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ON A CLASS OF PROJECTIVE MODULES OVER CENTRAL SEPARABLE ALGEBRAS

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In [5], DeMeyer extended one consequence of Wedderburn's theorem; that is, if R is a commutative ring with a finite number of maximal ideals (semi-local) and with no idempotents except 0 and 1 or if R is the ring of polynomials in one variable over a perfect field, then there is a unique (up to isomorphism) indecomposable finitely generated projective module over a central separable R-algebra A. Also, for this ring R, DeMeyer proved a structure theorem for a central separable R-algebra A. The purpose of this paper is to extend the above results of DeMeyer by using the Pierce's representation of a commutative ring with identity.

Throughout this paper, we assume that R is a commutative ring with identity, that all modules are left and unitary modules over a ring or an algebra. Let us recall some notations used in [6] and [7]. Let $B(R)$ denote the Boolean algebra of idempotents of R with addition $e \dotplus f = e+f-ef$ and multiplication $e * f = ef$ for any elements e and f in $B(R)$. Let Spec $B(R)$ be the set of maximal ideals of $B(R)$ and let U_e be the subset of Spec $B(R)$ such that $U_e = \{x$ with e in x and e fixed in $B(R)\}$. Then Spec $B(R)$ is a topological space with the basic open sets U_e. Furthermore, it is a compact, totally disconnected and Hausdorff topological space. Finally, let R_x denote R/xR for each x in Spec $B(R)$ and M_x denote $R_x \otimes_R M$ for a R-module M. A sheaf is defined whose base space is Spec $B(R)$ and whose stalks are R_x. Then the ring R is represented as a global cross section of this sheaf. We will employ the facts which were proved by D. Zelinsky and O. Villamayor in [7, §2]. We are interested in a class of rings R such that R_p is a semi-local ring for each p in Spec $B(R)$ (for example, a regular ring R in the sense of Von Neumann, see the remark in [4, p. 625]), or a polynomial ring $F[X]$ in one variable X with $B(R)=B(F)$ and F_p a field for each p in Spec $B(R)$ (for example, $F[X]$ with F a Boolean ring). We begin with extending Theorem 2 in [5].

LEMMA 1. *Let M and N be any two finitely generated projective and indecomposable modules over a central separable R-algebra A. If R is a polynomial ring $F[X]$ in one variable X over a commutative Noetherian ring F with 1 such that F_p is a perfect field and $B(R)=B(F)$, then the following statements are equivalent*: (a) $M \cong N$, (b) $M \cong N$ *as R-modules*, (c) $M_p \neq 0$ *and* $N_p \neq 0$ *for some p in* Spec $B(R)$.

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Proof. (a) ⇒ (b) is clear. For (a) ⇒ (c), suppose to the contrary that $M_p=0$ and $N_p=0$ for all p in Spec $B(R)$. Then $M=0$ and $N=0$ [7, (2.11)]. But M and N are always assumed nonzero then there is p in Spec $B(R)$ such that $M_p\neq 0$ and $N_p\neq 0$. For (c) ⇒ (a), since $R=F[X]$ is a polynomial ring in one variable X such that F_p is a perfect field and $B(R)=B(F)$, $R_p=F_p[X]$; and so there is only one isomorphic class of finitely generated projective and indecomposable A_p-modules [5, Theorem 2]. Assume the number of indecomposable submodules of M_p is less than that of N_p. We then have a homomorphism f from N_p onto M_p. The modules M and N are finitely generated and projective A-modules so f is lifted to a homomorphism f' from N into M. This gives $M=f'(N)+pM$ and so $(pM)_p=0$. But R is Noetherian and M is finitely generated then pM is finitely generated. Hence there is a neighborhood of p, U, such that $(pM)_q=0$ for each q in U. Let e be an idempotent of R with $1-e$ in q for all q in U. Then $U=\text{Spec } B(Re)$ and $e(pM)=0$. So, $eM=f'(eN)$. Thus the sequence is exact and splits, $0\to \ker(f')\to eN\to eM\to 0$. This implies that $eN\cong eM\oplus \ker(f')$. Noting that M and N are indecomposable A-modules we have $N=eN\cong eM=M$. (b) ⇒ (a) holds true by similar arguments.

With some minor modifications it is easy to extend Theorem 1 in [5].

LEMMA 2. *Let M and N be any two finitely generated projective and indecomposable modules over a central separable R-algebra A. If R is a commutative Noetherian ring with R_p a semi-local ring for each p in* Spec $B(R)$, *then the following statements are equivalent*: (a) $M\cong N$, (b) $M\cong N$ *as R-modules*, (c) $M_p\neq 0$ *and* $N_p\neq 0$ *for some p in* Spec $B(R)$.

A classification of all finitely generated projective and indecomposable modules over a central separable algebra can be obtained. From now on we assume that for each p in Spec $B(R)$ there is a finitely generated projective and indecomposable R-module M with $M_p\neq 0$.

THEOREM. *If R is given by Lemma* 1 *or* 2, *then the number of isomorphic classes of finitely generated projective and indecomposable modules over a central separable R-algebra A is finite.*

Proof. First we claim that all finitely generated and projective eR-modules are free for some idempotent e of R. Let M be any finitely generated projective and indecomposable R-module with $M_p\neq 0$ for some p in Spec $B(R)$. Since M_p is a free R_p-module, $M_p\cong\oplus\sum_{i=1}^{n}(R_p)_i$ for some integer n. But then M and $\oplus\sum_{i=1}^{n}(R)_i$ are finitely generated and projective R-modules with $M_p\cong(\oplus\sum_{i=1}^{n}(R)_i)_p$. By the proof of Lemma 1 we have an idempotent e of R and a neighborhood of p, U_e, such that $eM\cong e(\oplus\sum_{i=1}^{n}(R)_i)$. The module M is indecomposable so $n=1$. Thus $M=eM\cong eR$. On the other hand, let N be any finitely generated projective and indecomposable R-module with $N_q\neq 0$ for some q in U_e. Then $M\cong eR\cong N$. This follows because $M_q\cong(eR)_q\neq 0(U_e=\text{Spec } B(eR))$. Therefore all finitely generated and projective

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eR-modules are free. Let p vary over Spec $B(R)$ and cover Spec $B(R)$ with such U_e. Noting that Spec $B(R)$ is compact we have a finite subcover of $U_e, \{U_{e_1}, U_{e_2}, \ldots, U_{e_k}\}$, such that $R \cong \oplus \sum_{i=1}^{k} e_i R$ and all finitely generated and projective e_iR-modules are free for each i. Consequently, there is exactly one isomorphic class of finitely generated projective and indecomposable e_iA-modules for each i by Lemmas 1 and 2 and so the number of isomorphic classes of finitely generated projective and indecomposable modules over a central separable R-algebra A is finite.

For R given by Lemma 1 or 2, since $R \cong \oplus \sum_{i=1}^{k} e_i R$ and $A \cong \oplus \sum_{i=1}^{k} e_i A$ such that there is exactly one isomorphic class of finitely generated projective and indecomposable e_iA-modules, using the same proof as Corollaries 1 and 2 in [5] for each e_iA we have:

COROLLARY. *If the ring is given by Lemma* 1 *or* 2, *then* (a) *the Brauer group of* R, $G(R)$, *is isomorphic to a finite direct sum of Brauer groups*, $G(e_iR)$, *and* (b) *every class of* $G(e_iR)$ *contains a unique element* D *such that for any* A *equivalent to* D, A *is isomorphic to a matrix ring over* D *and* $D \cong eAe$ *for some idempotent of* A, e.

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