contact (κ, μ)-manifold, $\mu \neq 0$, such that the characteristic vector field is tangent to the submanifold is invariant. Some consequences of these results are then discussed.

2000 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C25, 53D10, 53D15.

1. Introduction. It is well known [**2**] that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R_{XY}\xi = 0$, where R is the curvature tensor. On the other hand, on a manifold M equipped with a Sasakian structure (φ, ξ, η, g), one has

$$R_{XY}\xi = \eta(Y)X - \eta(X)Y, \qquad X, Y \in \Gamma(TM). \tag{1.1}$$

As a generalization of both $R_{XY}\xi = 0$ and the Sasakian case (1.1), Blair, Koufogiorgos and Papantoniou [**4**] introduced the class of contact metric manifolds with contact metric structures (φ, ξ, η, g) which satisfy

$$R_{XY}\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y) \tag{1.2}$$

for all $X, Y \in \Gamma(TM)$, where κ and μ are real constants and $2h$ is the Lie derivative of φ in the direction ξ. A contact metric manifold belonging to this class is called a *contact* (κ, μ)*-manifold*. In fact there are many motivations for studying contact (κ, μ)-manifolds: the first is that, in the non-Sasakian case (that is for $\kappa \neq 1$), the condition (1.2) determines the curvature completely; moreover, while the values of κ and μ change, the form of (1.2) is invariant under $\mathcal{D}$-homothetic deformations [**4**]; finally, there is a complete classification of these manifolds, given in [**7**] by Boeckx, who proved also that any non-Sasakian contact (κ, μ)-manifold is locally homogeneous and strongly locally φ-symmetric [**5**], [**6**]. There are also non-trivial examples of contact (κ, μ)-manifolds, the most important being the unit tangent sphere bundle of a Riemannian manifold of constant sectional curvature with the usual contact metric structure.

An invariant submanifold of a contact (κ, μ)-manifold is a submanifold for which the structure tensor field φ maps tangent vectors into tangent vectors. Such a submanifold inherits a contact metric structure from the ambient space and it is in fact a contact (κ, μ)-manifold [**23**].

There is a well-known result of Kon that an invariant submanifold of a Sasakian manifold is totally geodesic, provided the second fundamental form of the immersion is covariantly constant [**15**]. In general, an invariant submanifold of a Sasakian manifold needs not to be totally geodesic. For example, the circle bundle (S, Q^n) over an n-dimensional complex quadric Q^n in a complex projective space $\mathbb{C}P^{n+1}$ is an invariant submanifold of a $(2n+3)$-dimensional Sasakian space form $S^{2n+3}(c)$ with $c > -3$, which is not totally geodesic [**24**, pp. 328–329]. Some necessary conditions for invariant submanifolds of contact (κ, μ)-manifolds (or particular cases of contact (κ, μ)-manifolds) to be totally geodesic are also found in some other papers (e.g. [**1**], [**14**], [**18**]). As a generalization of the result of Kon, in [**23**] it is proven that if the second fundamental form of an invariant submanifold in a contact (κ, μ)-manifold is covariantly constant then either $\kappa = 0$ or the submanifold is totally geodesic.

These circumstances motivate us to consider invariant submanifolds of non-Sasakian contact (κ, μ)-manifolds. In this paper we find in fact a much stronger result. Surprisingly, we prove that every invariant submanifold of a non-Sasakian contact (κ, μ)-manifold is totally geodesic (cf. Theorem 3.1). Conversely, we prove that every totally geodesic submanifold of a non-Sasakian contact (κ, μ)-manifold, with $\mu \neq 0$, such that the characteristic vector field is tangent to the submanifold is invariant. Finally, we discuss some examples and consequences of these results.

2. Contact (κ, μ)-manifolds.

A differentiable 1-form η on a $(2n+1)$-dimensional differentiable manifold M is called a *contact form* if $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, and M equipped with a contact form is a *contact manifold*. Since $d\eta$ has rank $2n$ on the Grassmann algebra $\bigwedge T_p^* M$ at each point $p \in M$, there exists a unique global vector field ξ, called the *characteristic vector field*, such that $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. Moreover, it is well known that M admits a Riemannian metric g and a $(1, 1)$-tensor field φ such that

$$\varphi\xi = 0, \ \eta \circ \varphi = 0, \quad \eta(X) = g(X, \xi), \quad \varphi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

$$d\eta(X, Y) = g(X, \varphi Y), \quad g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$. The structure (φ, ξ, η, g) is called a *contact metric structure* and the manifold M endowed with such a structure is called a *contact metric manifold*. In a contact metric manifold M, the $(1, 1)$-tensor field $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ is symmetric and satisfies

$$h\xi = 0, \quad \eta \circ h = 0, \quad h\varphi + \varphi h = 0, \quad \nabla \xi = -\varphi - \varphi h, \quad \mathrm{tr}(h) = \mathrm{tr}(\varphi h) = 0, \tag{2.3}$$

https://doi.org/10.1017/S0017089508004369 Published online by Cambridge University Press

where ∇ is the Levi-Civita connection of g. A contact metric manifold such that ξ is Killing (or equivalently $h = 0$, cf. [**3**], p. 65) is said to be *K-contact*, and a contact metric manifold satisfying (1.1) is said to be *Sasakian*. Any Sasakian manifold is K-contact, and in dimension 3 also the converse holds. A contact metric manifold satisfying (1.2) is called a *contact* (κ, μ)*-manifold*. In [**4**] it is proven that necessarily $\kappa \leq 1$ and that the class of contact (κ, μ)-manifolds contains the Sasakian manifolds for $\kappa = 1$. Moreover, if a contact (κ, μ)-manifold is K-contact then $\kappa = 1$ and it is Sasakian. Hence a non-Sasakian (κ, μ)-manifold cannot be K-contact. Examples of contact (κ, μ)-manifolds exist for all values of $\kappa \leq 1$ and $\mu \in \mathbb{R}$ (cf. [**4**, **7**]).

In a contact (κ, μ)-manifold the following properties hold [**4**]:

$$(\nabla_X \varphi)Y = g(X, Y + hY)\xi - \eta(Y)(X + hX),$$

$$(\nabla_X h)Y = ((1-\kappa)g(X, \varphi Y) + g(X, \varphi hY))\xi + \eta(Y)h(\varphi X + \varphi hX) - \mu\varphi hY,$$

for all $X, Y \in \Gamma(TM)$, from which, in particular, it follows that

$$\nabla_\xi h = \mu h \circ \varphi. \tag{2.4}$$

Moreover, $h^2 = (\kappa - 1)\varphi^2$ and the eigenvalues of h are 0, λ and $-\lambda$, where $\lambda = \sqrt{1-\kappa}$. The eigenspace relative to the eigenvalue 0 is $\{\xi\}$. Moreover, for $\kappa \neq 1$, the subbundle $\mathcal{D} = \ker(\eta)$ can be decomposed in the eigenspace distributions $\mathcal{D}_+$ and $\mathcal{D}_-$ relative to the eigenvalues λ and $-\lambda$, respectively. These distributions are orthogonal to each other and have dimension n.

Another approach to contact (κ, μ)-manifolds has been presented in [**10**], where the authors observed that in fact $\mathcal{D}_+$ and $\mathcal{D}_-$ define two conjugate (that is, they satisfy $\varphi\mathcal{D}_+ = \mathcal{D}_-$ and $\varphi\mathcal{D}_- = \mathcal{D}_+$) Legendrian foliations, so that any contact (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ is endowed with a canonical bi-Legendrian structure given by the mutually orthogonal integrable distributions $\mathcal{D}_+$ and $\mathcal{D}_-$. Then they proved the following characterization.

THEOREM 2.1 [**10**]. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold, which is not K-contact. Then $(M, \varphi, \xi, \eta, g)$ is a contact (κ, μ)-manifold if and only if it admits two orthogonal Legendrian distributions L and Q and a linear connection $\bar\nabla$ satisfying the following properties:*

(i) $\bar\nabla L \subset L, \quad \bar\nabla Q \subset Q,$

(ii) $\bar\nabla\eta = 0, \quad \bar\nabla d\eta = 0, \quad \bar\nabla g = 0, \quad \bar\nabla\varphi = 0, \quad \bar\nabla h = 0,$

(iii) $\bar T(X, Y) = 2d\eta(X, Y)\xi$ *for all* $X, Y \in \Gamma(\mathcal{D})$,
$\bar T(X, \xi) = [\xi, X_L]_Q + [\xi, X_Q]_L$ *for all* $X \in \Gamma(TM)$,

where $\bar T$ denotes the torsion tensor field of $\bar\nabla$ and X_L and X_Q are, respectively, the projections of X onto the sub-bundles L and Q of TM. Furthermore $\bar\nabla$ is uniquely determined, L and Q are integrable and coincide with the eigenspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ of the operator h.

The connection stated in Theorem 2.1 is in fact the *bi-Legendrian connection* [**8**] corresponding to the bi-Legendrian structure (L, Q). An explicit formula for $\bar\nabla$ in this case is the following (cf. [**9**], [**10**]):

$$\begin{aligned}\bar\nabla_X Y &= -(\varphi[X_+, \varphi Y_+])_+ - (\varphi[X_-, \varphi Y_-])_- + [X_-, Y_+]_+ + [X_+, Y_-]_- \\ &\quad + \eta(X)([\xi, Y_+]_+ + [\xi, Y_-]_-) + X(\eta(Y))\xi,\end{aligned} \tag{2.5}$$

where for any $X \in \Gamma(TM)$, X_+ and X_- denote, respectively, the components of X on the distributions $\mathcal{D}_+$ and $\mathcal{D}_-$, according to the decomposition $TM = \mathcal{D}_+ \oplus \mathcal{D}_- \oplus \{\xi\}$. On the contact distribution $\mathcal{D}$, the above connection is related to the Levi-Civita connection by

$$\bar{\bar{\nabla}}_X Y = \nabla_X Y - \eta(\nabla_X Y)\xi, \tag{2.6}$$

for all $X, Y \in \Gamma(\mathcal{D})$ [**10**].

3. The main results. Let M' be a submanifold in a manifold M equipped with a Riemannian metric g. The Gauss and Weingarten formulae are given respectively by

$$\nabla_X Y = \nabla'_X Y + B(X, Y), \tag{3.1}$$

$$\nabla_X N = - A_N X + \nabla^{\perp}_X N, \tag{3.2}$$

for $X, Y \in \Gamma(TM')$ and $N \in \Gamma\left(T^{\perp}M'\right)$. Here ∇' and $\nabla^{\perp}$ are the induced Riemannian and the induced normal connections on M' and on the normal bundle $T^{\perp}M'$, respectively, and B is the second fundamental form related to the shape operator A_N in the direction of N by

$$g(B(X, Y), N) = g(A_N X, Y).$$

A submanifold M' of a contact metric manifold $(M, \varphi, \xi, \eta, g)$ is called an *invariant submanifold* if for each $x \in M'$, $\varphi(T_x M') \subset T_x M'$. As a consequence, ξ becomes tangent to M' and M' inherits a contact metric structure by restriction (cf. [**3**]). Moreover, as it is proven in [**11**] and [**13**], any invariant submanifold of a contact metric manifold is minimal.

In [**23**], the authors studied invariant submanifolds of a contact (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$. In particular, they proved the following identities:

$$B(X, \xi') = 0, \tag{3.3}$$

$$B(X, \varphi Y) = B(\varphi X, Y) = \varphi B(X, Y), \tag{3.4}$$

where $X, Y \in \Gamma(TM')$, $N \in \Gamma\left(T^{\perp}M'\right)$ and ξ' represents the restriction of the characteristic vector field of M to M'. Furthermore, they proved that an invariant submanifold of a contact (κ, μ)-manifold is in turn a contact (κ, μ)-manifold. It is easy to show that h preserves the tangent spaces to M' and then $h' = \frac{1}{2}\mathcal{L}_{\xi'}\varphi'$ coincides with the restriction of h to M'. Moreover, h' has the same eigenvalues as h and the eigenspace distributions are given by $\mathcal{D}'_+ = \mathcal{D}_+ \cap TM'$ and $\mathcal{D}'_- = \mathcal{D}_- \cap TM'$.

Now, we prove the main result of this section.

THEOREM 3.1. *Every invariant submanifold of a non-Sasakian contact (κ, μ)-manifold is totally geodesic.*

Proof. Let M' be an invariant submanifold of a contact (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$. Since M' is in turn a contact (κ, μ)-manifold, by virtue of Theorem 2.1, M' admits an orthogonal bi-Legendrian structure (L', Q'), given just by $L' = \mathcal{D}'_+$ and $Q' = \mathcal{D}'_-$, and a unique linear connection $\bar{\nabla}'$ satisfying (i), (ii) and (iii) of Theorem 2.1. Then, taking into account (2.5) and the fact that M' is invariant, it is easy to see that the connection $\bar{\nabla}'$ is nothing but the connection induced by $\bar{\nabla}$ on M'. From this remark

https://doi.org/10.1017/S0017089508004369 Published online by Cambridge University Press

and from (2.6) it follows directly that $B(X, Y) = 0$ for all $X, Y \in \Gamma(\mathcal{D}')$. Moreover, by (3.3) we have $B(X, \xi') = 0$ for all $X \in \Gamma(TM')$ and this concludes the proof. □

Remark 3.2. Another way for proving Theorem 3.1 is the following. Let $X, Y \in \Gamma(TM')$. Since the tensor field h maps tangent vectors into tangent vectors, by (1.2) it follows that $R_{XY}\xi'$ is a vector field tangent to the submanifold. Then we have $(\nabla_X B)(Y, \xi') = (\nabla_Y B)(X, \xi')$ (cf. e.g. [**24**]), that is,

$$\nabla^{\perp}_X B(Y, \xi') - B(\nabla_X Y, \xi') - B(Y, \nabla_X \xi') = \nabla^{\perp}_Y B(X, \xi') - B(\nabla_Y X, \xi') - B(X, \nabla_Y \xi').$$

So taking into account (3.3) we get $B(X, \nabla_Y \xi') = B(Y, \nabla_X \xi')$ and then, by (2.3),

$$B(X, \varphi Y + \varphi h Y) = B(Y, \varphi X + \varphi h X). \tag{3.5}$$

Continuing the computation, from (3.4) and (3.5) it follows that

$$\varphi B(X, hY) = \varphi B(Y, hX). \tag{3.6}$$

Since for any $X, Y \in \Gamma(TM)$, $\eta(B(X, Y)) = 0$, from (2.1) and (3.6) we get

$$B(X, hY) = B(Y, hX). \tag{3.7}$$

Now using $X \in \Gamma(\mathcal{D}'_+)$ and $Y \in \Gamma(\mathcal{D}'_-)$ in (3.7), we have $\lambda B(X, Y) = -\lambda B(X, Y)$, from which $B(X, Y) = 0$ since $\lambda \neq 0$ because of the assumption that $(M, \varphi, \xi, \eta, g)$ is not Sasakian. The same conclusion is true if $X \in \Gamma(\mathcal{D}'_-)$ and $Y \in \Gamma(\mathcal{D}'_+)$. It remains to prove that $B(X, Y) = 0$ for $X, Y \in \Gamma(\mathcal{D}'_+)$ and $X, Y \in \Gamma(\mathcal{D}'_-)$. Let $X, Y \in \Gamma(\mathcal{D}'_+)$. Since $\varphi\mathcal{D}'_+ = \mathcal{D}'_-$, we can write $Y = \varphi Z$ with $Z \in \Gamma(\mathcal{D}'_-)$. Then, by (3.4), we have $B(X, Y) = B(X, \varphi Z) = \varphi B(X, Z) = 0$ because of our previous result. Analogously one can prove the assertion for $X, Y \in \Gamma(\mathcal{D}'_-)$.

Now we provide an example of an invariant submanifold of a non-Sasakian contact (κ, μ)-manifold.

Example 3.3. In [**4**] the authors proved that the tangent sphere bundle T_1M of a Riemannian manifold (M, G) of constant sectional curvature $c \neq 1$ with the standard contact metric structure is a non-Sasakian contact (κ, μ)-manifold with $\kappa = c(2 - c)$, $\mu = 2c$. Let M' be a totally geodesic submanifold of M. Then M' equipped with the induced Riemannian metric G' has constant sectional curvature c and also its tangent sphere bundle T_1M' is a contact (κ, μ)-manifold with $\kappa = c(2 - c)$, $\mu = 2c$. Thus it is reasonable to ask whether T_1M' is an invariant submanifold of T_1M. The answer is affirmative as we are going to see and then, due to Theorem 3.1, T_1M' is a totally geodesic submanifold of T_1M. We need to recall various constructions on the tangent bundle $\pi : TM \to M$ (for more details see, e.g., [**12**, **17**, **19**, **20**]). The *connection map* $K : TTM \to TM$ corresponding to the Levi-Civita connection ∇ of G (or any linear connection) is defined as follows: if U is a normal neighbourhood of a point p of M then the *canonical map* $\tau : \pi^{-1}(U) \to T_pM$ maps any $Z \in \pi^{-1}(U)$ to the vector $\tau(Z)$ obtained by parallel translation of Z along the only ∇-geodesic joining $\pi(Z)$ with p. For each $A \in T_Z TM$ we put

$$K(A) := \lim_{t \to 0} \frac{\tau(\zeta(t)) - Z}{t},$$

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where $\zeta : t \to \zeta(t)$ is a path in TM such that $\dot{\zeta}(0) = A$. It is well known that the linear connection ∇ induces a decomposition of the bundle TM in horizontal and vertical sub-bundles and that for each $Z \in TM$ the horizontal subspace $\mathcal{H}(T_Z TM)$ of $T_Z TM$ is nothing but the kernel of K and the vertical subspace $\mathcal{V}(T_Z TM)$ the kernel of π_*. Then, for any vector field X on M there exist unique horizontal and vertical lifts $X^{\mathcal{H}} \in \mathcal{H}(TM)$ and $X^{\mathcal{V}} \in \mathcal{V}(TM)$ such that at each $Z \in TM$ we have

$$\pi_* X_Z{}^{\mathcal{H}} = X_{\pi(Z)}, \quad K X_Z{}^{\mathcal{H}} = 0_{\pi(Z)}, \quad \pi_* X_Z{}^{\mathcal{V}} = 0_{\pi(Z)}, \quad K X_Z{}^{\mathcal{V}} = X_{\pi(Z)}.$$

We can also consider the 1-form β defined for each $Z \in TM$, $X \in TTM$ by

$$\beta(X)_Z := G(Z, \pi_* X).$$

The decomposition of TTM in its horizontal and vertical sub-bundles allows us to define an almost complex structure J on TM as $JX^{\mathcal{H}} := X^{\mathcal{V}}$, $JX^{\mathcal{V}} := -X^{\mathcal{H}}$ (cf. [**12**]). Using the classical procedure for a hypersurface of an almost Hermitian manifold, it is possible to construct a contact metric structure on the tangent sphere bundle $T_1 M$ as follows (cf. [**21**, **22**]). Let ν be the unit vector field on TM normal to $T_1 M$. We put

$$\xi := -2J\nu, \quad \eta := \frac{1}{2}\beta, \quad \varphi X := JX - \eta(X)\nu\,. \tag{3.8}$$

Moreover we consider the *Sasaki metric* g on $T_1 M$, defined for $X, Y \in TTM$ by

$$g(X, Y) := \frac{1}{4}\left(G(\pi_* X, \pi_* Y) + G(KX, KY)\right). \tag{3.9}$$

Then (φ, ξ, η, g) is a contact metric structure on $T_1 M$. The factors $\frac{1}{4}$ in (3.9), 2 and $\frac{1}{2}$ in (3.8) are necessary since we use the convention $2d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$ for any 1-form ω and for a contact metric manifold the equation $d\eta(\cdot, \cdot) = g(\cdot, \varphi\,\cdot)$ has to hold. Of course, all the constructions we have recalled can be repeated for the totally geodesic submanifold M': we shall label all the geometric objects relative to M' with prime. Moreover, by an abuse of notation, we will denote with the same symbol a vector field on M' and any of its extensions to M. We observe that if U is a normal neighbourhood in M then $U' = U \cap M'$ is a normal neighbourhood in M', hence for each pair p, q of points of U' the only geodesic in U' with respect to the Levi-Civita connection ∇' of G' joining p and q is also the unique geodesic in U with respect to ∇ joining p and q. Then we can conclude that the connection map $K' : TTM' \to TM'$ corresponding to ∇' is the restriction of K to TTM'. Then the following facts hold:

(i) we have the following relations regarding the horizontal and vertical bundles

$$\mathcal{H}(TM') = \mathcal{H}(TM) \cap TTM', \quad \mathcal{V}(TM') = \mathcal{V}(TM) \cap TTM';$$

(ii) the horizontal and vertical lifts of a vector field X on M' are the restrictions to M' of the horizontal and vertical lifts of X and the 1-form β' on M' is the restriction of β to M';

(iii) the almost complex structure J' defined on TTM' coincides with the almost complex structure induced by J on TTM';

(iv) since the vector field ν' normal to M' is the restriction of ν to M', the characteristic vector field of the contact metric structure $(\varphi', \xi', \eta', g')$ is the restriction of ξ to M' and hence ξ is tangent to M';

(v) if X is tangent to M' then, from $\varphi' X = J' X - \eta'(X)\nu'$ it follows that $\varphi X = \varphi' X$ is tangent to M'.

In particular, from (v) and Theorem 3.1, it follows that $T_1 M'$ is a totally geodesic invariant submanifold of $T_1 M$.

We try to generalize the previous example by showing that the contact (κ, μ)-manifolds are in fact the only totally geodesic invariant submanifolds of a given non-Sasakian contact (κ, μ)-manifold provided that $\mu \neq 0$.

THEOREM 3.4. *Let M' be a totally geodesic submanifold of a non-Sasakian contact (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ such that ξ is tangent to M'. Assume that $\mu \neq 0$. Then M' is an invariant submanifold of M.*

Proof. Since M' is totally geodesic, for each $X \in \Gamma(TM')$ and $p \in M'$ we have, by (1.2),

$$(R'_{\xi' X}\xi')_p = (R_{\xi X}\xi)_p = (\kappa(\eta(X)\xi - X) - \mu h X)_p$$

from which

$$(hX)_p = \frac{1}{\mu}(R_{X\xi}\xi - \kappa(\eta(X)\xi - X)_p \in T_p M'.$$

It follows that $(hX)_{|M'}$ is tangent to M', as well as $((\nabla_\xi h)X)_{|M'} = (\nabla_\xi hX - h\nabla_\xi X)_{|M'}$. By (2.4) we get $h^2(\varphi X) = \frac{1}{\mu} h((\nabla_\xi h)X)$, from which, using the formula $h^2 = (\kappa - 1)\varphi^2$, it follows that

$$\varphi X = \frac{1}{\mu(1-\kappa)} h((\nabla_\xi h)X).$$

Hence φX is tangent to M'. □

REMARK 3.5. Theorem 3.4 does not hold for $\kappa = \mu = 0$. Indeed, let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold satisfying $R_{XY}\xi = 0$ for all $X, Y \in \Gamma(TM)$. These manifolds have been deeply studied in [**2**] where the author proves that the distribution $\mathcal{D}_+ \oplus \{\xi\}$ is integrable and defines a totally geodesic foliation of M, where $\mathcal{D}_+$ is the eigenspace distribution corresponding to the eigenvalue $\lambda = 1$ of h. Thus its leaves give examples of totally geodesic submanifolds of M, which are not invariant because $\varphi\mathcal{D}_+ = \mathcal{D}_-$ so that φ maps tangent vectors into normal vectors. On the other hand, there are also examples of invariant submanifolds of contact metric manifolds satisfying $R_{XY}\xi = 0$. For instance, one is given just by Example 3.3 taking $c = 0$.

We conclude by recalling the notion of *contact* (κ, μ)-*space form*. For a unit vector X orthogonal to ξ, the sectional curvature $K(X, \varphi X)$ is called φ-sectional curvature. In [**16**], Koufogiorgos showed that if the φ-sectional curvature at a point p of a contact (κ, μ)-manifold $(M, \varphi, \xi, \eta, g)$ is independent of the φ-section at p, then it is constant. Moreover, he proved that a non-Sasakian contact (κ, μ)-manifold is of constant φ-sectional curvature c if and only if $\mu = \kappa + 1$; in this case $c = -2\kappa - 1$ and the contact (κ, μ)-manifold in question is referred as a contact (κ, μ)-space form and denoted by $M(c)$.

Returning to invariant submanifolds, we have in particular the following results:

PROPOSITION 3.6. *Any invariant submanifold of a non-Sasakian contact* (κ, μ)*-space form is in turn a contact* (κ, μ)*-space form.*

Proof. The condition for M to have constant φ-sectional curvature is $\mu = \kappa + 1$, which is also verified in M', since M' is a contact (κ, μ)-manifold with the same constants κ and μ as in M. □

PROPOSITION 3.7. *Let* M' *be an invariant submanifold of a non-Sasakian contact* (κ, μ)*-space form* $M(c)$*. Then the normal connection of* M' *is trivial if and only if* $c = 1$ *or, equivalently,* $\kappa = -1$.

Proof. It follows easily from Theorem 3.1 and [**23**, Theorem 5.1]. □

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