Infinitesimal Rigidity of Convex Polyhedra through the Second Derivative of the Hilbert–Einstein Functional

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Abstract. The paper presents a new proof of the infinitesimal rigidity of convex polyhedra. The proof is based on studying derivatives of the discrete Hilbert–Einstein functional on the space of "warped polyhedra" with a fixed metric on the boundary.

The situation is in a sense dual to using derivatives of the volume in order to prove the Gauss infinitesimal rigidity of convex polyhedra. This latter kind of rigidity is related to the Minkowski theorem on the existence and uniqueness of a polyhedron with prescribed face normals and face areas.

In the spherical space and in the hyperbolic-de Sitter space, there is a perfect duality between the Hilbert–Einstein functional and the volume, as well as between both kinds of rigidity.

We review some of the related work and discuss directions for future research.

1 Introduction

1.1 Infinitesimal Rigidity of Convex Polyhedra

Let $P \subset \mathbb{R}^3$ be a compact convex polyhedron; assume for simplicity that all faces of P are triangles. Then an *infinitesimal deformation* of P is an assignment of a vector q_i to each vertex p_i. Let $P(t)$ be a family of polyhedra with the same combinatorics as P and vertices $p_i(t) = p_i + tq_i$. An infinitesimal deformation is called *isometric* if the edge lengths of $P(t)$ remain constant in the first order of t at $t = 0$. There always exist *trivial* isometric infinitesimal deformations that are restrictions of infinitesimal isometries of $\mathbb{R}^3$.

> Every convex polyhedron P is infinitesimally rigid; that is, every isometric infinitesimal deformation of P is trivial.

This theorem was first stated by Dehn in [15]. Legendre–Cauchy's argument from [10] easily carries over to the infinitesimal setting. Other proofs are given in [3, 19, 37, 46, 52, 59].

Yet another proof is presented in this paper. In some aspects it is similar to Schlenker's proof [46] and to the Pogorelov–Volkov proof of global rigidity [38, 53]. The actual goal of this paper is to describe the general framework into which this argument fits and to indicate possible further developments.

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1.2 The Approach

Instead of deforming the embedding $P \subset \mathbb{R}^3$ by moving the vertices, we deform the metric in the interior of P. We choose a point $p_0 \in \text{int } P$ and subdivide P into pyramids with p_0 as the apex and faces of P as bases. By denoting $r_i = \|p_0 - p_i\|$, we start to vary lengths r_i, while leaving the lengths of boundary edges constant. As a result, the total dihedral angles ω_i around interior edges $p_0 p_i$ may become different from 2π, so that the polyhedron P becomes what we call a *warped polyhedron*.

Can one vary the lengths r_i so that the angles ω_i remain constant in the first order? One can always do this by moving the point r_0 inside P. It turns out that if these are the only possibilities, then the polyhedron P is infinitesimally rigid. Infinitesimal rigidity of P is equivalent to

$$(1.1) \qquad\qquad \dim \ker \left(\frac{\partial \kappa_i}{\partial r_j} \right) = 3,$$

where $\kappa_i = 2\pi - \omega_i$ is the *curvature* of the edge $p_0 p_i$. Our proof of the infinitesimal rigidity of convex polyhedra goes by determining the rank of the above Jacobian.

An immediate generalization of this approach is to consider an arbitrary triangulation of the polyhedron P and the Jacobi matrix $\left(\frac{\partial \kappa_{ij}}{\partial r_{kl}} \right)$, where r_{ij} and κ_{ij} are the length, respectively, curvature of the edge $p_i p_j$, and among the points p_i some are vertices of P, some lie on its edges or faces, and some in the interior. (As before, lengths of boundary edges are assumed constant.) Then infinitesimal rigidity of P is equivalent to

$$\dim \ker \left(\frac{\partial \kappa_{ij}}{\partial r_{kl}} \right) = 3m + n,$$

where m is the number of interior vertices, and n is the number of vertices interior to faces of P. In our joint paper with J.-M. Schlenker [28], we completely determined the signature of the above Jacobian at a critical point by showing that it has exactly m positive eigenvalues. In particular, the matrix has negative spectrum if all points p_i lie at vertices or on the edges of P. Note, however, that [28] uses the infinitesimal rigidity of convex polyhedra.

1.3 Hyperbolic Manifolds with Boundary and Hyperbolic Cone-manifolds

Similar to the previous subsection, consider a triangulation of a compact closed hyperbolic 3-manifold and the Jacobi matrix $\left(\frac{\partial \kappa_{ij}}{\partial r_{kl}} \right)$ of edge curvatures with respect to edge lengths.

Conjecture 1 *The matrix $\left(\frac{\partial \kappa_{ij}}{\partial r_{kl}} \right)$ has corank $3n$ and exactly n positive eigenvalues, where n is the number of vertices in the triangulation.*

The first part, corank equals $3n$, is known. It follows from the infinitesimal rigidity of compact closed hyperbolic manifolds in dimension greater than 2, the so-called Calabi-Weil rigidity [8, 57]. Therefore, if the arguments used in this paper could be extended to prove Conjecture 1, this would yield an elementary proof of Calabi–Weil rigidity. Such extension should also lead to a proof of the following conjecture, communicated to me by Schlenker, who proved a smooth analog in [45].

Conjecture 2 *Compact hyperbolic manifolds with convex polyhedral boundary are infinitesimally rigid.*

More generally, consider a compact closed manifold glued from hyperbolic simplices so that the total dihedral angles around the edges may be different from 2π. It is called a (triangulated) hyperbolic *cone-manifold*. From the works of C. D. Hodgson and S. P. Kerckhoff [23], R. Mazzeo and G. Montcouquiol [32], and H. Weiss [58], it is known that compact closed hyperbolic manifolds with cone angles less than 2π are infinitesimally rigid, that is, cannot be deformed so that their cone angles remain constant in the first order. Again, this can be reformulated in terms of the rank of the Jacobian of the map $r \mapsto \kappa$, and a generalization of our method would yield an alternative, elementary proof.

Note that the condition on cone angles is similar to the convexity condition for polyhedra. There exist infinitesimally flexible non-convex polyhedra as well as infinitesimally flexible cone-manifolds with some cone angles greater than 2π; see [9, 26].

A study of deformations of Euclidean cone-manifolds from a similar point of view is undertaken in [21].

1.4 The Discrete Hilbert–Einstein Functional

Here we sketch our proof of (1.1) for a star-like triangulation of a convex polyhedron P. Consider the function

$$\mathrm{HE}(r) = \sum_i r_i \kappa_i + \sum_{\{i,j\}} \ell_{ij} \lambda_{ij},$$

where ℓ_{ij} is the length of, and λ_{ij} is the exterior dihedral angle at the edge $p_i p_j$ of P. We call it the *discrete Hilbert–Einstein functional*, as it is the discrete analog of twice the total scalar curvature of P plus half of the total mean curvature of ∂P. The Schläfli formula implies that $\frac{\partial \mathrm{HE}}{\partial r_i} = \kappa_i$, therefore

$$(1.2) \qquad\qquad\qquad \frac{\partial \kappa_i}{\partial r_j} = \frac{\partial^2 \mathrm{HE}}{\partial r_i \partial r_j}.$$

In particular the left-hand side, is symmetric in i and j. This allows us to reformulate the infinitesimal rigidity in the following way:

$$(1.3) \qquad\qquad \text{if } \mathrm{HE}^{\cdot\cdot} = 0, \text{ then } \dot{r} \text{ is a trivial variation.}$$

Here $\mathrm{HE}^{\cdot\cdot} = \sum_{i,j} \frac{\partial^2 \mathrm{HE}}{\partial r_i \partial r_j} \dot{r}_i \dot{r}_j = \sum_i \dot{\kappa}_i \dot{r}_i$ is the second derivative of HE in the direction $\dot{r}$, and a trivial variation can be defined as the one that preserves boundary dihedral angles: $\dot{\lambda}_{ij} = 0$.

In order to prove (1.3), we rewrite $\mathrm{HE}^{\cdot\cdot}$ in a different form (equation (3.11)). Then Lemma 2.13 implies that $\mathrm{HE}^{\cdot\cdot}$ is non-positive, and vanishes only if $\dot{r}$ is trivial. But as $\mathrm{HE}^{\cdot\cdot}$ vanishes by assumption, the variation $\dot{r}$ must be trivial.

This argument is reminiscent of Koiso's proof [29] of the infinitesimal rigidity of Einstein manifolds under certain assumptions on the curvature operator. Koiso uses integration by parts to obtain two formulas for the second derivative of the Hilbert–Einstein functional. This yields an equation with zero on one side, while on the other side one has a non-positive quantity that vanishes only if the deformation is trivial. Note that our formula (3.11) is obtained from $\mathrm{HE}^{\cdot\cdot} = \sum_i \dot\kappa_i \dot r_i$ by a kind of discrete integration by parts.

Koiso's proof is an example of application of Bochner's technique: a second order differential operator is expressed as the sum of Laplacian and of a non-negative 0-th order operator. Thus our proof should be a particular manifestation of a *discrete Bochner technique*.

1.5 The Discrete Bochner Technique

Einstein manifolds in dimension 3 are manifolds of constant sectional curvature; therefore, Koiso's theorem contains Calabi–Weil rigidity of hyperbolic 3-manifolds as a special case. Weil also uses Bochner's technique; his approach is related to that of Koiso in the same way as moving vertices of a polyhedron P is related to deforming the metric inside P. The infinitesimal rigidity of hyperbolic cone-manifolds with cone angles less than 2π (see Subsection 1.3) is proved by extending Weil's arguments with the help of Cheeger's Hodge theory for singular spaces.

It would be natural if the infinitesimal rigidity of cone-manifolds could be re-proved by developing a discrete Bochner technique for this situation. Such a proof would not only be elementary, it would also provide a discrete-geometric counterpart to the original argument of Hodgson, Kerckhoff, and others.

Another theorem where the need for a discrete Bochner technique is felt is Cheeger's discrete analog [11] of the Bochner–Gallot–Meyer's vanishing theorem:

> If in a Euclidean cone-manifold of dimension d all cone angles are less than 2π, then it is a real homology d-sphere.

Again, the known proof uses Hodge theory for singular spaces.

It should be mentioned that R. Forman developed a *combinatorial Bochner technique* [20] that takes into account the combinatorics of a simplicial complex (or, more generally, cell complex), but not its geometry.

One of the components for a discrete Bochner technique should be the discrete Hodge theory based on Whitney forms; see, *e.g.*, [16]. The other component presumes some sort of discrete Riemannian geometry, still to be found.

1.6 Volume Derivatives and the Alexandrov–Fenchel Inequality

The proof of (1.1) sketched in Subsection 1.4 very much resembles a known proof of

$$(1.4) \qquad \dim \ker\left(\frac{\partial A_i}{\partial h_j}\right) = 3,$$

where h_i are lengths of perpendiculars dropped to the faces of a convex polyhedron Q from the origin, A_i are face areas, and the polyhedron is deformed by varying h_i while keeping the directions of face normals fixed.

In fact, equation (1.4), with d on the right-hand side, also holds for convex d-dimensional polyhedra. This is a key lemma in the proof of the Alexandrov–Fenchel inequalities.

It is geometrically clear that $A_i = \frac{\partial \, \text{Vol}}{\partial h_i}$, where $\text{Vol}(h)$ is the volume of a polyhedron with given face normals and support numbers h_i. Thus we have

$$\frac{\partial A_i}{\partial h_j} = \frac{\partial^2 \, \text{Vol}}{\partial h_i \partial h_j},$$

which looks similar to (1.2).

1.7 Duality between the Volume and Hilbert–Einstein Functional

The above analogy culminates in a striking identity,

$$\frac{\partial^2 \, \text{HE}}{\partial r_i \partial r_j} = \frac{\partial^2 \, \text{Vol}}{\partial h_i \partial h_j},$$

where the polyhedron Q on the right-hand side is polar dual to the polyhedron P on the left-hand side. We were able to prove this identity only by a direct computation. Note that functions $\text{HE}(r)$ and $\text{Vol}(h)$ have different natures: the former uses inverse trigonometric functions, while the latter is polynomial.

The situation is nicer in spherical and hyperbolic geometry. For a spherical convex polyhedron P and its polar dual P^* we have by [33]

$$\text{Vol}(P) + \frac{1}{2} \sum_{\{i,j\}} \ell_{ij} \lambda_{ij} + \text{Vol}(P^*) = \pi^2.$$

This implies that $S(P) + S^*(P^*) = 2\pi^2$, where S and S^* are functionals on the space of warped spherical polyhedra. Functionals S and S^* are the true analogs of HE and Vol, respectively. In particular, their variational properties are similar to those of their Euclidean counterparts. For more details, see Subsections 4.2 and 4.3. The hyperbolic case is similar, but the polar dual P^* of a hyperbolic polyhedron P lives in the de Sitter space; see Subsection 4.4.

1.8 Gauss Infinitesimal Rigidity and the Minkowski Theorem

Equation (1.4) expresses what we call *Gauss infinitesimal rigidity*:

If the support numbers h_i of a convex polyhedron vary in such a way that the face areas remain constant in the first order, then the polyhedron undergoes a parallel translation.

This theorem was stated and proved by Alexandrov [3, Chapter XI]. Minkowski proved what we would call the global Gauss rigidity: if two convex polyhedra have the same face normals and face areas, then they differ only by a translation. This is the uniqueness part of the *Minkowski theorem*:

> Given unit vectors ν_i that span $\mathbb{R}^3$ and positive numbers C_i such that $\sum_i C_i \nu_i = 0$, there exists a convex polyhedron, unique up to translation, with face areas C_i and outward face normals ν_i.

More generally, Minkowski's existence and uniqueness theorem holds in $\mathbb{R}^d$.

1.9 Alexandrov Theorem on Existence of a Convex Polyhedron with a Given Metric on the Boundary

A counterpart to the Minkowski theorem in dimension 3 is the *Alexandrov theorem* [2]:

> Given a Euclidean cone-metric g on the 2-sphere $\mathbb{S}^2$ with all cone angles less than 2π, there exists a convex polyhedron in $\mathbb{R}^3$ with g as the intrinsic metric on the boundary.

In a joint paper [6] with A. Bobenko, we gave a new proof of the Alexandrov theorem. The polyhedron P is obtained by constructing a family of warped polyhedra $P_t, 0 \leq t \leq 1$ where the curvatures κ_i tend to 0 as t tends to 1. The local existence of such a family is based on the following property of the Jacobian of the map $r \mapsto \kappa$:

$$\dim \ker \left(\frac{\partial \kappa_i}{\partial r_j} \right) = 0, \quad \text{if } 0 < \kappa_i < \delta_i \text{ for all } i.$$

Here δ_i is the angular defect of the i-th cone point on $(\mathbb{S}^2, g)$. For the global existence one has to make sure that polyhedra do not degenerate in the process of deformation. Note that the triangulation of the boundary of P may change in the process of deformation. See Subsection 5.3 for more details.

2 Gauss Rigidity of Convex Polyhedra

2.1 The Theorem

A set $(Q_t)_{0 \leq t < \varepsilon}$ of compact convex polyhedra is called a *linear family* if each Q_t is obtained from $Q = Q_0$ by translating the planes of the faces, with the translation vector of each face depending linearly on t.

In this section, a proof of the following theorem is presented.

Theorem 2.1 *Assume that the area of each face in the linear family (Q_t) is constant in the first order of t. Then all Q_t are translates of Q.*

This theorem is proved by Alexandrov [3] using two different methods. The proof given here is essentially the one from [3, Chapter XI, § 3], but presented in a self-consistent way. Our purpose is to reveal that the argument is based on certain variational properties of the volume in order to make the relationship with Section 3 more straightforward.

Let us introduce some notation. Let $F_1, F_2, \ldots, F_n$ be the faces of Q, and let ν_i be the outer unit normal to the face F_i. Then the equation of the plane of the face F_i is

$$\mathrm{span}(F_i) = \{x \in \mathbb{R}^3 \mid \langle x, \nu_i \rangle = h_i^0\},$$

where h_i^0 is the signed distance from the coordinate origin $0 \in \mathbb{R}^3$ to the plane spanned by F_i. The numbers (h_i^0) are called *support parameters* of the polyhedron Q. We have

$$(2.1) \qquad Q = \{x \in \mathbb{R}^3 \mid \langle x, \nu_i \rangle \leq h_i^0, \ i = 1, \ldots, n\} =: Q(h^0).$$

A linear family of polyhedra is obtained when ν_i are fixed and the support parameters change linearly:

$$Q_t := Q(h^0 + tu),$$

where $u \in \mathbb{R}^n$ is some fixed vector, and $t \in [0, \varepsilon)$ for a sufficiently small ε.

Let A_i be the area of the face F_i. Denote by $DA_i(u)$ the derivative of A_i in the direction u:

$$DA_i(u) := \frac{d}{dt}\Big|_{t=0} A_i(h + tu) = \sum_i \frac{\partial A_i}{\partial h_i} u_i.$$

Usually we denote u by $\dot{h}$ and use the notation $\dot{A}_i := DA_i(\dot{h})$.

Lemma 2.2   *Theorem* 2.1 *is equivalent to each of the following two statements:*

(i)   *If* $\dot{A}_i = 0$ *for all* $i \in \{1, \ldots, n\}$, *then there exists a vector* $a \in \mathbb{R}^n$ *such that* $\dot{h}_i = \langle \nu_i, a \rangle$ *for all* i.

(ii) *The Jacobian of the map* $(h_i) \mapsto (A_i)$ *has corank* 3:

$$\dim \ker\left(\frac{\partial A_i}{\partial h_j} \right)\Big|_{h=h^0} = 3.$$

Proof   Take any $a \in \mathbb{R}^n$ and consider the translate $Q(h^0) + ta$ of the polyhedron (2.1). Then $Q(h^0) + ta$ has the same face normals. It is easy to see that its support parameters are equal to $h_i^0 + t\langle \nu_i, a \rangle$, so that we have

$$Q(h^0) + ta = Q(h^0 + t\dot{h}), \ \text{where } \dot{h}_i = \langle \nu_i, a \rangle$$

Thus Lemma 2.2(i) is equivalent to Theorem 2.1. The assumption $\dot{A}_i = 0$ means that the face areas are constant in the first order, and the conclusion means that all members of the family are translates of Q.

Consider the linear subspace

$$L = \{u \in \mathbb{R}^n \mid \exists a \in \mathbb{R}^3 \ \text{such that } u_i = \langle \nu_i, a \rangle \ \text{for all } i\}$$

Since the vectors $(\nu_i)_{i=1}^n$ span $\mathbb{R}^3$, we have $\dim L = 3$. Also we have

$$L \subset \ker\left(\frac{\partial A_i}{\partial h_j} \right),$$

because, by the previous paragraph, changing h^0 by an element of L corresponds to translation of the polyhedron and thus preserves A_i. Thus the second statement of the lemma is equivalent to $\ker(\frac{\partial A_i}{\partial h_j}) = L$. On the other hand, by definition of the directional derivative we have

$$\dot{A}_i = 0 \,\forall i \Rightarrow \dot{h} \in \ker\left(\frac{\partial A_i}{\partial h_j}\right).$$

Thus (ii) is also equivalent to $\dot{A}_i = 0$ for all $i \Rightarrow \dot{h} \in L$, which is the first statement. The lemma is proved. ■

2.2 The Approach

Recall that we put

$$Q(h) := \{x \in \mathbb{R}^3 \mid \langle x, \nu_i \rangle \le h_i, \ i = 1, \dots, n\}.$$

The set $Q(h)$ is a convex polyhedron for all $h \in \mathbb{R}^n$, but it may have fewer than n faces or even be empty. Let $U \subset \mathbb{R}^n$ be a neighborhood of h^0 such that $Q(h)$ has n faces for all $h \in U$. Consider the function $\mathrm{Vol}\colon U \to \mathbb{R}$, where $\mathrm{Vol}(h)$ is the volume of the polyhedron $Q(h)$.

Lemma 2.3 *The function* Vol *is continuously differentiable on U with*

$$(2.2) \qquad\qquad \frac{\partial \mathrm{Vol}}{\partial h_i} = A_i,$$

where $A_i(h)$ is the area of the i-th face of the polyhedron $Q(h)$.

Proof Equation (2.2) is geometrically obvious; as we shift the plane of the i-th face by ε, we glue to (or cut from) Q a convex slice of thickness ε. One side of the slice has area A_i, the other side has area $A_i + O(\varepsilon)$. Hence the volume of the slice is $\varepsilon A_i + o(\varepsilon)$, and (2.2) follows.

Since its partial derivatives A_i are continuous, the function Vol is continuously differentiable. ■

Equation (2.2) implies that the Jacobian of the map $(h_i) \mapsto (A_i)$ equals the matrix of the second differential of the function Vol. Here by the second differential we mean a symmetric bilinear form

$$D^2 \mathrm{Vol}(u, v) := \frac{d}{dt}\Big|_{t=0} D_{h+tv} \mathrm{Vol}(u) = \sum_{i,j} \frac{\partial^2 \mathrm{Vol}}{\partial h_i \partial h_j} u_i v_j.$$

This yields the following reformulation of Theorem 2.1.

The second differential of the volume at $h = h^0$ has corank 3:

$$\dim \ker(D^2_{h^0} \mathrm{Vol}) = 3.$$

In order to prove Theorem 2.1, we will compute the second variation of the volume in two different ways and compare the formulas obtained. Here the *second variation* Vol¨ is the quadratic form associated with D^2 Vol:

$$\text{Vol}^{\cdot\cdot} := D^2 \text{Vol}(\dot{h}, \dot{h}).$$

Equivalently,

$$\text{Vol}^{\cdot\cdot} = \frac{d^2}{dt^2}\Big|_{t=0} \text{Vol}(h + t\dot{h}).$$

2.3 Orthoscheme Decomposition

Recall that F_i is the face of $Q(h)$ with outer normal ν_i. If F_i and F_j share an edge, denote this edge by F_{ij}.

Let q_i be the foot of the perpendicular dropped from $0 \in \mathbb{R}^3$ to the plane spanned by F_i. For every pair of adjacent faces, drop perpendiculars from the points q_i and q_j to the line spanned by F_{ij}. Their common foot will be denoted by $q_{ij} = q_{ji}$. Finally, denote by q_{ijk} the vertex of $Q(h)$ where faces F_i, F_j, and F_k meet.

Denote by h_{ij} the signed length of the segment $q_i q_{ij}$, the sign being positive if q_i lies on the same side from the edge F_{ij} as the polygon F_i. Similarly, let h_{ijk} be the signed length of the segment $q_{ij} q_{ijk}$. See Figure 1.
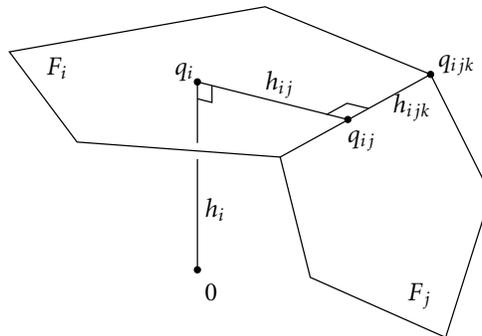
Figure 1: Definition of h_{ij} and h_{ijk}.

From now on, we assume that the polyhedron $Q(h^0)$ is *simple, i.e.,* that exactly three faces meet at each of its vertices. Then, by choosing a neighborhood U of h^0 appropriately small, we can ensure that all polyhedra $Q(h)$ with $h \in U$ are combinatorially isomorphic (under identification of faces with equal outward normals). The case of a non-simple $Q(h^0)$ is a bit subtle, and we explain in Subsection 2.6 how it can be treated.

Under this assumption, the functions

$$h_{ij}\colon U \to \mathbb{R}, \qquad h_{ijk}\colon U \to \mathbb{R},$$

are linear; see equation (2.9). If $F_i \cap F_j = \varnothing$ or $F_i \cap F_j \cap F_k = \varnothing$, then the corresponding functions h_{ij} and h_{ijk} are not defined or can be put identically zero.

Lemma 2.4 *For all $h \in U$, we have*

$$(2.3) \qquad \mathrm{Vol}(h) = \frac{1}{6} \sum_{i,j,k} h_i h_{ij} h_{ijk}.$$

Proof Denote by ℓ_{ij} the length of the edge F_{ij}. Then we have

$$(2.4) \qquad \ell_{ij} = h_{ijk} + h_{ijl},$$

where q_{ijk} and q_{ijl} are the endpoints of F_{ij}. By substituting this into

$$(2.5) \qquad A_i = \frac{1}{2} \sum_j h_{ij} \ell_{ij}$$

and substituting the result into

$$(2.6) \qquad \mathrm{Vol}(h) = \frac{1}{3} \sum_i h_i A_i,$$

we obtain (2.3).

Alternatively, the right-hand side of (2.3) can be seen as the sum of signed volumes of the orthoschemes $0 q_i q_{ij} q_{ijk}$. ∎

Remark 2.5 Equations (2.5) and (2.6) hold also in the case when some of the h_{ij} or h_i are negative (that is when q_i lies outside F_i or 0 lies outside Q), see [48, Lemma 5.1.1].

Note that since h_{ij} and h_{ijk} are linear functions of h, formula (2.3) expresses $\mathrm{Vol} \colon U \to \mathbb{R}$ as a third degree homogeneous polynomial in $(h_i)_{i=1}^n$.

2.4 First and Second Variations of the Volume and of the Face Areas

We start with an auxiliary lemma.

Lemma 2.6 *For an arbitrary variation $\dot{h} \in \mathbb{R}^n$ we have*

$$(2.7) \qquad \dot{h}_i h_{ij} + \dot{h}_j h_{ji} = h_i \dot{h}_{ij} + h_j \dot{h}_{ji},$$

$$(2.8) \qquad \dot{h}_{ij} h_{ijk} + \dot{h}_{ik} h_{ikj} = h_{ij} \dot{h}_{ijk} + h_{ik} \dot{h}_{ikj}.$$

Here $\dot{h}_{ij}$ and $\dot{h}_{ijk}$ denote derivatives in the direction of $\dot{h}$.

Proof By [48, Equation 5.1.3], we have

$$(2.9) \qquad h_{ij} = h_j \operatorname{cosec} \varphi_{ij} - h_i \cot \varphi_{ij}, \quad h_{ji} = h_i \operatorname{cosec} \varphi_{ij} - h_j \cot \varphi_{ij},$$

these formulas holding independently of the signs of h_i, h_j, h_{ij}, h_{ji}. Note that the angle φ_{ij} is constant. By differentiation we obtain (2.7).

Equation (2.8) is proved in the same way, by expressing h_{ijk} and h_{ikj} through h_{ij} and h_{ik}. ∎

Alternatively, equation (2.7) can be proved by differentiating the equation

$$\varphi_{ij} = \arctan \frac{h_{ij}}{h_i} + \arctan \frac{h_{ji}}{h_j}$$

(valid if $h_i, h_j \neq 0$) and using $h_i^2 + h_{ij}^2 = h_j^2 + h_{ji}^2$.

Lemma 2.7 *The first variations of the face areas are given by*

$$(2.10) \qquad \dot{A}_i = \sum_{j,k} \dot{h}_{ij} h_{ijk} = \sum_{j,k} h_{ij} \dot{h}_{ijk}.$$

Proof Equations (2.5) and (2.4) imply

$$\dot{A}_i = \frac{1}{2} \sum_{j,k} (\dot{h}_{ij} h_{ijk} + h_{ij} \dot{h}_{ijk})$$

$$= \frac{1}{2} \sum_{\{j,k\}} (\dot{h}_{ij} h_{ijk} + \dot{h}_{ik} h_{ikj}) + \frac{1}{2} \sum_{\{j,k\}} (h_{ij} \dot{h}_{ijk} + h_{ik} \dot{h}_{ikj}),$$

where $\sum_{\{j,k\}}$ denotes the sum over unordered pairs of j and k. By using (2.8) and converting back to the sum over ordered pairs, we obtain (2.10). ∎

Lemma 2.8 *The first variation of the volume is given by either of the following formulas:*

$$(2.11) \qquad \mathrm{Vol}^{\cdot} = \sum_i \dot{h}_i A_i,$$

$$(2.12) \qquad \mathrm{Vol}^{\cdot} = \frac{1}{2} \sum_i h_i \dot{A}_i.$$

Proof From (2.6) we have

$$(2.13) \qquad \mathrm{Vol}^{\cdot} = \frac{1}{3} \sum_i \dot{h}_i A_i + \frac{1}{3} \sum_i h_i \dot{A}_i.$$

By using (2.5), regrouping, then using (2.7), regrouping back, and finally applying Lemma 2.7, we obtain

$$2\sum_i \dot{h}_i A_i = \sum_i \dot{h}_i \sum_j h_{ij}\ell_{ij} = \sum_{i,j,k} \dot{h}_i h_{ij} h_{ijk} = \sum_{\{i,j\},k} (\dot{h}_i h_{ij} + \dot{h}_j h_{ji}) h_{ijk}$$

$$= \sum_{\{i,j\},k} (h_i \dot{h}_{ij} + h_j \dot{h}_{ji}) h_{ijk} = \sum_{i,j,k} h_i \dot{h}_{ij} h_{ijk} = \sum_i h_i \sum_{j,k} \dot{h}_{ij} h_{ijk}$$

$$= \sum_i h_i \dot{A}_i.$$

Substituting this in (2.13) yields (2.11) and (2.12). ∎

Remark 2.9 Note that (2.11) is equivalent to (2.2). Similarly, the first equation in Lemma 2.7 says that $\frac{\partial A_i}{\partial h_{ij}} = \ell_{ij}$, which is geometrically obvious. This provides an alternative, more geometric approach to Lemmas 2.8 and 2.7.

Lemma 2.10 *The second variations of the face areas are given by*

$$\ddot{A}_i = \sum_{j,k} \dot{h}_{ij} \dot{h}_{ijk}.$$

Proof This follows from Lemma 2.7, since $\ddot{h}_{ij} = \ddot{h}_{ijk} = 0$ because of their linearity in h. ∎

Lemma 2.11 *The second variation of the volume is given by either of the following two formulas:*

$$(2.14) \qquad\qquad \mathrm{Vol}^{\cdot\cdot} = \sum_i \dot{h}_i \dot{A}_i,$$

$$(2.15) \qquad\qquad \mathrm{Vol}^{\cdot\cdot} = \sum_i h_i \ddot{A}_i.$$

Proof By differentiating (2.11) and taking into account $\ddot{h}_i = 0$, we obtain (2.14). Differentiating (2.12) yields

$$\mathrm{Vol}^{\cdot\cdot} = \frac{1}{2}\sum_i \dot{h}_i \dot{A}_i + \frac{1}{2}\sum_i h_i \ddot{A}_i.$$

Combining this with (2.14) yields (2.15). ∎

Remark 2.12 Another way to write the first and the second variations of Vol is

$$\mathrm{Vol}^{\cdot} = \frac{1}{2}\sum_{i,j,k} \dot{h}_i h_{ij} h_{ijk} = \frac{1}{2}\sum_{i,j,k} h_i \dot{h}_{ij} h_{ijk} = \frac{1}{2}\sum_{i,j,k} h_i h_{ij} \dot{h}_{ijk},$$

$$\mathrm{Vol}^{\cdot\cdot} = \sum_{i,j,k} \dot{h}_i \dot{h}_{ij} h_{ijk} = \sum_{i,j,k} h_i \dot{h}_{ij} \dot{h}_{ijk} = \sum_{i,j,k} \dot{h}_i h_{ij} \dot{h}_{ijk}.$$

2.5 Proof of Theorem 2.1

The proof of Theorem 2.1 is based on the following key lemma.

Lemma 2.13 *Let $\dot{h}$ be such that $\dot{A}_i = 0$ for some i. Then we have*

$$(2.16) \qquad \ddot{A}_i \le 0.$$

Furthermore, equality in (2.16) *holds only if $\dot{\ell}_{ij} = 0$ for all edges bounding the face F_i.*

This lemma will be proved later in this subsection.

Proof of Theorem 2.1 We will prove Theorem 2.1 using the reformulation given in Lemma 2.2. if $\dot{h}$ is such that $\dot{A}_i = 0$ for all i, then $\dot{h}_i = \langle a, \nu_i \rangle$ for some $a \in \mathbb{R}^3$.

If $\dot{A}_i = 0$ for all i, then (2.14) implies that

$$(2.17) \qquad \mathrm{Vol}^{\cdot\cdot} = \sum_i \dot{h}_i \dot{A}_i = 0.$$

On the other hand, by Lemma 2.13 we have $\ddot{A}_i \le 0$. Furthermore, without loss of generality we can assume that $h_i > 0$ for all i (just choose the origin $0 \in \mathbb{R}^3$ inside Q). Hence (2.15) implies

$$\mathrm{Vol}^{\cdot\cdot} = \sum_i h_i \ddot{A}_i \le 0.$$

By comparing this with (2.17), we deduce that $\ddot{A}_i = 0$ for all i. Then by the second part of Lemma 2.13, we have $\dot{\ell}_{ij} = 0$ for all edges of Q. It follows easily that $\dot{h}$ translates the polyhedron as a rigid body, thus $\dot{h}_i = \langle a, \nu_i \rangle$ for some $a \in \mathbb{R}^n$. Theorem 2.1 is proved. ∎

Fix an index i and consider the function $A_i \colon U \to \mathbb{R}$. It can be written as a function of (h_{ij}). Here j varies over all faces of Q adjacent to the i-th face. Without loss of generality assume that $j = 1, \ldots, m$ and introduce new variables

$$g_j = h_{ij}, \quad j = 1, \ldots, m.$$

Formally speaking, we consider a linear map

$$\Phi \colon \mathbb{R}^n \to \mathbb{R}^m, \qquad (h_i) \mapsto (g_j) = (h_{ij}).$$

The map A_i factors through Φ and induces a map $A \colon V \to \mathbb{R}$, where $V = \Phi(U) \subset \mathbb{R}^m$ is a neighborhood of $g^0 = \Phi(h^0)$. We have

$$A = \frac{1}{2} \sum_{j=1}^m g_j \ell_j,$$

where $\ell_j = \ell_{ij}$ are linear functions of g.

By our construction, $g_j = h_{ij}$ are the support parameters of the face F_i with respect to the projection of 0 to $\mathrm{span}(F_i)$. We have

$$(2.18) \qquad \frac{\partial A}{\partial g_j} = \ell_j, \quad \text{hence } \frac{\partial^2 A}{\partial g_j \partial g_k} = \frac{\partial \ell_j}{\partial g_k}.$$

Lemma 2.14 *The second differential D^2A of the area function on the space of convex m-gons with fixed edge directions has signature $(+_1, 0_2, -_{m-3})$. Furthermore, D^2A at a point g takes a positive value on the vector g.*

Proof First, let us show that

$$\text{(2.19)} \qquad \dim \ker D^2A = 2.$$

By (2.18), D^2A coincides with the Jacobian of the map $g \mapsto \ell$. Thus

$$\text{(2.20)} \qquad \dot g \in \ker D^2A \iff \dot\ell = 0.$$

But $\dot\ell = 0$ implies that $\dot g$ is induced by a parallel translation of the polygon. Such deformations form a 2-dimensional space, and (2.19) follows.

Consider the space $\mathcal{Q}_m$ of all convex m-gons. A convex polygon is determined by its edge normals (μ_j) and support parameters (g_j). The numbers (g_j) must satisfy a system of linear inequalities (with coefficients depending on (μ_j)) expressing the fact that all edge lengths are non-negative. For every collection (μ_j) of m different unit vectors positively spanning $\mathbb{R}^2$, the collection $(g_j = 1)$ satisfies this system, as (μ, g) corresponds to a circumscribed polygon. Thus $\mathcal{Q}_m$ retracts to the space of configurations (μ_j) of m unit vectors positively spanning $\mathbb{R}^2$. It follows that $\mathcal{Q}_m$ is connected.

Because of (2.19), the rank of D^2A is constant over the space $\mathcal{Q}_m$. As $\mathcal{Q}_m$ is connected, it follows that the signature of D^2A is also constant. Thus in order to determine the signature at the point we need, it suffices to compute it at a point we like.

Let us compute the matrix of D^2A. Let $\alpha_{j,j+1}$, $j = 1, \dots, m$, be the exterior angle between j-th and $(j+1)$-st side ($j+1$ taken modulo m). Similarly to (2.9), we have

$$\ell_j = g_{j,j-1} + g_{j,j+1}$$
$$= (g_{j-1} \operatorname{cosec} \alpha_{j-1,j} - g_j \cot \alpha_{j-1,j}) + (g_{j+1} \operatorname{cosec} \alpha_{j,j+1} - g_j \cot \alpha_{j,j+1}),$$

which implies that

$$\frac{\partial \ell_j}{\partial g_j} = -(\cot \alpha_{j-1,j} + \cot \alpha_{j,j+1}),$$

$$\frac{\partial \ell_j}{\partial g_{j+1}} = \frac{\partial \ell_{j+1}}{\partial g_j} = \frac{1}{\sin \alpha_{j,j+1}}.$$

Note that the matrix of D^2A does not depend on g. This is because A is a homogeneous polynomial of second degree in (g_j). By the same reason we have

$$D^2A(g,g) = 2A > 0,$$

which proves the second statement in the lemma.

To determine the signature of D^2A, put $\alpha_{j,j+1} = \frac{2\pi}{m}$. Then we have

$$D^2A = \frac{1}{\sin\frac{2\pi}{m}} \begin{pmatrix} -2\cos\frac{2\pi}{m} & 1 & 0 & \cdots & 1 \\ 1 & -2\cos\frac{2\pi}{m} & 1 & \cdots & 0 \\ 0 & 1 & -2\cos\frac{2\pi}{m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -2\cos\frac{2\pi}{m} \end{pmatrix}.$$

The spectrum of this matrix is

$$\left\{ \frac{2(\cos\frac{2\pi k}{m} - \cos\frac{2\pi}{m})}{\sin\frac{2\pi}{m}}, k = 1, \ldots, m \right\},$$

which contains exactly one positive (for $k = m$) and two zero ($k = 1$ and $k = m - 1$) eigenvalues. The lemma is proved. ∎

Proof of Lemma 2.13 Change the variables for the function A_i with the help of the map (2.5). Let $\dot{g} = d\Phi(\dot{h})$ be the variation of g induced by $\dot{h}$. By assumption, we have

$$\sum_j \dot{g}_j \ell_j = \dot{A} = 0.$$

On the other hand, since (ℓ_j) are linear functions of g, we have

$$\ell_j = \sum_k \frac{\partial \ell_j}{\partial g_k} g_k,$$

which results in

$$D^2A(\dot{g}, g) = \sum_{j,k} \frac{\partial \ell_j}{\partial g_k} \dot{g}_j g_k = \sum_j \dot{g}_j \ell_j = 0.$$

That is, the vectors $\dot{g}, g \in \mathbb{R}^m$ are mutually orthogonal with respect to the symmetric bilinear form D^2A. By Lemma 2.14, g is a positive vector for D^2A, and D^2A is negative semidefinite on the orthogonal complement to g. Thus we have

$$\ddot{A} = D^2A(\dot{g}, \dot{g}) \le 0.$$

If $\ddot{A} = 0$, then $\dot{g} \in \ker D^2A$, which by (2.20) implies that $\dot{\ell}_j = 0$ for all j. The lemma is proved. ∎

Remark 2.15 In general, the second differential does not behave well under change of variables. However, it does in our case, because the map (2.5) is linear. Therefore the second variation $\ddot{A}_i$ in Lemma 2.13 is associated with the second differential D^2A from Lemma 2.14:

$$\ddot{A}_i = D^2A(d\Phi(\dot{h}), d\Phi(\dot{h})) = D^2A(\dot{g}, \dot{g}),$$

which was implicitly used in the above proof.

2.6   The Case of a Non-simple Polyhedron Q

In the course of our proof we assumed that the polyhedron Q is simple; that is, each of its vertices belongs to exactly three faces. This assumption implies that the combinatorics of Q is preserved when its support parameters vary slightly.

If the polyhedron $Q = Q(h^0)$ is non-simple, then $Q(h)$ may be combinatorially different from Q even for h close to h^0. A neighborhood U of $h^0 \in \mathbb{R}^n$ is subdivided into cells (U^Δ) with Δ indexing simple combinatorial types of perturbed polyhedra. On each cell, the function Vol is a third degree homogeneous polynomial in (h_i),

$$\mathrm{Vol}(h) = V^\Delta(h) \text{ for } h \in U^\Delta,$$

so that function Vol is piecewise polynomial in a neighborhood of h^0. It can be shown that

$$\mathrm{Vol} \in C^2(U), \quad A_i \in C^1(U).$$

But A_i may fail to be C^2, which is bad because second differentials of face areas play a key role in Subsection 2.5.

This problem can be resolved as follows. Choose any simple combinatorics Δ from a neighborhood of h^0, and work with V^Δ and A_i^Δ instead of Vol and A_i. Geometrically this means that we view our non-simple polyhedron $Q(h^0)$ as a member of a family of simple polyhedra, with some edge lengths vanishing; by varying h^0 in U, we allow negative edge lengths to appear. One can check that all arguments in Subsection 2.5 go through.

This modification proves Theorem 2.1 for non-simple polyhedra.

3   Metric Rigidity of Convex Polyhedra

3.1   The Theorem

Let $P \subset \mathbb{R}^3$ be a convex polyhedron with vertices $p_1, \ldots, p_n$. Assume that P is simplicial; that is, all of its faces are triangles. Subsection 3.8 explains how our arguments change in the non-simplicial case.

Definition 3.1   An *infinitesimal deformation* of P is a collection of vectors $q_i \in \mathbb{R}^3$, $i = 1, \ldots, n$. Each q_i is thought of as a vector applied at the point p_i.

An infinitesimal deformation (q_i) of a polyhedron P is called *isometric* if

$$(3.1) \qquad \frac{d}{dt}\Big|_{t=0} \|(p_i + t q_i) - (p_j + t q_j)\| = 0,$$

for all edges $p_i p_j$ of P. In other words lengths of all edges do not change in the first order as vertices (p_i) move linearly with velocities (q_i).

A simple computation shows that condition (3.1) is equivalent to

$$\langle p_i - p_j, q_i - q_j \rangle = 0.$$

Definition 3.2 An infinitesimal deformation (q_i) of P is called *trivial*, if the map $p_i \mapsto q_i$ is the restriction of an infinitesimal isometry of $\mathbb{R}^3$:

$$q_i = Ap_i + b, \quad A \in \mathfrak{so}(3), b \in \mathbb{R}^3.$$

An infinitesimal isometry of $\mathbb{R}^3$ preserves in the first order distances between *all* pairs of points; therefore, every trivial infinitesimal deformation of P is isometric.

Definition 3.3 A polyhedron P is called *infinitesimally rigid* if every isometric infinitesimal deformation of P is trivial.

Theorem 3.4 (Legendre–Cauchy–Dehn) *Every convex polyhedron in $\mathbb{R}^3$ is infinitesimally rigid.*

This is usually referred to as Dehn's theorem. Let us explain why we prefer a different attribution. Cauchy [10] proved a global rigidity statement: two convex polyhedra with the same combinatorics and pairwise isometric faces are congruent. His proof was based on ideas presented by Legendre in [31, note XII, pp. 321–334]. Dehn was the first to state and prove the infinitesimal rigidity theorem in [15]. However, in the footnote on the first page of [59], Weyl remarks that the argument in [10] carries over word for word to yield a proof of Theorem 3.4 and reproaches Dehn for not citing Cauchy. Note also that the Cauchy's "arm lemma" is more immediate in the infinitesimal context.

3.2 A Reformulation

Without loss of generality we may assume that the coordinate origin $0 \in \mathbb{R}^3$ lies in the interior of P. For every vertex p_i of P, put $r_i^0 = \|p_i\|$. The length of an edge $p_i p_j$ will be denoted by $\ell_{ij} = \|p_i - p_j\|$. Triangles $0p_i p_j$, where $p_i p_j$ ranges over all edges of P, cut the polyhedron P into triangular pyramids. These pyramids have a common apex at 0, their bases are faces of P. Let $r \in \mathbb{R}^n$ be a point close to $r^0 = (r_i^0)_{i=1}^n$. Change lateral edge lengths of the pyramids from r_i^0 to r_i while keeping base edge lengths ℓ_{ij} fixed. A metric space glued from the new collection of pyramids (by the old gluing rules) is called a *warped polyhedron*. A warped polyhedron is not embeddable in $\mathbb{R}^3$ in general, because the sum ω_i of all dihedral angles at an edge $0p_i$ may be different from 2π, for some i. Denote by $\kappa_i = 2\pi - \omega_i$ the *curvature* of the warped polyhedron at the edge $0p_i$. This yields a C^∞-map

$$U \to \mathbb{R}^n, \qquad r \mapsto \kappa,$$

where $U \subset \mathbb{R}^n$ is a sufficiently small neighborhood of r^0.

Lemma 3.5 *A convex polyhedron P is infinitesimally rigid if and only if*

$$\dim \ker \left(\frac{\partial \kappa_i}{\partial r_j} \right) \bigg|_{r=r^0} = 3.$$

Proof We will establish a correspondence between isometric infinitesimal deformations of P and elements of the kernel of $J_r^\kappa = \left(\frac{\partial \kappa_i}{\partial r_j}\right)$.

Let q be an infinitesimal isometric deformation of P and let $p_i^t = p_i + tq_i$ be the corresponding linear motions of the vertices. Put

(3.2) $$r_i^t = \|p_i^t\|, \quad \ell_{ij}^t = \|p_i^t - p_j^t\|.$$

This defines a family of warped polyhedra, this time with non-constant metric on the boundary. By Definition 3.1 and since the polyhedron remains embedded in $\mathbb{R}^3$, we have

$$\dot{\ell}_{ij} := \frac{d}{dt}\bigg|_{t=0} \ell_{ij}^t = 0, \quad \kappa_i^t = 0$$

for all i, j. Modify the deformation (3.2) by putting $\ell_{ij}^t = \ell_{ij}$. Although the functions κ_i^t might not be identically zero anymore, their time derivatives at $t = 0$ will not change:

$$\ell_{ij}^t = \ell_{ij} \Rightarrow \dot{\kappa}_i = 0.$$

It follows that $\dot{r} \in \ker J_r^\kappa$.

In particular, if q is an infinitesimal rotation around an axis through the origin, then we have $r_i^t = r_i$, and thus $\dot{r} = 0$. If q is a parallel translation, then $\dot{r} \neq 0$ is a non-trivial element in $\ker J_r^\kappa$. One can show that translations in linearly independent directions in $\mathbb{R}^3$ produce linearly independent vectors $\dot{r} \in \mathbb{R}^n$. Hence trivial infinitesimal deformations of P give rise to a 3-dimensional subspace of $\ker J_r^\kappa$.

In the opposite direction, let us associate with $\dot{r} \in \ker J_r^\kappa$ an isometric infinitesimal deformation of P. Let $p_1 p_2 p_3$ be a face of P. Choose a vector q_1 collinear with $0p_1$ such that when p_1 moves with the velocity q_1, its distance from 0 changes with speed $\dot{r}_1$:

$$q_1 = \dot{r}_1 \frac{p_1}{\|p_1\|}.$$

Choose q_2 collinear with the plane $0p_1 p_2$ so as to satisfy the conditions on variations of r_2 and ℓ_{12}:

$$\left\langle q_2, \frac{p_2}{\|p_2\|} \right\rangle = \dot{r}_2, \quad \langle p_1 - p_2, q_1 - q_2 \rangle = 0.$$

And choose q_3 so as to satisfy the conditions on variations of r_3, ℓ_{13}, and ℓ_{23}:

(3.3) $$\left\langle q_3, \frac{p_3}{\|p_3\|} \right\rangle = \dot{r}_3, \quad \langle p_1 - p_3, q_1 - q_3 \rangle = 0, \quad \langle p_2 - p_3, q_2 - q_3 \rangle = 0.$$

Now proceed to an adjacent face $p_1 p_3 p_4$ and determine q_4 by conditions similar to (3.3). Continue the face path $(p_1 p_i p_{i+1})$ around the vertex p_1 until it closes at $p_1 p_k p_2$. On the vector q_k there will be four conditions, the fourth being the stability of ℓ_{k2}. It is easy to see that this condition follows from the first three due to

$$\dot{\kappa}_1 := \sum_i \frac{\partial \kappa_1}{\partial r_i} \dot{r}_i = 0.$$

so that q_k is well defined. In a similar way we can find q_i for all i, all closing conditions being satisfied due to $\dot{\kappa} = 0$.

Note that we made some voluntary choices at the beginning, with q_1 and q_2, but starting from q_3 everything was forced. It follows that the infinitesimal deformation q is determined by $\dot{r}$ uniquely up to an infinitesimal rotation.

As we observed, trivial infinitesimal deformations of P generate a 3-dimensional subspace of $\ker J_r^{\kappa}$. If there is a non-trivial isometric infinitesimal deformation, then the corresponding $\dot{r} \in \ker J_r^{\kappa}$ lies outside this subspace, as the map $q \mapsto \dot{r}$ identifies only deformations that differ by an infinitesimal rotation. Thus $\dim \ker J_r^{\kappa} = 3$ implies infinitesimal rigidity. Vice versa, if $\dim \ker J_r^{\kappa} > 3$, then there is an $\dot{r} \in \ker J_r^{\kappa}$ not coming from a trivial infinitesimal deformation. Thus it determines a non-trivial isometric infinitesimal deformation q. That is, infinitesimal rigidity implies $\dim \ker J_r^{\kappa} = 3$. ∎

3.3 The Discrete Hilbert–Einstein Functional

In Section 3.2 we introduced warped polyhedra that have curvatures (κ_i) along their radial edges $0p_i$. Recall that the boundary edge lengths (ℓ_{ij}) are kept constant, while the radial edge lengths $r = (r_i)_{i=1}^n$ are allowed to vary in a neighborhood U of a point $r^0 \in \mathbb{R}^n$. Denote by $P(r)$ the warped polyhedron with radial edge lengths r. Let λ_{ij} be the exterior dihedral angle at the boundary edge $p_i p_j$. Note that (λ_{ij}), as well as (κ_i) are functions of r.

Definition 3.6 The *discrete Hilbert–Einstein functional* of a warped polyhedron $P(r)$ is

$$\mathrm{HE}(r) = \sum_i r_i \kappa_i + \sum_{\{i,j\}} \ell_{ij} \lambda_{ij}.$$

By $\sum_{\{i,j\}}$ we denote the sum over all unordered pairs of i and j, so that every boundary edge is taken once.

Lemma 3.7 *We have*

$$(3.4) \qquad\qquad \frac{\partial \mathrm{HE}}{\partial r_i} = \kappa_i.$$

Lemma 3.7 can be derived from (and is equivalent to) the Schläfli formula for Euclidean polyhedra. We will prove it by a different method in Subsection 3.5.

Corollary 3.8 *A convex polyhedron P is infinitesimally rigid if and only if the second differential of the Hilbert–Einstein functional has corank* 3 *at the point* r^0:

$$\dim \ker(D^2 \mathrm{HE})|_{r=r^0} = 3.$$

Proof In view of Lemma 3.7, the matrix of the second differential of HE is the Jacobi matrix of the map $r \mapsto \kappa$. Thus Corollary 3.8 is simply a reformulation of Lemma 3.5. ∎

3.4 The Total Curvature of a Warped Spherical Polygon

In a warped polyhedron $P(r)$, consider the spherical link S_i of the vertex p_i. By definition, S_i is a complex of spherical triangles obtained from trihedral angles at p_i by intersecting them with a unit sphere centered at p_i. In our case, these triangles are glued cyclically around a common vertex, forming a total angle of $2\pi - \kappa_i$. The angles at the boundary vertices of S_i are equal to the dihedral angles of $P(r)$, that is, to $\pi - \lambda_{ij}$. The lengths of radial edges in S_i are equal to ρ_{ij}, where ρ_{ij} is the angle at the vertex p_i in the triangle $0p_ip_j$; see Figure 2. Denote

$$s_{ij} = \cos \rho_{ij}.$$
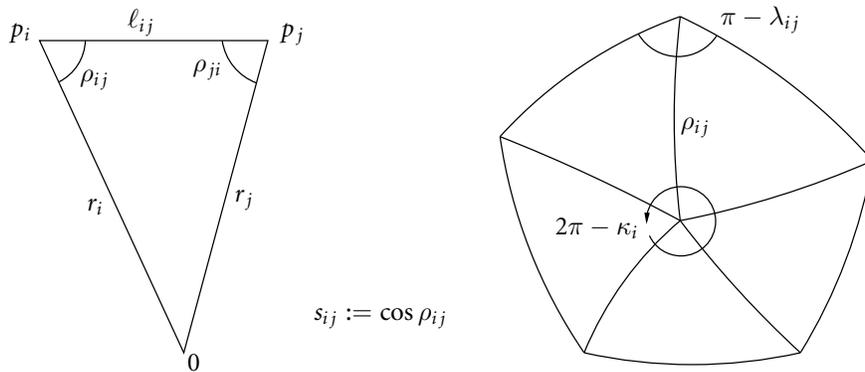
Figure 2: The triangle $0p_ip_j$ and the spherical link S_i of the vertex p_i.

A complex of spherical triangles similar to S_i (a set of triangles glued cyclically around a common vertex) will be called a *warped spherical polygon*. Note that S_i is in addition *convex*, that is, the angles at its boundary vertices are less than or equal to π.

Definition 3.9   The *total curvature K_i* of the warped spherical polygon S_i in Figure 2, right, is defined as

$$K_i = \kappa_i + \sum_j s_{ij}\lambda_{ij}.$$

Lemma 3.10   *We have*

$$(3.5) \qquad\qquad \mathrm{HE} = \sum_i r_iK_i.$$

Proof   The orthogonal projection of 0 to the line p_ip_j splits the edge p_ip_j into two segments of lengths $r_i\cos\rho_{ij}$ and $r_j\cos\rho_{ji}$. Thus we have

$$(3.6) \qquad\qquad \ell_{ij} = r_is_{ij} + r_js_{ji}.$$

(If the projection of 0 lies outside the edge, the formula remains valid.) By substituting (3.6) in the definition of HE, we obtain

$$\text{HE} = \sum_i r_i \kappa_i + \sum_{\{i,j\}} (r_i s_{ij} + r_j s_{ji}) \lambda_{ij} = \sum_i r_i \kappa_i + \sum_{i,j} r_i s_{ij} \lambda_{ij}$$

$$= \sum_i r_i \left(\kappa_i + \sum_j s_{ij} \lambda_{ij} \right) = \sum_i r_i K_i,$$

and the lemma is proved. ∎

3.5 Variations of HE and K_i

Let $r \in U$ and let $\dot{r} \in \mathbb{R}^n$ be an arbitrary variation of r. Denote by

$$\text{HE}^{\cdot} = \frac{d}{dt}\Big|_{t=0} \text{HE}(r + t\dot{r})$$

the derivative of the function HE in the direction $\dot{r}$. Similar notations $\dot{K}_i$, $\dot{s}_{ij}$, $\dot{\lambda}_{ij}$ are used for the directional derivatives of other functions of r.

Lemma 3.11 *The first variation of the total curvature of the spherical link S_i of the vertex p_i in $P(r)$ is given by*

$$\dot{K}_i = \sum_j \dot{s}_{ij} \lambda_{ij}.$$

Proof The warped polygon S_i is made of spherical triangles, see Figure 2. Consider a triangle with edge lengths ρ_{ij}, ρ_{ik}. Its third side has a constant length, as it is equal to the angle $p_j p_i p_k$ on the boundary of the polyhedron $P(r)$. By applying Lemma A.2, we obtain

(3.7) $$\dot{\alpha}_{ijk} + \dot{\beta}_{ij} s_{ij} + \dot{\gamma}_{ik} s_{ik} = 0,$$

where α_{ijk} is the angle between the ρ_{ij} and ρ_{ik} sides and β_{ij}, γ_{ik} are the two other angles. As we have

$$\sum_{jk} \alpha_{ijk} = 2\pi - \kappa_i, \quad \beta_{ij} + \gamma_{ij} = \pi - \lambda_{ij},$$

summing (3.7) over all triangles yields

(3.8) $$\dot{\kappa}_i + \sum_j s_{ij} \dot{\lambda}_{ij} = 0.$$

It follows that

$$\dot{K}_i = \dot{\kappa}_i + \sum_j (\dot{s}_{ij} \lambda_{ij} + s_{ij} \dot{\lambda}_{ij}) = \sum_j \dot{s}_{ij} \lambda_{ij},$$

and the lemma is proved. ∎

Lemma 3.12 *For every edge $p_i p_j$, we have*

$$\dot{r}_i s_{ij} + \dot{r}_j s_{ji} + r_i \dot{s}_{ij} + r_j \dot{s}_{ji} = 0.$$

Proof This follows from (3.6) and $\ell_{ij} = 0$. ∎

Lemma 3.13 *The first variation of the Hilbert–Einstein functional of a warped polyhedron $P(r)$ is given by*

$$(3.9) \qquad\qquad \text{HE}^{\cdot} = \sum_i \dot{r}_i \kappa_i.$$

Proof The proof follows by differentiating (3.5) and applying Lemmas 3.11 and 3.12:

$$\text{HE}^{\cdot} = \sum_i \dot{r}_i K_i + \sum_i r_i \dot{K}_i = \sum_i \dot{r}_i \Big(\kappa_i + \sum_j s_{ij} \lambda_{ij} \Big) + \sum_i r_i \sum_j \dot{s}_{ij} \lambda_{ij}$$

$$= \sum_i \dot{r}_i \kappa_i + \sum_{i,j} (\dot{r}_i s_{ij} + r_i \dot{s}_{ij}) \lambda_{ij}$$

$$= \sum_i \dot{r}_i \kappa_i + \sum_{\{i,j\}} (\dot{r}_i s_{ij} + \dot{r}_j s_{ji} + r_i \dot{s}_{ij} + r_j \dot{s}_{ji}) \lambda_{ij} = \sum_i \dot{r}_i \kappa_i. \qquad ∎$$

Lemma 3.7 follows from Lemma 3.13, as (3.4) is just a reformulation of (3.9).

Recall that the *second variation* of HE is the quadratic form in $(\dot{r}_i)$ associated to the second differential of HE:

$$\text{HE}^{\cdot\cdot} = D^2 \,\text{HE}(\dot{r}, \dot{r}) = \sum_{i,j} \frac{\partial^2 \text{HE}}{\partial r_i \partial r_j} \dot{r}_i \dot{r}_j = \frac{d^2}{dt^2}\Big|_{t=0} \text{HE}(r + t\dot{r}).$$

Lemma 3.14 *The second variation of HE is given by either of the following two formulas:*

$$(3.10) \qquad\qquad \text{HE}^{\cdot\cdot} = \sum_i \dot{r}_i \dot{\kappa}_i,$$

$$(3.11) \qquad\qquad \text{HE}^{\cdot\cdot} = \sum_i r_i \sum_j \dot{s}_{ij} \dot{\lambda}_{ij}.$$

Proof Equation (3.10) is a direct consequence of (3.9). In order to prove (3.11), transform (3.10) by using (3.8) and Lemma 3.12:

$$\text{HE}^{\cdot\cdot} = \sum_i \dot{r}_i \dot{\kappa}_i = -\sum_i \dot{r}_i \sum_j s_{ij} \dot{\lambda}_{ij} = -\sum_{\{i,j\}} (\dot{r}_i s_{ij} + \dot{r}_j s_{ji}) \dot{\lambda}_{ij}$$

$$= \sum_{\{i,j\}} (r_i \dot{s}_{ij} + r_j \dot{s}_{ji}) \dot{\lambda}_{ij} = \sum_i r_i \sum_j \dot{s}_{ij} \dot{\lambda}_{ij}. \qquad ∎$$

3.6 Proof of Theorem 3.4

The proof is based on the following lemma.

Lemma 3.15 *Let $\dot{r}$ be such that $\dot{\kappa}_i = 0$ for some i. Then we have*

$$(3.12) \qquad\qquad \sum_j \dot{s}_{ij}\dot{\lambda}_{ij} \le 0.$$

Equality in (3.12) *holds only if $\dot{\lambda}_{ij} = 0$ for all j.*

The proof of this lemma requires a detailed study of the second variation of K_i with respect to the variables (s_{ij}). This is done in Subsection 3.7, which ends with the proof of Lemma 3.15. Now we prove Theorem 3.4 assuming the validity of Lemma 3.15.

Proof of Theorem 3.4 We prove Theorem 3.4 in the reformulation obtained in Lemma 3.5. It suffices to show that if $\dot{r}$ is such that $\dot{\kappa}_i = 0$ for all i, then $\dot{r}$ is induced by a trivial isometric deformation of P.

If $\dot{\kappa}_i = 0$ for all i, then (3.10) implies

$$\mathrm{HE}^{\cdot\cdot} = \sum_i \dot{r}_i \dot{\kappa}_i = 0.$$

On the other hand, equation (3.11) and Lemma 3.15 imply

$$\mathrm{HE}^{\cdot\cdot} = \sum_i r_i \sum_j \dot{s}_{ij}\dot{\lambda}_{ij} \le 0,$$

because $r_i > 0$ for all i. Thus we must have

$$\sum_j \dot{s}_{ij}\dot{\lambda}_{ij} = 0$$

for all i. By Lemma 3.15, this happens only if $\dot{\lambda}_{ij} = 0$ for all edges p_ip_j. This means that the isometric infinitesimal deformation that corresponds to $\dot{r}$ does not change the dihedral angles of the polyhedron P in the first order. It follows that this infinitesimal deformation is trivial. ∎

3.7 The Second Variation of the Total Curvature of a Warped Spherical Polygon

In Subsection 3.4, we defined warped spherical polygons and their total curvature. Warped spherical polygons appeared as spherical links S_i of vertices of a warped polyhedron $P(r)$; see Figure 2. Consequently, the total curvature of S_i was viewed as a function of $r \in \mathbb{R}^n$.

In this subsection, we study warped spherical polygons on their own. We preserve the notations on the right of Figure 2, but suppress the index i. Without loss of generality, assume that the index j numbering the radial edges varies from 1 to m. As

before, lengths of boundary edges are assumed constant, so that a warped spherical polygon is determined by m parameters

$$s_j = \cos \rho_j, \quad j = 1, \ldots, m,$$

with s varying in a neighborhood V of $s^0 \in \mathbb{R}^m$. The point s^0 has the property $\kappa(s^0) = 0$, where κ (a former κ_i) is the angular defect of the interior vertex of the warped polygon.

Consider the total curvature of a warped spherical polygon as a function of s:

$$(3.13) \qquad K: V \to \mathbb{R}, \qquad K(s) = \kappa + \sum_j s_j \lambda_j.$$

Lemma 3.16 *The first and second variations of K as a function of s are given by*

$$(3.14) \qquad \dot{K} = \sum_j \dot{s}_j \lambda_j,$$

$$(3.15) \qquad \ddot{K} = \sum_j \dot{s}_j \dot{\lambda}_j.$$

Proof Equation (3.14) follows from Lemma 3.11, as the form of the first variation does not depend on the choice of variables. (Actually, in the proof of Lemma 3.11 we deal with coordinates (s_{ij}), so we implicitly use the invariance of the first variation.)

Equation (3.15) is a direct consequence of (3.14). Compare this with (3.10). ∎

We have abused notation here. In (3.15), double dots denote the second variation in the variables (s_j), while before we used them to denote the second variation in variables (r_i). Do not ever try to substitute (3.15) into (3.11)!

Lemma 3.17 *The second differential $D_{s^0}^2 K$ of the function* (3.13) *at the point s^0 has signature $(+_1, 0_2, -_{m-3})$. Besides, the associated quadratic form is positive on the vector s^0:*

$$D_{s^0}^2 K(s^0, s^0) > 0.$$

Proof First let us show that

$$\dim \ker D_{s(0)}^2 K = 2.$$

Lemma 3.16 implies that

$$\frac{\partial K}{\partial s_j} = \lambda_j, \text{ hence } \frac{\partial^2 K}{\partial s_j \partial s_k} = \frac{\partial \lambda_j}{\partial s_k},$$

so that the matrix of $D^2 K$ coincides with the Jacobi matrix of the map $s \mapsto \lambda$. Thus

$$\dot{s} \in \ker D^2 K \iff \dot{\lambda} = 0;$$

that is, if and only if the variation $\dot{s}$ of radial edge lengths induces a zero variation of angles between boundary edges. It is not hard to show that at $s = s^0$ such variations form a 2-dimensional space, namely they come from moving the interior vertex while the boundary of the polygon remains fixed.

Let $\mathcal{P}_m$ be the space of all convex spherical m-gons with a marked interior point (*i.e.*, convex warped spherical m-gons with $\kappa = 0$). It is easy to see that $\mathcal{P}_m$ is connected. Every element of $\mathcal{P}_m$ can be viewed as a point s^0 in a space of warped polygons with a fixed boundary metric, thus has a matrix $D^2_{s^0}K$ associated with it. By the previous paragraph, the rank of $D^2_{s^0}K$ is constant over $\mathcal{P}_m$. Since $\mathcal{P}_m$ is connected, and $D^2_{s^0}K$ depends continuously on an element of $\mathcal{P}_m$, the signature is constant as well. Thus in order to compute the signature of $D^2_{s^0}K$ for all warped spherical m-gons with $\kappa = 0$, it suffices to do this for *one* such m-gon.

Let us compute the matrix of $D^2_{s^0}K$. Let $\alpha_{j,j+1}$ be the angle between two consecutive radial edges. Then we have by Lemma A.1

$$\frac{\partial \lambda_j}{\partial s_j} = -\frac{\cot \alpha_{j-1,j} + \cot \alpha_{j,j+1}}{\sin^2 \rho_j},$$

$$\frac{\partial \lambda_j}{\partial s_{j+1}} = \frac{\partial \lambda_{j+1}}{\partial s_j} = \frac{1}{\sin \alpha_{j,j+1} \sin \rho_j \sin \rho_{j+1}}.$$

We compute the signature of $D^2_{s^0}K$ for $\alpha_{j,j+1} = \frac{2\pi}{m}$ and $\rho_j = \rho \in (0, \frac{\pi}{2})$. In this case

$$D^2_{s^0}K = \frac{1}{\sin \frac{2\pi}{m} \sin^2 \rho} \begin{pmatrix} -2\cos \frac{2\pi}{m} & 1 & 0 & \cdots & 1 \\ 1 & -2\cos \frac{2\pi}{m} & 1 & \cdots & 0 \\ 0 & 1 & -2\cos \frac{2\pi}{m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -2\cos \frac{2\pi}{m} \end{pmatrix}.$$

This matrix has spectrum

$$\left\{ \frac{2(\cos \frac{2\pi k}{m} - \cos \frac{2\pi}{m})}{\sin \frac{2\pi}{m} \sin^2 \rho}, k = 1, \ldots, m \right\},$$

which contains exactly one positive eigenvalue (for $k = m$). The first part of Lemma 3.17, concerning the signature, is proved.

The positivity of the form $D^2_{s^0}K$ on the vector s^0 follows from Lemma 3.18. ∎

Let $S \subset \mathbb{S}^2$ be a convex spherical polygon with a distinguished point p_0 in its interior. The polar dual S^* of S is the intersection of all hemispheres centered at vertices of S. We define the *Euclidean polar dual* $S^*_{\mathbb{E}}$ of S as the projection of S^* from the center of $\mathbb{S}^2$ to the plane tangent to $\mathbb{S}^2$ at p_0; see Figure 3. Note that if some of the distances ρ_j are bigger than $\frac{\pi}{2}$, then the point p_0 lies outside the spherical and Euclidean polar duals.
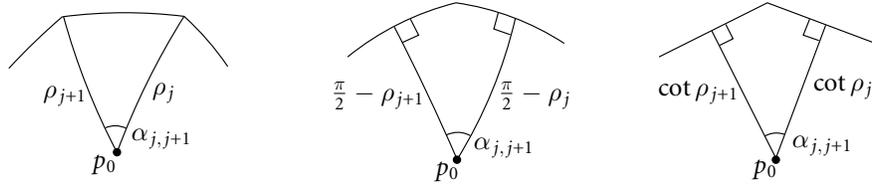
Figure 3: Fragments of S, S^*, and $S^*_\mathbb{E}$.

Lemma 3.18 *Let S be a convex spherical polygon with a marked interior point p_0. Let (s^0_j) be distances from p_0 to the vertices of S. Consider the space of warped spherical polygons obtained from S by varying (s_j) while keeping the boundary edge lengths fixed. Then we have*

$$D^2_{s^0} K(s^0, s^0) = 2\,\mathrm{Area}(S^*_\mathbb{E}),$$

where $S^*_\mathbb{E}$ is the Euclidean polar dual to S; see the definition before the lemma.

Proof Draw the perpendiculars from p_0 to the sides of S^*. They cut the spherical polygon S^* into quadrilaterals, each with a pair of opposite right angles (quadrilaterals may be self-intersecting). The signed lengths of the perpendiculars equal $\frac{\pi}{2} - \rho_j$, with angles $\alpha_{j,j+1}$ between them. The Euclidean polygon $S^*_\mathbb{E}$ has a similar decomposition, with the same angles $\alpha_{j,j+1}$ but with perpendiculars of lengths $\cot \rho_j$; see Figure 3. Equation (2.9) gives us the lengths of the other two sides of a $(j, j+1)$-quadrilateral, so that we can compute its area:

$$\frac{1}{2}\left(\cot \rho_j \frac{\cot \rho_{j+1} - \cot \rho_j \cos \alpha_{j,j+1}}{\sin \alpha_{j,j+1}} + \cot \rho_{j+1} \frac{\cot \rho_j - \cot \rho_{j+1} \cos \alpha_{j,j+1}}{\sin \alpha_{j,j+1}} \right).$$

By summing over j and performing simple transformations, we obtain

$$\mathrm{Area}(S^*_\mathbb{E}) = \frac{1}{2} \sum_{j,k} \frac{\partial \lambda_j}{\partial s_k} s^0_j s^0_k = \frac{1}{2} D^2_{s^0} K(s^0, s^0),$$

and the lemma is proved. ∎

Proof of Lemma 3.15 Suppress the index i and consider $\kappa = \kappa_i$ and $\lambda_j = \lambda_{ij}$ as functions of $(s_j) = (s_{ij})$. Due to invariance of the first differential, upper dots on the left hand side of (3.12) can be viewed as variations with respect to the variables (s_j), and we find ourselves in the setting of the present subsection.

By assumption, we have $\dot\kappa = 0$ for the variation $\dot s$ at the point s^0. Together with equation (3.8) (or (3.14)) this implies that

$$\sum_j s^0_j \dot\lambda_j = -\dot\kappa = 0.$$

On the other hand,

$$\sum_j s_j^0 \dot\lambda_j = \sum_{j,k} s_j^0 \frac{\partial \lambda_j}{\partial s_k} \dot s_k = \sum_{j,k} s_j^0 \frac{\partial^2 K}{\partial s_j \partial s_k} \dot s_k = D_{s^0}^2 K(s^0, \dot s).$$

Thus we have

$$D_{s^0}^2 K(s^0, \dot s) = 0,$$

which means that the vectors s^0 and $\dot s$ are mutually orthogonal with respect to the symmetric bilinear form $D_{s^0}^2 K$. By Lemma 3.17, s^0 is a positive vector for $D_{s^0}^2 K$, and $D_{s^0}^2 K$ is negative semidefinite on the orthogonal complement to s^0. It follows that

$$D_{s^0}^2 K(\dot s, \dot s) \le 0.$$

Since, by (3.15),

$$\sum_j \dot s_j \dot\lambda_j = \ddot K = D_{s^0}^2 K(\dot s, \dot s),$$

the inequality (3.12) in Lemma 3.15 follows. Equality $\sum_j \dot s_j \dot\lambda_j = 0$ means that $\dot s \in \ker D_{s^0}^2 K$. This implies $\dot\lambda_j = 0$ for all j, as established in the first lines of the proof of Lemma 3.17. ∎

3.8 The Case of a Non-simplicial Polyhedron P

First of all, if P has some non-triangular faces, our definition of an isometric infinitesimal deformation (the second half of Definition 3.1) has to be modified. If we only require stability of lengths of edges, then, say, the cube would be considered as infinitesimally flexible. There are two possibilities. Either one requires that each face of P is moved by (q_i) as a rigid plate. Or, one subdivides each face into triangles by non-crossing diagonals and requires stability of lengths also for the diagonals. The latter class of infinitesimal deformations is a priori larger than the former and can be shown to be independent of the choice of subdividing diagonals.

We choose the second possibility. Thus, $P = P(r^0)$ can be viewed as a simplicial polyhedron with some dihedral angles equal to π. As we vary r in a neighborhood U of r^0, these dihedral angles can become less than π. Undaunted by this, we carry out our arguments for non-convex warped polyhedra as well. As we come to warped spherical polygons, we also consider non-strictly convex and "slightly non-convex" ones. Everything goes through, including Lemma 3.17. For the positivity of the form $D_{s^0}^2 K$ on the vector s^0 we need only positivity of the area of the corresponding Euclidean polar dual. This is fulfilled, as the polygon S is a convex (albeit non-strictly) spherical polygon.

Thus a simple modification of the definitions allows us to extend the argument to non-simplicial polyhedra.

4   Duality

4.1   Duality Between Second Derivatives of Vol and HE

There is an apparent duality between constructions and arguments in Sections 2
and 3. This is the same kind of duality as between Cauchy's proof of Theorem 3.4
in [10] and Alexandrov's proof of Theorem 2.1 in [3, Chapter XI]. Cauchy studies a
deformation of the spherical link S_i of a vertex p_i, marks with + or − the j-th vertex
of S_i if the angle (λ_{ij}) increases or decreases during the deformation, and shows that
either at least four sign changes occur as one goes along the boundary of S_i, or no
vertex of S_i is marked, *i.e.*, the dihedral angles at p_i are stable. Then a combinatorial
argument involving the Euler characteristic of a polyhedron shows that there are no
markings at all. Alexandrov considers variations of edge lengths along the boundary
of a face F_i, marks lengthening and shortening edges, and follows Cauchy's argument.

In this paper, new similarities show up. Deformations of Q in Section 2 are gov-
erned by support parameters (h_i), while deformations of P in Section 3 are governed
by radii (r_i). There are functions $\mathrm{Vol}(h)$ and $\mathrm{HE}(r)$ with similar variational proper-
ties, and infinitesimal rigidity is proved by showing that

$$\dim \ker(D^2\,\mathrm{Vol}) = 3 \quad \text{and} \quad \dim \ker(D^2\,\mathrm{HE}) = 3.$$

The following lemma strengthens this analogy by making it quantitative rather than
qualitative.

Lemma 4.1   *Let $P \subset \mathbb{R}^3$ be a convex polyhedron with vertices $p_1, \ldots, p_n$ and such
that 0 lies in the interior of P. Let $Q = P^*$ be the polar dual of P, that is*

$$Q = \{x \in \mathbb{R}^3 \mid \langle x, \nu_i \rangle \le h_i^0, i = 1, \ldots, n\},$$

where

$$\nu_i = \frac{p_i}{\|p_i\|} = \frac{p_i}{r_i^0}, \quad h_i^0 = \frac{1}{r_i^0}.$$

Then we have

(4.1) $$D_{h^0}^2\,\mathrm{Vol} = D_{r^0}^2\,\mathrm{HE}\,.$$

Proof   A direct computation [6, Section 3] shows that

$$\frac{\partial^2\,\mathrm{HE}}{\partial r_i \partial r_j} = \frac{\cot \beta_{ij} + \cot \gamma_{ij}}{\ell_{ij} \sin \rho_{ij} \sin \rho_{ji}} = \frac{\partial^2\,\mathrm{Vol}}{\partial h_i \partial h_j},$$

$$\frac{\partial^2\,\mathrm{HE}}{\partial r_i^2} = -\sum_j \cos \varphi_{ij} \frac{\cot \beta_{ij} + \cot \gamma_{ij}}{\ell_{ij} \sin \rho_{ij} \sin \rho_{ji}} = \frac{\partial^2\,\mathrm{Vol}}{\partial h_i^2}.$$

Here, angles β_{ij} and γ_{ij} were defined in the proof of Lemma 3.11, φ_{ij} is the angle at
0 in the triangle $0p_ip_j$, and ℓ_{ij} and ρ_{ij} are as in Figure 2. The lemma is proved.   ∎

An immediate consequence of Lemma 4.1 is that each of Theorems 2.1 and 3.4 implies the other one (*e.g.,* one can replace Subsections 3.3 to 3.7 by Lemma 4.1). On the other hand, a proof of identity (4.1) by a direct computation is more a question than an answer. There is no obvious reason why (4.1) should hold; for example, the functions $HE(r^0 + x)$ and $Vol(h^0 + x)$ are by no means equal.

The next subsection partially "demystifies" identity (4.1) by pointing out a close relationship between HE and Vol in the context of polar duality in the 3-sphere.

4.2 Duality in Spherical Geometry

Let $P \subset \mathbb{S}^3$ be a convex spherical polyhedron (*i.e.,* the intersection of a finite number of hemispheres that contains no pair of antipodal points). Define the *polar dual* of P as

$$P^* = \{x \in \mathbb{S}^3 \mid \langle x, y \rangle \leq 0 \text{ for all } y \in P\}.$$

(We put $\mathbb{S}^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\}$, and let $\langle \cdot, \cdot \rangle$ be the scalar product in $\mathbb{R}^4$.) Then P^* is also a finite intersection of hemispheres (take one hemisphere for every vertex of P). We assume that $\dim P = 3$; then P^* contains no pair of antipodal points. Let $p_i p_j$ be an edge of P. In P^* there is a dual edge lying in the intersection of hyperspheres polar to p_i and p_j. Denote by ℓ_{ij} the length of the edge $p_i p_j$, and by ℓ_{ij}^* the length of the dual edge.

Theorem 4.2 (McMullen [33]) *For every* 3*-dimensional spherical polyhedron P and its polar dual P* the following identity holds:*

$$(4.2) \qquad Vol(P) + \frac{1}{2} \sum_{\{i,j\}} \ell_{ij} \ell_{ij}^* + Vol(P^*) = \pi^2.$$

The proof is based on Lemma 4.3.

We have $\ell_{ij} = \lambda_{ij}^*$ and $\ell_{ij}^* = \lambda_{ij}$, where λ_{ij} is the exterior dihedral angle of P at the edge $p_i p_j$, and λ_{ij}^* is the exterior dihedral angle of P^* at the dual edge. The first equation holds because the distance between p_i and p_j is equal to the exterior dihedral angle between the hyperspheres polar to p_i and p_j. The second holds, since $(P^*)^* = P$ and $p_i p_j$ is dual of its dual. It follows that the summand in the middle of (4.2) is the discrete total mean curvature of ∂P and at the same time the discrete total mean curvature of ∂P^*.

More generally, let P be a convex polyhedron in the d-sphere $\mathbb{S}^d$ for arbitrary $d \geq 1$, and let F be a face of P. We define the dual face of P^* as

$$F^\perp = \{x \in P^* \mid \langle x, y \rangle = 0 \text{ for all } y \in F\}.$$

We have $\dim F^\perp + \dim F = d - 1$. Every polyhedron is considered to be its own face and also to have $\varnothing$ as a (-1)-dimensional face. Then clearly

$$\varnothing^\perp = P^*, \quad P^\perp = \varnothing.$$

Define the norm of a k-dimensional face as its k-volume divided by the k-volume of the k-dimensional sphere:

$$\|F\| = \frac{\mathrm{Vol}_k(F)}{\mathrm{Vol}_k(\mathbb{S}^k)}.$$

As $\mathrm{Vol}_0(\mathbb{S}^0) = 2$, the norm of a vertex equals $\frac{1}{2}$; we also put $\|\varnothing\| = 1$.

Lemma 4.3 *For every convex d-dimensional polyhedron $P \subset \mathbb{S}^d$ the following identities hold:*

(4.3) $$\sum_F \|F\| \cdot \|F^\perp\| = 1,$$

(4.4) $$\sum_F (-1)^{\dim F} \|F\| \cdot \|F^\perp\| = 0.$$

Here, the summation extends over all faces of P, including $\varnothing$ and P itself.

Proof Denote by $F \times F^\perp$ the convex hull of $F \cup F^\perp$ in $\mathbb{S}^d$. For every F, this is a d-dimensional spherical polyhedron, and we have

$$\|F \times F^\perp\| = \|F\| \cdot \|F^\perp\|,$$

since $\mathrm{span}(F)$ and $\mathrm{span}(F^\perp)$ are mutually orthogonal subspaces of $\mathbb{R}^{d+1}$. It is not hard to see that the sphere decomposes as a union of polyhedra with disjoint interiors:

$$\mathbb{S}^d = \bigcup_F (F \times F^\perp).$$

Thus we have

$$\mathrm{Vol}_d(\mathbb{S}^d) = \sum_F \mathrm{Vol}_d(F \times F^\perp),$$

which implies equation (4.3).

Equation (4.4) is proved in a similar way, by replacing P^* with $-P^*$. The family of polyhedra $F \times -F^\perp$, each counted with multiplicity $(-1)^{\dim F}$, forms a covering with total multiplicity 0 over every point of $\mathbb{S}^d$. Therefore

$$\sum_F (-1)^{\dim F} \mathrm{Vol}_d(F \times F^\perp) = 0,$$

and equation (4.4) follows. Figure 4 illustrates the case $d = 1$. ∎

Proof of Theorem 4.2 Since $\mathrm{Vol}_3(\mathbb{S}^3) = 2\pi^2$ and $\mathrm{Vol}_1(\mathbb{S}^1) = 2\pi$, for $d = 3$ formulas (4.3) and (4.4) yield

$$\mathrm{Vol}(P) + \frac{\pi}{4}\mathrm{Area}(\partial P) + \frac{1}{2}\sum_{\{i,j\}} \ell_{ij}\ell_{ij}^* + \frac{\pi}{4}\mathrm{Area}(\partial P^*) + \mathrm{Vol}(P^*) = 2\pi^2,$$

$$\mathrm{Vol}(P) - \frac{\pi}{4}\mathrm{Area}(\partial P) + \frac{1}{2}\sum_{\{i,j\}} \ell_{ij}\ell_{ij}^* - \frac{\pi}{4}\mathrm{Area}(\partial P^*) + \mathrm{Vol}(P^*) = 0.$$
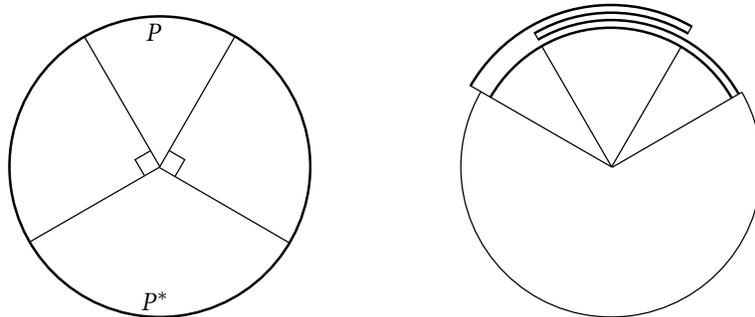
Figure 4: To the proof of Lemma 4.3.

By adding and subtracting these two formulas, we obtain

$$\text{(4.5)} \qquad \text{Vol}(P) + \frac{1}{2} \sum_{\{i,j\}} \ell_{ij} \ell_{ij}^* + \text{Vol}(P^*) = \pi^2,$$

$$\text{Area}(\partial P) + \text{Area}(\partial P^*) = 4\pi$$

Theorem 4.2 is proved. ∎

Remark 4.4 Equation (4.5) is the discrete Gauss–Bonnet theorem for surfaces in $\mathbb{S}^3$. Equation (4.3) is the Steiner formula for the volume of the $\frac{\pi}{2}$-neighborhood of P. Equation (4.2) is an analog of the Weyl tube formula.

Lemma 4.3 was proved by McMullen in [33]. Section 4 of [33] contains also references to smooth analogs of formulas (4.2) and (4.5). The main reference is Herglotz [22] (see also [27, Subsection 6.4]), who mentions that Fenchel has proved formulas (4.2) and (4.5) by induction on dimension. There seems to be no written account of Fenchel's proof.

Remark 4.5 Points of P^* are poles of great spheres disjoint from the interior of P. This gives Theorem 4.2 an integral-geometric interpretation: a random great sphere intersects P with the probability

$$\frac{1}{\pi^2} \left(\text{Vol}(P) + \frac{1}{2} \sum_{\{i,j\}} \ell_{ij} \lambda_{ij} \right).$$

See also [43, Chapter 17, §5, Note 1].

Remark 4.6 Milnor on the last page of [35] suggests another way of proving equation (4.2), by deforming P to a point and integrating the Schläfli formula. It would be interesting to see whether the Schläfli formula can be derived from (4.2).

4.3   Gauss and Metric Rigidity for Convex Spherical Polyhedra

The following are the spherical analogs of Theorems 2.1 and 3.4.

Theorem 4.7   *Let $Q \subset \mathbb{S}^3$ be a simple convex polyhedron, and let (Q_t) be a deformation of Q that keeps all dihedral angles constant in the first order. Then Q_t is congruent to Q in the first order.*

This theorem extends to non-simple convex spherical polyhedra, if one requires stability of dihedral angles between the planes of any two faces having a vertex in common. A less restrictive generalization allows change of combinatorics and appearance of negative edge lengths; see Subsection 2.6.

To substantiate the analogy between Theorems 2.1 and 4.7, note that directions of face normals of a Euclidean polyhedron determine its dihedral angles, and that dihedral angles of a simple polyhedron determine angles of its faces and thus, for a spherical polyhedron, face areas.

Theorem 4.8   *Let $P \subset \mathbb{S}^3$ be a simplicial convex spherical polyhedron, and let (P_t) be a deformation of P that keeps all edge lengths constant in the first order. Then P_t is congruent to P in the first order.*

A generalization can be stated for non-simplicial polyhedra by subdividing non-triangular faces; see Subsection 3.8.

Note that Theorems 4.7 and 4.8 are equivalent: just put $Q_t = P_t^*$.

One can prove Theorem 4.8 following the arguments used in Section 3 to prove its Euclidean analog. Warped spherical polyhedra are defined in an obvious way. In place of the Hilbert–Einstein functional consider the functional

$$S(P) := 2\operatorname{Vol}(P) + \sum_i r_i \kappa_i + \sum_{\{i,j\}} \ell_{ij} \lambda_{ij},$$

where $P = P(r)$ is a warped polyhedron with radii (r_i). Schläfli's formula implies that

$$\frac{\partial S}{\partial r_i} = \kappa_i.$$

The functional S can be described as a discrete gravity action with non-zero cosmological constant, *cf.* [27, Subsection 6.5].

When we want to copy the approach of Section 2 to prove Theorem 4.7, it undergoes more substantial changes and becomes closer to the approach to the dual theorem. Let q_0 be an interior point of Q. By dropping perpendiculars $q_0 q_i$ to the faces and then perpendiculars $q_i q_{ij}$ to the edges of Q, we cut Q into "bricks". A brick is a polyhedron combinatorially equivalent to the cube and with six right angles at the edges not incident to the vertices q_0 and q_{ijk}. A brick is uniquely determined by lengths h_i, h_j, h_k of edges adjacent to q_0 and by dihedral angles at the edges adjacent to q_{ijk} (just choose q_0 as a unique point at distances h_i, h_j, h_k from the planes of a given trihedral angle). We vary "support parameters" (h_i) of a brick decomposition

of Q while leaving dihedral angles of Q constant. As a result, singularities around the edges $q_0 q_i$ appear. Denote by κ_i the curvatures at $q_0 q_i$ and put

$$S^*(Q) := 2\operatorname{Vol}(Q) + \sum_i h_i \kappa_i.$$

Then Schläfli's formula again implies that $\frac{\partial S^*}{\partial h_i} = \kappa_i$. As deformations of Q that keep dihedral angles constant can be identified with vectors $\dot{h}$ such that $\dot{\kappa} = 0$, Theorem 4.7 reduces to a statement about the kernel of $D^2 S^*$.

Lemma 4.9 *Let P be a warped spherical polyhedron, and P^* be its polar dual. Then we have*

$$S(P) + S^*(P^*) = 2\pi^2.$$

Proof Let $\mathbb{S}^3_\kappa$ be a spherical cone manifold obtained by taking a warped product of $[0, \pi]$ with the spherical link S_0 of the warped polyhedron P. (The manifold $\mathbb{S}^3_\kappa$ is a 3-sphere with n singular meridians of curvatures (κ_i); meridians are arranged as the cone singularities in the link of the point $p_0 \in P$.) Both warped polyhedra P and P^* can be embedded in $\mathbb{S}^3_\kappa$.

Similarly to Lemma 4.3 and Theorem 4.2, we can show that

$$\operatorname{Vol}(P) + \frac{1}{2} \sum_{\{i,j\}} \ell_{ij} \ell^*_{ij} + \operatorname{Vol}(P^*) = \frac{\pi}{2} \left(2\pi - \sum_i \kappa_i \right),$$

the right-hand side being half the volume of $\mathbb{S}^3_\kappa$. On the other hand, radii of P and heights of P^* are related by $h_i = \pi - r_i$. Therefore,

$$S(P) + S^*(P^*) = 2\operatorname{Vol}(P) + \sum_{\{i,j\}} \ell_{ij} \ell^*_{ij} + 2\operatorname{Vol}(P^*) + \sum_i (r_i + h_i)\kappa_i$$

$$= \pi(2\pi - \sum_i \kappa_i) + \pi \sum_i \kappa_i = 2\pi^2,$$

and the lemma is proved. ∎

Thus, in the spherical case the duality between Gauss and metric rigidity of convex polyhedra is perfect, and their proofs that use the second derivatives of S^* and S are simply the same.

4.4 Duality in the Hyperbolic-de Sitter Geometry

Consider the hyperboloid model of the hyperbolic space

$$\mathbb{H}^3 = \{x \in \mathbb{R}^4 \mid \|x\|_{3,1} = -1, x_0 > 0\},$$

where $\|x\|_{3,1} = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ is the Minkowski scalar product. The orthogonal complement $x^\perp \subset \mathbb{R}^4$ does not intersect $\mathbb{H}^3$, but it intersects the one-sheeted hyperboloid

$$\mathrm{d}\mathbb{S}^3 = \{x \in \mathbb{R}^4 \mid \|x\|_{3,1} = 1\}$$

that is a model of the *de Sitter space*. For every convex polyhedron $P \subset \mathbb{H}^3$, define its polar dual as

$$P^* = \{x \in d\mathbb{S}^3 \cup \mathbb{H}^3_- \mid \langle x, y \rangle_{3,1} \leq 0\},$$

where $\mathbb{H}^3_-$ is the antipodal copy of $\mathbb{H}^3$. In particular, ∂P^* is a convex polyhedral surface in $d\mathbb{S}^3$. Hyperbolic-de Sitter duality was used by Rivin [41] to study geometry of hyperbolic polyhedra.

The Minkowski scalar product induces a semi-Riemannian metric on $d\mathbb{S}^3$, and the boundary of P^* is space-like with respect to this metric. Between lengths and angles in polar objects in $\mathbb{H}^3$ and $d\mathbb{S}^3$ there is a correspondence similar to that in $\mathbb{S}^3$; see Subsection 4.2. As a consequence, Gauss (respectively, metric) rigidity of a convex polyhedron in $\mathbb{H}^3$ is equivalent to metric (respectively, Gauss) rigidity of its polar dual in $d\mathbb{S}^3$.

A variational approach to the metric rigidity of convex hyperbolic polyhedra is based on the discrete gravity action

$$S(P) := -2 \operatorname{Vol}(P) + \sum_i r_i \kappa_i + \sum_{\{i,j\}} \ell_{ij} \lambda_{ij},$$

where P is a warped hyperbolic polyhedron. Similarly to the previous subsection, we have

$$S(P) + S^*(P^*) = 0,$$

where S^* is the gravity action without the boundary term. This is implied by the following analog of Theorem 4.2.

Theorem 4.10   *For every convex polyhedron $P \subset \mathbb{H}^3$ and its polar dual $P^* \subset d\mathbb{S}^3 \cup \mathbb{H}^3_-$ the following identity holds:*

$$(4.6) \qquad \operatorname{Vol}(P) - \frac{1}{2} \sum_{\{i,j\}} \ell_{ij} \ell_{ij}^* + \operatorname{Vol}(P^*) = 0.$$

Note that P^* is the union of a whole hyperbolic space $\mathbb{H}^3_-$ and an infinite end of $d\mathbb{S}^3$. However, there is a consistent way to define a finite measure $\operatorname{Vol}(P^*)$ [7, 13].

Similarly to Milnor's approach in the spherical case, Theorem 4.10 is proved in [50] by integrating the Schläfli formula.

Remark 4.11   Formula (4.6) has the following integral-geometric interpretation: the (motion-invariant and appropriately normalized) measure of the set of all planes that intersect a convex body $P \subset \mathbb{H}^3$ equals

$$\frac{1}{2} \sum_{\{i,j\}} \ell_{ij} \lambda_{ij} - \operatorname{Vol}(P).$$

In particular, this quantity is monotone under inclusion and always positive.

Integral non-euclidean geometry is dealt with in [43, Chapter 17]. A proof of Theorems 4.2 and 4.10 indicated in [43, Chapter 17, §5, Note 1] is by first obtaining a similar formula in the smooth case and then going to the limit $\varepsilon \to 0$ for the boundary of ε-neighborhood of P.

4.5 Shearing vs. Bending

There is another kind of duality between Theorems 2.1 and 3.4, relating metric rigidity of P with Gauss rigidity of P, rather than with that of P^*.

An infinitesimal isometric deformation of P can be described by assigning to each face of P an infinitesimal isometry of $\mathbb{R}^3$, so that these isometries agree on the edges. (Faces move as rigid plates joined by hinges.) An infinitesimal isometry of $\mathbb{R}^3$ is an infinitesimal screw motion, that is, a vector field

$$\xi(x) = \eta \times x + \tau, \quad \eta, \tau \in \mathbb{R}^3.$$

Let $\xi_i = (\eta_i, \tau_i)$ be the infinitesimal isometry associated with a face F_i. Let $F_{ij} = F_i \cap F_j$ be an edge of P. The condition $\xi_i|_{F_{ij}} = \xi_j|_{F_{ij}}$ is equivalent to

$$(\eta_i - \eta_j) \times x = \tau_j - \tau_i \quad \text{for all } x \in F_{ij},$$

which implies that the vector $\eta_i - \eta_j$ is parallel to F_{ij}:

$$(4.7) \qquad \eta_i - \eta_j \parallel F_{ij} \quad \text{for all edges } F_{ij}.$$

Now, instead of rotating each face F_i according to the vector η_i, let us translate the plane of F_i by η_i (consider a plane as a set of points rather than as a geometric figure, so that the horizontal component of η_i cannot be neglected). By (4.7), the two translations of $\mathrm{span}(F_{ij})$ differ by a vector parallel to F_{ij}. This implies that the boundary of F_i is translated together with F_i, except that some new edges may appear. Namely, if F_i and F_k had only a vertex in common, the translations can split this vertex and create an edge between F_i and F_k. If there is no vertex splitting at all, then the polyhedron P is translated as a rigid body.

Even if vertex splitting happens, the area of each face changes by $o(t)$, if we do translations by $t\eta_i$. It follows that putting $\dot{h}_i = \langle \eta_i, \nu_i \rangle$ we obtain $\dot{A}_i = 0$ for all i, in the notation of Subsection 2.1.

Thus, to every isometric infinitesimal deformation of P there corresponds an infinitesimal Gauss image preserving deformation of P. The correspondence can be inverted by solving the above equations for (t_i).

This observation was made by Weyl in [59], inspired by a work of Blaschke [5] where a similar correspondence for smooth surfaces was indicated. See also [3, Chapter XI, §3]. A similar correspondence is known for ideal hyperbolic polyhedra, where it is sometimes described as the duality between shearing and bending. Indeed, with (d_i) viewed as rotations, the difference $d_i - d_j$ represents the change of the dihedral angle between F_i and F_j, whereas if (d_i) are viewed as translations, then $d_i - d_j$ is the shift of F_i along F_{ij} relative to F_j.

4.6 Statics and Polarity

Infinitesimal rigidity of a polyhedron is equivalent to static rigidity of its 1-skeleton (with diagonals of non-triangular faces added); see [25, 42]. Basically, statics deals

with duals of vector spaces of isometric, respectively trivial, infinitesimal deformations.

In [60, Theorems 3.1 and 3.2], Whiteley establishes a correspondence between statics of a 3-dimensional polyhedron and its polar dual. The polarity can be taken with respect to any quadric. This reflects the projective nature of static (and hence infinitesimal) rigidity; see the next subsection.

4.7 Projective Invariance and Infinitesimal Pogorelov Maps

Infinitesimal rigidity of a bar-and-joint framework in $\mathbb{R}^d$ is invariant under projective transformations. This fact follows from a projective formulation of statics, see [14, 25].

Projective invariance provides a link between infinitesimal rigidity in $\mathbb{R}^d$, $\mathbb{S}^d$, and $\mathbb{H}^d$: an infinitesimally rigid framework in $\mathbb{R}^d$ remains infinitesimally rigid when viewed as a framework in a projective model of $\mathbb{S}^d$ or $\mathbb{H}^d$. The correspondences arising between infinitesimal isometric deformations are called *infinitesimal Pogorelov maps*; see [25, 39].

5 Miscellaneous Remarks

5.1 The Regge Action and the Steiner Formula

The sum $\sum_i r_i \kappa_i$ appearing in the definition of the discrete Hilbert–Einstein functional for warped polyhedra makes sense for an arbitrary compact closed manifold glued from Euclidean simplices. It was introduced in [40] and is sometimes called the Regge action. It was shown in [12] that $\sum_i r_i \kappa_i$ converges to a constant times the total scalar curvature if a sequence of piecewise Euclidean manifolds converges in some good sense to a smooth manifold.

The sum $\frac{1}{2} \sum_{\{i,j\}} \ell_{ij} \lambda_{ij}$ is a discrete analog of the total mean curvature. It appears as a coefficient at ε^2 in the formula for the volume of an ε-neighborhood of a convex polyhedron (as the total mean curvature does for convex bodies with smooth boundary). This was noticed by Jakob Steiner [49]. The convergence of $\sum_{\{i,j\}} \ell_{ij} \lambda_{ij}$ to a constant times the total mean curvature follows from the continuity of mixed volumes with respect to the Hausdorff distance; see [48].

5.2 Mixed Volumes and the Alexandrov–Fenchel Inequality

Our constructions in Section 2 are closely related to the theory of mixed volumes; see *e.g.,* [48]. In particular, the proof of Theorem 2.1 is copied from (a part of) the proof of Alexandrov–Fenchel inequalities [1, 4, 48]. The relation between mixed volumes and derivatives of the volume is apparent from the formula

$$\mathrm{Vol}(h + tk) = \mathrm{Vol}(h) + 3t\,\mathrm{Vol}(h, h, k) + 3t^2\,\mathrm{Vol}(h, k, k) + t^3\,\mathrm{Vol}(k),$$

which can serve as the definition of mixed volumes $\mathrm{Vol}(h, h, k)$ and $\mathrm{Vol}(h, k, k)$. (One has to assume that polyhedra $Q(h)$ and $Q(k)$ are combinatorially isomorphic in order that addition of support parameters h and k correspond to Minkowski addition

of polyhedra.) Right-hand sides of the formulas in Remark 2.12 give alternative expressions for mixed volumes.

To prove Theorem 2.1, it is sufficient to show that the symmetric bilinear form $D^2 \mathrm{Vol}$ has corank 3. The proof of the Alexandrov–Fenchel inequalities goes further. It establishes that $D^2 \mathrm{Vol}$ has signature $(+_1, 0_3, -_{n-4})$, and in general shows by induction that the signature is $(+_1, 0_d, -_{n-d-1})$ for d-dimensional polyhedra with n facets.

5.3 Existence Theorems

Infinitesimal rigidity can sometimes be used to prove existence theorems. Assume that an object is infinitesimally rigid with respect to some parameters. If the space of objects and the space of parameters have equal dimension, then the inverse function theorem provides us with local existence and uniqueness of an object with given parameters. In other words, the map {object} $\mapsto$ {parameters} is a local homeomorphism. A local homeomorphism with good topological properties is a global homeomorphism, which implies that any given set of parameters defines a unique object.

This method is used in [3] to prove the following two theorems.

Theorem 5.1 (Minkowski)   *Let $\nu_1, \nu_2, \ldots, \nu_n \in \mathbb{R}^3$ be unit vectors spanning $\mathbb{R}^3$, and let $C_1, C_2, \ldots, C_n$ be positive numbers such that*

$$\sum_i C_i \nu_i = 0.$$

Then there is a unique convex polyhedron in $\mathbb{R}^3$ with outer face normals (ν_i) and respective face areas (C_i).

Theorem 5.2 (Alexandrov)   *Let g be a Euclidean cone-metric on $\mathbb{S}^2$ (every such metric can be obtained by gluing a set of Euclidean triangles). Assume that the angles at all cone points are less than 2π. Then there is a unique convex polyhedron in $\mathbb{R}^3$ such that g is an intrinsic metric of its boundary.*

Note that in both theorems the combinatorial structure (which pairs of faces are adjacent, which pairs of cone points are joined by edges) is not given in advance and is practically impossible to determine without finding the polyhedron in question.

Properties of the functionals Vol and HE suggest a variational approach to both theorems. Indeed,

$$\frac{\partial}{\partial h_i}\left(\mathrm{Vol}(h) - \sum_i h_i C_i \right) = A_i(h) - C_i$$

implies that the polyhedron in the Minkowski theorem is a critical point of the function $\mathrm{Vol}(h) - \sum_i h_i C_i$ on the set of polyhedra with outer face normals (ν_i). Similarly,

$$\frac{\partial \, \mathrm{HE}}{\partial r_i} = \kappa_i$$

implies that the polyhedron in the Alexandrov theorem is a critical point of HE on the set of warped polyhedra. As stated in Subsection 5.2, the second differential D^2 Vol has only one positive eigenvalue. By a happy coincidence, Vol is concave on the hyperplane $\{\sum_i h_i C_i = 1\}$. At the maximum point (whose existence can be shown by a simple trick) face areas A_i are proportional to C_i, so that a scaling yields the desired polyhedron. This proof is due to Minkowski himself, [36], see also [3, Chapter VII, §2].

The situation is more complicated with HE and the Alexandrov theorem. The discrete Hilbert–Einstein functional on warped polyhedra is neither concave nor convex, moreover the signature of its second differential is non-constant. Nevertheless, similarly to Subsection 3.7 one can show that $\ker D^2 \text{HE} = \{0\}$, provided that $\kappa_i > 0$ and the areas of Euclidean polar duals of the vertex links are positive. This allows us to apply the inverse function theorem to the map $r \mapsto \kappa$ and to find the desired polyhedron by constructing a family of warped polyhedra with $\kappa \to 0$. This proof is given in [6].

Y. Volkov, a student of Alexandrov, studied the discrete Hilbert–Einstein functional with the aim of finding a variational proof of Alexandrov's theorem similar to that of Minkowski's theorem. His proof in [53, 56]) does not use the function HE, but is similar in spirit. The polyhedron is found by minimizing the sum of radii (r_i) over all warped polyhedra (this time with apex at a boundary vertex) with negative (κ_i). See also [55], which is reprinted in the Appendix to [3], where Volkov studies HE in order to give some a priori bounds for the embedding problem.

In some theorems of Alexandrov or Minkowski type, the functional HE, respectively Vol, happens to be concave. This is the case with the Alexandrov convex cap theorem [24] and with the Alexandrov and Minkowski-type theorems for convex hyperbolic cusps [17, 18]. However, in all these cases proofs of existence of a maximum point are quite difficult.

5.4   Related Work

Our proof of infinitesimal rigidity of convex polyhedra in Section 3 is related to works of Pogorelov and Volkov [38, 54] and of Schlenker [46]. In [46, Section 3] an alternative proof of Lemma 3.17 can be found.

Duality between metric rigidity of P and Gauss rigidity of P^* involving theory of mixed volumes appears in P. Filliman's work [19]. Lee generalized this duality [30] to higher dimensions, relating it with McMullen's theory of weights on polytopes [34]. See also the work of Tay, White, and Whiteley [51].

The approach of Section 3 was extended by Schlenker to star-shaped polyhedra with vertices in convex position [47]. A further development of this technique is given in [28]. There the signature of D^2 HE for an arbitrary triangulation of a convex polyhedron is determined.

5.5   Directions for Future Research

Problem 1   Can equation (4.1) be obtained as a limiting case of Theorem 4.2?

Problem 2   Give a proof of Theorem 4.10 similar to McMullen's proof of Theorem 4.2. All prerequisites are probably contained in [13]. See also [22].

Lemma 4.1 relates metric rigidity of P and Gauss rigidity of P^*; Subsection 4.6 relates metric rigidity of P and P^*, and Subsection 4.5 relates Gauss and metric rigidity of P^*. In the smooth case, this forms a part of Darboux's wreath [27, Subsection 5.3]. Sauer [44] found a discrete analog of Darboux's wreath for polyhedral surfaces with quadrilateral faces and four-valent vertices.

Problem 3   Is there a Darboux's wreath for polyhedral surfaces of arbitrary combinatorics?

Problems 4–7 are discussed in more detail in Subsections 1.3–1.5.

Problem 4   Reprove the Calabi–Weil rigidity of compact closed hyperbolic 3-manifolds by showing

$$\dim \ker\left(\frac{\partial \kappa_{ij}}{\partial r_{kl}}\right) = 3n,$$

where n is the number of vertices of a triangulation of a hyperbolic manifold.

Problem 5   In a similar way, prove the infinitesimal rigidity of compact hyperbolic 3-manifolds with convex polyhedral boundary.

Problem 6   In a similar way, reprove the infinitesimal rigidity of compact hyperbolic cone-manifold with all cone angles less than 2π.

Problem 7   Reprove Cheeger's vanishing theorem with a discrete Bochner technique.

The next problem generalizes Problems 4 and 6 to higher dimensions.

Problem 8   Define discrete Einstein manifolds in dimensions greater than 3 and prove their infinitesimal rigidity under suitable assumptions.

To approach this problem, one can start from the following consideration. Gauss infinitesimal rigidity holds for convex polyhedra in all dimensions. Convex polyhedra of higher dimensions are also infinitesimally rigid, in fact they are "too rigid". An alternative extension of the infinitesimal rigidity theorem to higher dimensions could be infinitesimal rigidity in the class of discrete Einstein metrics with a given restriction to the boundary. To define discrete Einstein metrics for a star-like triangulation of P, consider infinitesimal deformations $\dot{h} = u$ of P^* that preserve volumes of facets in the first order. Putting $\dot{r} = u$ should define an infinitesimal Einstein deformation of P.

Problem 9   Define infinitesimal Einstein deformations of warped convex polyhedra in dimension $d > 3$ and prove Einstein infinitesimal rigidity of convex polyhedra.

A Some Trigonometry

The first two lemmas concern a spherical triangle with side lengths a, b, c, and values α, β, γ of the respective opposite angles.

Lemma A.1   *The partial derivatives of the angles as functions of side lengths are:*

$$\frac{\partial \alpha}{\partial a} = \frac{1}{\sin b \sin \gamma} = \frac{1}{\sin c \sin \beta}, \quad \frac{\partial \alpha}{\partial b} = -\frac{\cot \gamma}{\sin b}.$$

Proof The lemma can be proved with straightforward calculation using spherical cosine and sine laws.   ∎
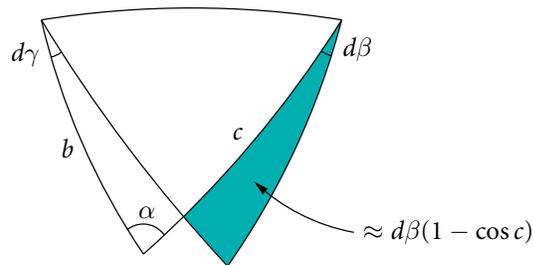
Figure 5: Variation of the area of a spherical triangle when one of the side lengths is preserved.

Lemma A.2   *Let the side lengths b and c change with velocities $\dot{b}$ and $\dot{c}$, while the length a remains constant. Then the corresponding variations of the angles satisfy the equation*

(A.1) $$\dot{\alpha} + \dot{\beta} \cos c + \dot{\gamma} \cos b = 0.$$

Proof This can be proved by tedious computations using Lemma A.1. We give an alternative geometric argument.

Denote by area the area of the triangle. As area $= \alpha + \beta + \gamma - \pi$, equation (A.1) is equivalent to

$$\dot{\text{area}} = \dot{\beta}(1 - \cos c) + \dot{\gamma}(1 - \cos b).$$

The latter is proved in Figure 5 (recall that the area of a spherical cap of radius c equals $2\pi(1 - \cos c)$).   ∎

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