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JACOBI MATRICES AND THE SPECTRUM OF THE NEUMANN OPERATOR ON A FAMILY OF RIEMANN SURFACES

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ABSTRACT. The Neumann operator is an operator on the boundary of a smooth manifold which maps the boundary value of a harmonic function to its normal derivative. The spectrum of the Neumann operator is studied on the curves bounding a family of Riemann surfaces. The Neumann operator is shown to be isospectral to a direct sum of symmetric Jacobi matrices, each acting on $l^2(\mathbb{Z})$. The Jacobi matrices are shown to be isospectral to generators of bilateral, linear birth-death processes. Using the connection between Jacobi matrices and continued fractions, it is shown that the eigenvalues of the Neumann operator must solve a certain equation involving hypergeometric functions. Study of the equation yields uniform bounds on the eigenvalues and also the asymptotics of the eigenvalues as the curves degenerate into a wedge of circles.

1. **Introduction.** In this paper, we shall study the spectral properties of the Neumann operator on a certain family of curves which degenerate into a wedge of circles.

The Neumann operator is a pseudodifferential operator living on the boundary of a smooth domain. The spectral properties of the Neumann operator have not been studied until relatively recently [K], [G-M], [E]. Nevertheless, understanding these spectral properties may yield some insight into the general relationship between geometry and spectra. In particular, study of the spectrum of the Neumann operator on a degenerating family of curves might yield some insight into spectral problems arising in the study of adiabatic limits (*cf.* [C]). Furthermore, the spectra of certain simple perturbations of the Neumann operator can be used to study the relationship between the spectrum of the Dirichlet Laplacian and the spectrum of the Neumann Laplacian on a broad class of domains [F], [B-F-K].

The spectral properties of the Neumann operator on the family of curves studied in this paper are also of interest for the following reason. It will be shown that the Neumann operator on these curves is isospectral to a direct sum of Jacobi matrices, each acting on $l^2(\mathbb{Z})$. These Jacobi matrices are isospectral to generators of bilateral, linear birth-death processes. Thus the results in this paper complement the results in [K-M], where the spectra of unilateral, linear birth-death processes are studied.

The Neumann operator will be defined as follows. Let Ω be a compact, bordered Riemann surface which is biholomorphically equivalent to the unit disk, and suppose the boundary, $\partial\Omega$, is a simple closed curve. Endow Ω with a Riemann metric. Let $u \in$

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$C^\infty(\partial\Omega)$. Then it is well known that there is a unique harmonic extension $\tilde{u}$ of u to the interior Ω.

We define the Neumann operator, $N_{\partial\Omega}$, on $\partial\Omega$ as the map

$$u \longrightarrow \partial_\eta \tilde{u}|_{\partial\Omega}.$$

Here ∂_η is the outward normal derivative at the boundary.

We state some of the known properties of $N_{\partial\Omega}$.

Using Green's formula it is easy to show the following:

LEMMA 1. *$N_{\partial\Omega}$ is symmetric and non-negative.*

Unlike the Laplacian, $N_{\partial\Omega}$ is not a differential operator. In fact, let x be a local coordinate of $\partial\Omega$ induced by an arclength parametrization of $\partial\Omega$. Let ξ be the cotangent variable corresponding to x. The following is proven in [E]:

PROPOSITION 1. *$N_{\partial\Omega}$ is a linear, first-order elliptic classical pseudodifferential operator. In the coordinates (x,ξ), the symbol $n(x,\xi)$ has the following complete asymptotic expansion:*

$$n(x,\xi) \sim |\xi|.$$

For simplicity we denote the self-adjoint extension of $N_{\partial\Omega}$ again by $N_{\partial\Omega}$.

By the basic spectral theory of pseudodifferential operators on compact manifolds, it follows that $N_{\partial\Omega}$ has a non-negative discrete spectrum. Denote the spectrum of $N_{\partial\Omega}$, including multiplicities, by $\sigma(N_{\partial\Omega})$.

The following analogue of Weyl's law, due to Victor Guillemin and Richard Melrose, is proven in [E]:

THEOREM 1. *Let the perimeter of Ω be L. Then*

$$\sigma(N_{\partial\Omega}) = \{(2\pi/L)|k| + O(|k|^{-\infty})\ ;\ k \in \mathbb{Z}\}.$$

Here $O(|k|)^{-\infty})$ means vanishing more rapidly than the reciprocal of any polynomial.

For example, it is easy to compute the eigenvalues of $N_{\partial\mathbb{D}}$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. Let $\theta \in [0, 2\pi)$ denote the usual arclength parametrization of $\partial\mathbb{D}$, so that $\{e^{\imath m\theta}\}$ is a basis of $L^2(\partial\mathbb{D})$, the set of square integrable functions on $\partial\mathbb{D}$. Then the harmonic extension of $e^{\imath m\theta}$ to $\mathbb{D}$ is

$$z^m = r^m e^{\imath m\theta} \text{ for } m \geq 0,$$
$$\bar{z}^{-m} = r^{|m|} e^{\imath m\theta} \text{ for } m < 0.$$

Applying the normal derivative at $\partial\mathbb{D}$ one obtains

$$N_{\partial\mathbb{D}}(e^{\imath m\theta}) = |m| e^{\imath m\theta}. \tag{1.1}$$

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Thus $\sigma(N_{\partial\mathbb{D}}) = \{|k| \,;\, k \in \mathbb{Z}\}$.

In Section 2 of this paper we will define a family of ramified Riemann surfaces denoted $\mathbb{D}[c,q]$, parametrized by $q \in \mathbb{N}$, $\mathbb{N}$ being the set of strictly positive integers, and $c \in [0,1)$. For $c = 0$, $\mathbb{D}[c,q]$ is the Riemann surface of the function $z^{1/q}$. For $c > 0$, $\mathbb{D}[c,q]$ is a ramified q-fold cover of $\mathbb{D}$, with the ramification point projecting to a point $p \in \mathbb{D}$ with $|p| = c$. $\mathbb{D}[c,q]$ is assigned a Riemann metric away from the ramification point by defining the q-fold covering map to be a local isometry.

The boundary of $\mathbb{D}[c,q]$ is a smooth closed curve which degenerates into a wedge of circles as $c \longrightarrow 1$. Let $N(c,q)$ denote the Neumann operator on the boundary. The spectrum of $N(c,q)$ will be the subject of study in this paper.

One of the key properties of $\mathbb{D}[c,q]$ is that it is biholomorphically equivalent to $\mathbb{D}$. Thus, using the properties of biholomorphisms, the problem

$$N(c,q)u = \lambda u, \quad u \in C^\infty(\partial\mathbb{D}[c,q])$$

pulls back to the problem

$$M_g \circ N_{\partial\mathbb{D}} v = \lambda v, \quad v \in C^\infty(\partial\mathbb{D}).$$

Here M_g denotes multiplication by the function g, with

$$g(\theta) = \frac{1 + c^2 - 2c\cos q\theta}{q(1-c^2)}. \tag{1.2}$$

Fix q, and let v_m be such that $v_m(\theta) = e^{\imath m\theta}$, and let

$$E_d = \operatorname{span}\{v_{nq+d} \,;\, n \in \mathbb{Z}\}. \tag{1.3}$$

Then $L^2(\partial\mathbb{D}) = \oplus_{d=0}^{q-1} E_d$ provides a decomposition of $L^2(\partial\mathbb{D})$ into subspaces invariant under $M_g \circ N_{\partial\mathbb{D}}$. Thus one can study the spectrum of $M_g \circ N_{\partial\mathbb{D}}$ restricted to E_d.

In the case $d = 0$, geometric considerations show that the spectrum of $M_g \circ N_{\partial\mathbb{D}}$ restricted to E_d is $\{|k| \,;\, k \in \mathbb{Z}\}$.

In the case $d \neq 0$, let $\{e_\jmath\}$ be the standard basis of $l^2(\mathbb{Z})$. Then one can identify E_d with $l^2(\mathbb{Z})$ using the map $v_{qm+d} \longrightarrow e_m$. The matrix representing $M_g \circ N_{\partial\mathbb{D}}$ restricted to E_d is isospectral to the following symmetric Jacobi matrix which we label B_d^c:

$$\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \ddots & \frac{c}{1-c^2}\sqrt{\pi_{n-2}\pi_{n-1}} & & & & \\ & \ddots & \frac{1+c^2}{1-c^2}\pi_{n-1} & \frac{c}{1-c^2}\sqrt{\pi_{n-1}\pi_n} & & & \\ & & \frac{c}{1-c^2}\sqrt{\pi_{n-1}\pi_n} & \frac{1+c^2}{1-c^2}\pi_n & \frac{c}{1-c^2}\sqrt{\pi_n\pi_{n+1}} & & \\ & & & \frac{c}{1-c^2}\sqrt{\pi_n\pi_{n+1}} & \frac{1+c^2}{1-c^2}\pi_n & \ddots & \\ & & & & \frac{c}{1-c^2}\sqrt{\pi_{n+1}\pi_{n+2}} & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{1.4}$$

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where

$$\pi_n = |n + d/q|.$$

The theory of symmetric Jacobi matrices has been studied in connection with the classical moment problem, with Lie theory, with orthogonal polynomials, and with birth-death processes. It will be shown in Section 4 that the matrix B_d^c is isospectral to the generator of a birth-death process acting on a weighted l^2 space; this generator is represented by:

$$(1.5)\qquad -\begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & \ddots & \frac{c}{1-c^2}\pi_{n-2} & & & & \\ & \ddots & \frac{-1-c^2}{1-c^2}\pi_{n-1} & \frac{c}{1-c^2}\pi_{n-1} & & & \\ & & \frac{c^2}{1-c^2}\pi_n & \frac{-1-c^2}{1-c^2}\pi_n & \frac{c}{1-c^2}\pi_n & & \\ & & & \frac{c^2}{1-c^2}\pi_{n+1} & \frac{-1-c^2}{1-c^2}\pi_{n+1} & \ddots & \\ & & & & \frac{c^2}{1-c^2}\pi_{n+2} & \ddots & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}.$$

To study the spectrum of B_d^c, we first need to prove a result in the theory of Jacobi matrices. Let A be a symmetric bilateral Jacobi matrix, and suppose the entries to the superdiagonal of A are non-zero, and that the spectrum of A is discrete. Then it was proven in [I-L-M-V] that the spectrum consists of the poles of the following three functions meromorphic in z:

$$\langle e_0, (A - zI)^{-1} e_0 \rangle,\ \langle e_0, (A - zI)^{-1} e_1 \rangle,\ \langle e_1, (A - zI)^{-1} e_1 \rangle.$$

Here $\langle *, * \rangle$ is the inner product on $l^2(\mathbb{Z})$. In Section 3 of this paper it is shown that actually the spectrum of A consists of the poles of $\langle e_0, (A - zI)^{-1} e_0 \rangle$. In fact for any i, j, the poles of $\langle e_i, (A - zI)^{-1} e_j \rangle$ coincide with the poles of $\langle e_0, (A - zI)^{-1} e_0 \rangle$.

It is a standard fact in the theory of Jacobi matrices that $\langle e_0, (A - zI)^{-1} e_0 \rangle^{-1}$ can be expressed as a sum of continued fractions. In the case where $A = B_d^c$, these continued fractions can be expressed explicitly in terms of some 2-1 hypergeometric functions. This is because B_d^c is closed related to the Jacobi matrices connected with associated Meixner polynomials, and the continued fraction representation of the resolvent has been computed explicitly in terms of 2-1 hypergeometric functions in these cases [M-R].

Thus one obtains an equation involving 2-1 hypergeometric functions whose solution set is precisely the spectrum of B_d^c. This equation is given in Theorem 12.

Applying the theory of 2-1 hypergeometric functions to Theorem 12, one obtains the main results in this paper:

THEOREM 2. *$\sigma(B_d^c) \cap \mathbb{N} = \emptyset$, for $0 < c < 1$, $d \neq 0$.*

It follows from the theory of Jacobi matrices that the eigenvalues of B_d^c have multiplicity one, and it follows from analytic perturbation theory that the eigenvalues are analytic functions of c. Let $\{\lambda_i(c)\}$ be the set of eigenvalues on B_d^c. Then the theorem above provides bounds on $\lambda_i(c)$:

COROLLARY 1. *Let $m \in \mathbb{N}$, and suppose $\lambda_n(0) \in (m, m+1)$. Then $\lambda_n(c) \in (m, m+1)$ for $0 < c < 1$.*

For the lowest eigenvalues the bounds can be improved on:

THEOREM 3. *Let $q > 2$. $\{d/q, 1 - d/q\} \cap \sigma(B_c^d) = \emptyset$ for $0 < c < 1$.*

In the special case $q = 2$, $d = 1$, the bounds can be improved as follows:

THEOREM 4. $\{n - 1/2 \,;\, n \in \mathbb{N}\} \cap \sigma(B_c^d) = \emptyset$ *for* $0 < c < 1$.

It is clear from Equation (1.4) that $\sigma(B_d^0) = \{i - 1 + d/q, i - d/q \;;\; i \in \mathbb{N}\}$. It is convenient for the spectrum of B_d^c to be split into two subsets $\{\lambda_i^+(c)\}$ and $\{\lambda_i^-(c)\}$, with $i \in \mathbb{N}$, such $\lambda_i^+(0) = i - d/q$ and $\lambda_i^-(0) = i - 1 + d/q$. Then in another application of Theorem 12, one obtains the asymptotics of the eigenvalues as $c \longrightarrow 1$:

THEOREM 5. *Suppose $d \neq 0$. Then, as $c \longrightarrow 1$,*
1. $\lambda_1^-(c) \sim 1/\left(2 \ln \frac{c^2}{1-c^2}\right)$.
2. $\lambda_i^-(c) \sim i + 1/\left(\ln \frac{c^2}{1-c^2}\right)$, $i > 1$.
3. $\lambda_i^+(c) \sim i - (1-c)(1-d/q)(d/q)i(2i+1)$.

As a consequence of Theorem 5, one can make an interesting observation on the relationship between the spectrum of the Neumann operator and the underlying topology. As the underlying curve deforms into a wedge of q copies of $\partial\mathbb{D}$, Neumann operator spectrally decouples in the following sense:

COROLLARY 2. *As $c \longrightarrow 1$, the spectrum of $N(c, q)$ converges pointwise to the spectrum of the Neumann operator associated to the disjoint union of q copies of $\partial\mathbb{D}$.*

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2. **The underlying geometry and preliminary results.** Let q be a positive integer and $h: \mathbb{D} \longrightarrow \mathbb{D}$ be a ramified q-fold cover. In particular, let $h = f \circ Q$, where $f: \mathbb{D} \longrightarrow \mathbb{D}$ is a bihilomorphism of the disk, and $Q(z) = z^q$. Let $p = f(0)$. Then it is well known that there exists:
1. a ramified Riemann surface $\mathbb{D}[p', q]$ with ramification point p',
2. a biholomorphism $\psi_{p'}: \mathbb{D} \longrightarrow \mathbb{D}[p', q]$ such that $\psi_{p'}(0) = p'$,
3. a holomorphism $\pi_{p'}: \mathbb{D}[p', q] \longrightarrow \mathbb{D}$ which is a q-fold covering map away from p', and under which $\pi_{p'}(p') = p$,
4. and a commuting diagram:

(I)
$$\begin{array}{ccc} & & \mathbb{D}[p', q] \\ & \overset{\psi_{p'}}{\nearrow} & \downarrow \pi_{p'} \\ \mathbb{D} & \xrightarrow{h} & \mathbb{D} \end{array} .$$

Clearly, $\mathbb{D}[p', q]$ is unique up to biholomorphism.

One can metrize $\mathbb{D}[p, q]$ be defining $\pi_{p'}$ to be local isometry away from p'. The metric then vanishes at p'. Note that such a metric is compatible with the conformal structure away from p'.

Because $\mathbb{D}[p', q]$ is unique up to biholomorphism and $\pi_{p'}$ is a local isometry, it is easy to show the following two lemmas:

LEMMA 2. *$\mathbb{D}[p', q]$ defined as above is unique up to isometry.*

LEMMA 3. *If $p_1, p_2 \in \mathbb{C}$ are such that $|p_1| = |p_2|$, then $\mathbb{D}[p_1', q]$ and $\mathbb{D}[p_2', q]$ are isometric.*

It follows from Lemma 3 that one can rewrite, without ambiguity, $\mathbb{D}[p', q]$ as $\mathbb{D}[c, q]$, with $c = |p| \in [0, 1)$.

Let $f\colon \mathbb{D} \longrightarrow \mathbb{D}$ be defined as above. Then f induces a biholomorphism $F\colon \mathbb{D}[0, q] \longrightarrow \mathbb{D}[c, q]$ such that the following diagram commutes.

(II)
$$\begin{tikzcd}
& \mathbb{D}[0,q] \arrow[r, "F"] \arrow[d, "\pi_0"] & \mathbb{D}[c,q] \arrow[d, "\pi_c"] \\
\mathbb{D} \arrow[ur, "\psi_0"] \arrow[r, "Q"] & \mathbb{D} \arrow[r, "f"] & \mathbb{D}
\end{tikzcd}$$

Here the ramification point in $\mathbb{D}[c, q]$ gets mapped by π_c to $c \in [0, 1) \subset \mathbb{C}$.

Denote the boundary of $\mathbb{D}[c, q]$ by $\partial\mathbb{D}[c, q]$. It is clear that the Neumann operator on $\partial\mathbb{D}[c, q]$, denoted $N(c, q)$, exists. To study its spectrum, we will pull $N(c, q)$ back to an operator on $\partial\mathbb{D}$. The biholomorphism $F \circ \psi_0\colon \mathbb{D} \longrightarrow \mathbb{D}[c, q]$, when restricted to $\partial\mathbb{D}$, is a diffeomorphism onto $\partial\mathbb{D}[c, q]$. For convenience, let us denote this restriction by $F \circ \psi_0$.

PROPOSITION 2. *$(F \circ \psi_0)^* N(c, q) = M_g \circ N_{\partial\mathbb{D}}$, where M_g denotes multiplication by the function g, and*

$$g(\theta) = \frac{1 + c^2 - 2c\cos(q\theta)}{q(1 - c^2)}. \tag{2.1}$$

PROOF. Note that harmonic functions composed with holomorphisms remain harmonic.

Also note that since $(F \circ \psi_0)$ is angle preserving, normal derivatives on $\partial\mathbb{D}[c, q]$ pull back to normal derivatives on $\partial\mathbb{D}$. In fact, let $x \in \partial\mathbb{D}[c, q]$, and let η_x be the outward unit normal vector to $\partial\mathbb{D}[c, q]$ at x. Similarly, let ν_θ be the outward unit normal vector to $\partial\mathbb{D}$ at θ, with $\theta = (F \circ \psi_0)^{-1}(x)$. Then by Diagram II one has

$$\begin{aligned}(F \circ \psi_0)^* \eta_x &= (f \circ Q)^* (\pi_c)_* \eta_x \\ &= \big(1/q\,|f'(q\theta)|\big)\nu_\theta.\end{aligned}$$

To compute $|f'(\theta)|$, recall that f was defined as a disk-preserving biholomorphism such that $f(0) = c$; thus f must be of the form

$$f(z) = e^{i\alpha}\frac{z - ce^{i\psi}}{1 - ce^{-i\psi}z},$$

with $\alpha + \psi = \pi$. By Lemma 3 we can assume $\psi = 0$ and $\alpha = \pi$. Thus

$$|f'(\theta)| = \frac{1-c^2}{1+c^2-2c\cos\theta}. \tag{2.2}$$

The proposition now follows from the definition of the Neumann operator.

Thus $N(c,q)$ is isospectral to $M_g \circ N_{\partial\mathbb{D}}$. Let

$$0 = \lambda_0(c) < \lambda_1(c) \leq \lambda_2(c) \leq \cdots$$

be the eigenvalues of $N(c,q)$.

One can prove that the spectrum of $N(c,q)$ is a non-constant function of c by studying that zeta function of $N(c,q)$. Recall that the zeta function has as its formal definition

$$\zeta(z) = \sum_{\lambda_\iota(c)\neq 0} \big(\lambda_\iota(c)\big)^{-z}.$$

By Theorem 1, $\zeta(z)$ is analytic for $\mathrm{Re}(z) > 1$. For $\mathrm{Re}(z) \leq 1$, $\zeta(z)$ can be defined by meromorphic extension, and is explicitly computable at the negative integers [E]. In particular, one has

$$\zeta(-2) = \frac{(q^2-1)c^2}{1-c^2}. \tag{2.3}$$

Equation (2.3) shows the following:

PROPOSITION 3. *Let $q > 1$. Then $c \neq c' \Rightarrow \sigma\big(N(c,q)\big) \neq \sigma\big(N(c',q)\big)$.*

There is one subset of $\sigma\big(N(c,q)\big)$ which is independent of c, as we now show. Let v_m be such that $v_m(\theta) = e^{\iota m\theta}$, $\theta \in \partial\mathbb{D}$. Thus $\{v_m\}_{m=-\infty}^{\infty}$ forms a basis of $L^2(\partial\mathbb{D})$. For $d \in \mathbb{Z}$, let E_d be defined as in (1.3).

Then $L^2(\partial\mathbb{D}) = \oplus_{d=0}^{q-1} E_d$, and by (2.1) each E_d is invariant under $M_g \circ N_{\partial\mathbb{D}}$.

PROPOSITION 4. $\sigma\big((M_g \circ N_{\partial\mathbb{D}})|_{E_0}\big) = \{|k| \; ; k \in \mathbb{Z}\}$.

PROOF. Recall that $N_{\partial\mathbb{D}} v_m = |m| v_m$. Since π_c is a holomorphism, and a local isometry near the boundary, it follows that

$$N(c,q)(\pi_c)^* v_k = |k|(\pi_c)^* v_k.$$

Thus

$$M_g \circ N_{\partial\mathbb{D}}(\pi_c \circ F \circ \psi_0)^* v_k = |k|(\pi_c \circ F \circ \psi_0)^* v_k.$$

But by Diagram II, $\pi_c \circ F \circ \psi_0 = f \circ Q$, and clearly $(f \circ Q)^* L^2(\partial\mathbb{D}) = E_0$. Thus $\{(\pi_c \circ F \circ \psi_0)^* v_k \; ; m \in \mathbb{Z}\}$ forms a basis of eigenvectors of E_0, with $\{|k| \; ; k \in \mathbb{Z}\}$ as eigenvalues.

3. **Jacobi matrices and the continued fraction representation of the resolvent.** Let $\ell^2(\mathbb{Z}^+)$ be the set of all square summable complex sequences, and let $\{e_n\}_{n=0}^{\infty}$ be the standard basis of $\ell^2(\mathbb{Z}^+)$. If $u = \sum_{n=0}^{\infty} u_n e_n$, and $v = \sum_{n=0}^{\infty} v_n e_n$, then the inner product is defined by $\langle u, v\rangle = \sum_{n=0}^{\infty} u_n \bar{v}_n$.

Consider the symmetric Jacobi matrix

$$A = \begin{pmatrix} a_0 & b_1 & 0 & \\ b_1 & a_1 & b_2 & \ddots \\ 0 & b_2 & a_2 & \ddots \end{pmatrix}, \quad bn \neq 0, \tag{3.1}$$

acting $\ell^2(\mathbb{Z}^+)$. Suppose A represents a closed operator on a minimal domain in $\ell^2(\mathbb{Z}^+)$.

PROPOSITION 5 [A]. *A is self-adjoint if and only if* $\sum_{n=1}^{\infty} b_n^{-1} = \infty$.

Associated to A one has the difference equation

$$b_{n+1}X_{n+1}(\lambda) + (a_n - \lambda)X_n(\lambda) + b_n X_{n-1}(\lambda) = 0, \quad n \geq 0. \tag{3.2}$$

Here we assume $b_0 = 1$. Assuming $X_n \neq 0$ for $n \geq 0$, one gets

$$b_n \frac{X_{n-1}(\lambda)}{X_n(\lambda)} = (\lambda - a_n) + \cfrac{-b_{n+1}^2}{b_{n+1}\cfrac{X_n(\lambda)}{X_{n+1}(\lambda)}}. \tag{3.3}$$

Repeated application of formula (3.3) gives, formally at least,

$$\frac{X_{-1}(\lambda)}{X_0(\lambda)} = (\lambda - a_0) + \cfrac{-b_1^2}{\lambda - a_1 - \cfrac{b_2^2}{\lambda - a_2 - \cdots}}. \tag{3.4}$$

The continued fraction on the right hand side is expressed by the compact notation

$$\mathop{K}_{n=1}^{\infty} \frac{-b_n^2}{\lambda - a_n}. \tag{3.5}$$

To discuss the convergence of the continued fraction, it is necessary to define the subdominant solution. The subdominant solution of (3.2), denoted $X_n^s(\lambda)$, is a solution such that for any linearly independent solution $X_n(\lambda)$ of (3.2),

$$\lim_{n\to\infty} \frac{X_n^s(\lambda)}{X_n(\lambda)} = 0.$$

The subdominant solution $X_n^s(\lambda)$ is also referred to as the *minimal solution*. It is well known from the classical moment problem [A] that a sufficient condition for the existence of a subdominant solution is for A to be self-adjoint.

The relationship between the continued fraction and the subdominant solution is exhibited in the following theorem of Pincherle [G]:

THEOREM 6. *The continued fraction (3.5) converges (possibly to ∞) if and only if X_n^s exists, and in this case,*

$$\frac{X_{-1}^s(\lambda)}{X_0^s(\lambda)} = (\lambda - a_0) + \mathop{K}_{n=1}^{\infty} \frac{-b_n^2}{\lambda - z_n}.$$

The continued fraction is also related to the resolvent $(A - \lambda I)^{-1}$ by the following formula [M-R]:

$$\langle e_0, (\lambda I - A)^{-1} e_0 \rangle = \frac{1}{a_0 - \lambda + K_{n=1}^{\infty} \frac{-b_n^2}{a_n - \lambda}}. \tag{3.6}$$

Much of the theory of unilateral symmetric Jacobi matrices extends easily to the bilateral case. Let $\ell^2(\mathbb{Z})$ be the set of square-summable bilateral complex sequences, with $\{e_i\}_{i=-\infty}^{\infty}$ the standard basis. If $u = \sum_{i=-\infty}^{\infty} u_i e_i$ and $v = \sum_{j=-\infty}^{\infty} v_j e_j$, then the inner product on $\ell^2(\mathbb{Z})$ is defined by $\langle u, v \rangle = \sum_{i=-\infty}^{\infty} u_i \bar{v}_i$.

Suppose we have a closed, symmetric operator with minimal domain on $\ell^2(\mathbb{Z})$ represented by the bilateral infinite Jacobi matrix

$$A = \begin{pmatrix} \ddots & \ddots & b_{-1} & & & \\ & b_{-1} & a_{-1} & b_0 & & \\ & & b_0 & a_0 & b_1 & \\ & & & b_1 & a_1 & b_2 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \begin{array}{l} a_n = \langle e_n, A e_n \rangle \in \mathbb{R}, \\ b_n = \langle e_{n+1}, A e_n \rangle \neq 0. \end{array}$$

PROPOSITION 6 [A]. *If $\sum_{n=0}^{\infty} 1/b_n = \infty$ and $\sum_{n=-\infty}^{-\infty} 1/b_n = \infty$, then A is self-adjoint.*

Suppose that A is self-adjoint. We have the following extensions of the unilateral Jacobi matrix theory:

THEOREM 7 I) [M-R]. *For every $i, j \in \mathbb{Z}$, $\langle e_i, (\lambda I - A)^{-1} e_j \rangle$ has a universal continued fraction representation. In particular for $\lambda \notin \sigma(A)$,*

$$\langle e_i, (\lambda I - A)^{-1} e_i \rangle = \frac{1}{\lambda - a_i + \mathop{K}_{n=1}^{\infty} \frac{-b_{n+i}^2}{\lambda - a_{n+i}} + \mathop{K}_{n=1}^{\infty} \frac{-b_{1-n+i}^2}{\lambda - a_{-n+i}}}, \tag{3.7}$$

and

$$\langle e_0, (\lambda I - A)^{-1} e_1 \rangle = \frac{1}{(\lambda - a_0)\left(\lambda - a_1 + K_{n=1}^{\infty} \frac{-b_{n+1}^2}{a_{n+1}}\right) - \frac{b_0^2\left(\lambda - a_1 + \mathop{K}_{n=1}^{\infty} \frac{-b_{n+1}^2}{\lambda - a_{n+1}}\right)}{\lambda - a_{-1} + \mathop{K}_{n=-1}^{-\infty} \frac{-b_n^2}{a_{n-1}}}}. \tag{3.8}$$

II) [I-L-M-V]. λ_0 is an eigenvalue of A if and only if $\langle e_n, (zI - A)^{-1} e_m \rangle$ has a pole at $z = \lambda_0$ for $m, n = 0$ or 1.

Part II of this theorem can be improved as follows:

THEOREM 8 *λ_0 is an eigenvalue of A if and only if $\langle e_0, (zI - A)^{-1} e_0\rangle$ has a pole at $z = \lambda_0$*

PROOF To show that $z = \lambda_0$ is a pole of $\langle e_0, (zI - A)^{-1} e_0\rangle$ if and only if it is a pole of $\langle e_1, (zI - A)^{-1} e_1\rangle$, it suffices to use (3 7) to show that

$$\langle e_1, (zI - A)e_1\rangle^{-1}$$
$$= \lambda - a_1 + \mathop{K}_{n=1}^{\infty} \frac{-b_{n+1}^2}{\lambda - a_{n+1}} - \cfrac{b_1^2}{\langle e_0, (zI - A)^{-1} e_0\rangle^{-1} + \cfrac{b_1^2}{\lambda - a_1 + \mathop{K}_{n=1}^{\infty} \frac{-b_{n+1}^2}{\lambda - a_{n+1}}}}$$

Similarly, one can use (3 8) to show that $z = \lambda_0$ is a pole of $\langle e_0, (zI - A)^{-1} e_0\rangle$ if and only if it is a pole of $\langle e_0, (zI - A)^{-1} e_1\rangle$

We conclude this section with the following classical result

THEOREM 9 [A] *All eigenvalues of A have multiplicity one*

4 An equation characterising the eigenvalues.

We apply the results of the previous section to $M_g \circ N_{\partial\mathbb{D}}$ Fix $q > 1$ and $0 < d < q$ Relabel $M_g \circ N_{\partial\mathbb{D}}|_{E_d}$ as N_c Using the map $v_{nq+d} \longrightarrow e_n$, E_d is identified with $\ell^2(\mathbb{Z})$ Under this identification, N_c is represented by the matrix

$$(4\ 1)\qquad \begin{pmatrix} \frac{c}{1-c^2}|n-1+d/q| & & \\ \frac{1+c^2}{1-c^2}|n-1+d/q| & \frac{c}{1-c^2}|n+d/q| & \\ \frac{c}{1-c^2}|n-1+d/q| & \frac{1+c^2}{1-c^2}|n+d/q| & \frac{c}{1-c^2}|n+1+d/q| \\ & \frac{c}{1-c^2}|n+d/q| & \frac{1+c^2}{1-c^2}|n+1+d/q| \\ & & \frac{c}{1-c^2}|n+1+d/q| \end{pmatrix},$$

which we denote again by N_c

LEMMA 4 *The matrix N_c is isospectral to the matrix B_d^c given in (1 4)*

PROOF Consider the weighted ℓ^2-space $\ell^2(\mathbb{Z}, \pi_n)$, where the inner-product is defined by $(u, v) = \sum_{n=-\infty}^{\infty} \pi_n u_n \bar{v}_n$, with $\pi_n = |n + d/q|$

Let N_c denote the matrix (4 1) acting on $\ell^2(\mathbb{Z})$, and let N_c' denote the matrix (4 1) acting on $\ell^2(\mathbb{Z}, \pi_n)$ Suppose $v = (v_m) \in \ell^2(\mathbb{Z})$ satisfies $(N_c - \lambda)v = 0$ By the ellipticity of $M_g \circ N_{\partial\mathbb{D}}$, $|v_n|$ vanishes rapidly as $|n| \longrightarrow \infty$ Hence $v \in \ell^2(\mathbb{Z}, \pi_n)$, and so λ is an eigenvalue N_c' Similarly, if λ is an eigenvalue of N_c', then λ is an eigenvalue of N_c Thus the spectra of N_c and N_c' coincide

Let $f_n = \pi_n^{-1/2}(\ldots,0,0,1,0,0,\ldots)^t$, so that $\{f_n\}_{n=-\infty}^{\infty}$ forms an orthonormal basis of $\ell^2(\mathbb{Z},\pi_n)$. Define $L = \ell^2(\mathbb{Z},\pi_n) \longrightarrow \ell^2(\mathbb{Z})$ by $L(f_n) = e_n$. Let B_d^c: $\ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z})$ be densely defined by:

$$B_d^c e_n = \frac{c}{1-c^2}\sqrt{|n-1+d/q|\,|n+d/q|}e_{n-1} + \frac{1+c^2}{1-c^2}|n+d/q|e_n \\ + \frac{c}{1-c^2}\sqrt{|n+d/q|\,|n+1+d/q|}e_{n+1}.$$

It is easily verified that

$$(L^{-1}B_d^c Lf_n, f_m) = (N_c' f_n, f_m),$$

and so $L^{-1}B_d^c L = N_c'$. This proves the lemma.

THEOREM 10. *The operator B_d^c is unitarily equivalent to the matrix (1.5) acting on a weighted l^2 space.*

PROOF. Let $\pi_n' = c^{-2n}d/|qn+d|$, and let $l^2(\mathbb{Z},\pi_n')$ be the weighted l^2 space whose inner product is given by $(u,v) = \sum_{n=-\infty}^{\infty}\pi_n' u_n \bar{v}_n$.

Let A be the operator on $l^2(\mathbb{Z},\pi_n')$ whose representation, with respect to the basis $\{e_n\ ;\ n \in \mathbb{Z}\}$, is the matrix (1.5). Let U be the unitary transformation on $l^2(\mathbb{Z},\pi_n')$ given by $U_{e_n} = (-1)^n e_n$, and let $A' = -U^{-1}AU$. Then arguing as in the previous lemma, we see that A' is unitarily equivalent to B_d^c.

We will now study the spectrum of B_d^c. By Proposition 6, B_d^c is self-adjoint, and by Theorem 9, its eigenvalues have multiplicity one. Furthermore, it is easily verified that B_d^c is analytic in c in the sense of Kato, and so one has [R-S]:

PROPOSITION 7. *Let $\lambda_i(c)$ be defined as in Section 2. Then for $0 \le c < 1$, $\lambda_i(c)$ is analytic in c.*

THEOREM 11. $\sigma(B_d^c) = \sigma(B_{(q-d)}^c)$.

PROOF. By Theorem 7,

$$\langle e_0,(z-B_d^c)^{-1}e_0\rangle^{-1} = z - \frac{d(1+c^2)}{q(1-c^2)} + \mathop{K}_{n=1}^{\infty} \frac{\frac{-c^2}{(1-c^2)^2}|n-d/q|\,|n-1-d/q|}{z-|n-d/q|\left(\frac{1+c^2}{1-c^2}\right)} \\ + \mathop{K}_{n=1}^{\infty} \frac{\frac{-c^2}{(1-c^2)^2}|-n+1-d/q|\,|-n-d/q|}{z-|-n-d/q|\left(\frac{1+c^2}{1-c^2}\right)} \\ = \langle e_{-1},(z-B_{(q-d)}^c)^{-1}e_{-1}\rangle^{-1}. \tag{4.2}$$

The theorem now follows from Theorem 8.

Theorem 11 shows that we can assume without loss of generality that $d \le q/2$.

The continued fraction in (4.2) is closely related to the continued fractions derived from the difference equations associated to Meixner polynomials. Because the subdominant solution for the Meixner case has been computer explicitly, [M-R], one obtains a new equation characterising the eigenvalues of B_d^c which involves 2-1 hypergeometric functions.

Recall that a 2-1 hypergeometric function is defined as follows:

$$(4.3)\qquad {}_2F_1\begin{pmatrix} a, b, z \\ c \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

with $(d)_n = \Gamma(d+n)/\Gamma(d)$. The 2-1 hypergeometric function is analytic in z for $|z| < 1$ and continuous in z for $|z| \leq 1$ provided $\mathrm{Re}(c-a-b) > 0$.

The following identities will be frequently used in this paper: Let $n = 0, 1, 2, \ldots$. Then

$$(4.4)\qquad \begin{aligned} &\lim_{c \to -n} {}_2F_1\begin{pmatrix} a, b, z \\ c \end{pmatrix} \Big/ \Gamma(c) \\ &\quad = \frac{(a)_{n+1}(b)_{n+1}}{(n+1)!} z^{n+1} {}_2F_1\begin{pmatrix} a+n+1, b+n+1, z \\ n+2 \end{pmatrix} \ ; \ ([\text{S.A}], 15.1.2) \end{aligned}$$

$$(4.5)\qquad {}_2F_1\begin{pmatrix} a, b, 1 \\ c \end{pmatrix} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \text{ if } c > \mathrm{Re}(a+b) \ ; \ ([\text{S.A}], 15.1.20)$$

$$(4.6)\qquad {}_2F_1\begin{pmatrix} a, b, z \\ c \end{pmatrix} = (1-z)^{-a} {}_2F_1\begin{pmatrix} a, c-b, z/(z-1) \\ c \end{pmatrix} \ ; \ ([\text{S.A}], 15.3.4)$$

$$(4.7)\qquad {}_2F_1\begin{pmatrix} a, b, z \\ c \end{pmatrix} = (1-z)^{c-a-b} {}_2F_1\begin{pmatrix} c-a, c-b, z \\ c \end{pmatrix}. \ ([\text{S.A}], 15,3.3)$$

Finally, for $|\arg(-z)| < \pi$, $|z| < 1$, $(c-a) \notin \mathbb{Z}$, there is ([G-R], 9.154):

$$(4.8)\qquad \begin{aligned} &\frac{\Gamma(z)\Gamma(a+1)}{\Gamma(c)} {}_2F_1\begin{pmatrix} a, a+1, z \\ c \end{pmatrix} \\ &\quad = (-z)^{-a-1} \sum_{n=0}^{\infty} \frac{\Gamma(a+1+n)\Gamma(2-c+a+n)}{n!\,(n+1)!} z^{-n} g(n) \\ &\qquad + (-z)^{-a}\Gamma(a)\Gamma(1-c+a), \end{aligned}$$

with $g(n) = \ln(-z) + \pi \cot\big(\pi(c-a)\big) + \psi(n+2) + \psi(n+1) - \psi(a+n+1) - \psi(2-c+a+n)$.

THEOREM 12. *$\lambda \in \sigma(B_d^c)$ if and only if*

$$(4.9)\qquad \begin{aligned} &(d/q-\lambda)(1-d/q-\lambda)\, {}_2F_1\begin{pmatrix} -1+d/q, d/q, -c^2/(1-c^2) \\ d/q-\lambda \end{pmatrix} \\ &\quad {}_2F_1\begin{pmatrix} -d/q, 1-d/q, -c^2/(1-c^2) \\ 1-d/q-\lambda \end{pmatrix} \\ &\quad - d/q(1-d/q)\frac{c^2}{(1-c^2)}\, {}_2F_1\begin{pmatrix} d/q, 1+d/q, -c^2/(1-c^2) \\ 1+d/q-\lambda \end{pmatrix} \\ &\quad {}_2F_1\begin{pmatrix} 1-d/q, 2-d/a, -c^2/(1-c^2) \\ 2-d/q-\lambda \end{pmatrix} = 0. \end{aligned}$$

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REMARK. *If $\lambda = \lambda_0$ is such that any of the hypergeometric functions on the left-hand side of (4.9) is singular, then the left hand side of (4.9) should be understood as the limit as $\lambda \longrightarrow \lambda_0$.*

PROOF. By (4.2),

$$\langle e_0, (z - B_d^c)^{-1} e_0 \rangle^{-1} = \mathrm{CF}_1(z) + \frac{\left(-\frac{d}{q}\right)\left(1 - \frac{d}{q}\right)\frac{c^2}{(1-c^2)^2}}{\mathrm{CF}_2(z)}, \tag{4.10}$$

with

$$\mathrm{CF}_1(z) = z - \frac{d}{q}\frac{(1+c^2)}{(1-c^2)} + \operatorname*{K}_{n=1}^{\infty} \frac{\frac{-c^2}{(1-c^2)^2}(n + d/q)(n - 1 + d/q)}{z - (n + d/q)\frac{1+c^2}{1-c^2}}, \tag{4.11}$$

and

$$\mathrm{CF}_2(z) = z - (1 - d/q)\frac{1+c^2}{1-c^2} + \operatorname*{K}_{n=1}^{\infty} \frac{\frac{-c^2}{(1-c^2)^2}(n - d/q)(n + 1 - d/q)}{z - (n + 1 - d/q)\frac{1+c^2}{1-c^2}}. \tag{4.12}$$

$\mathrm{CF}_1(z)$ is the continued fraction associated to the difference equation:

$$\begin{aligned}\frac{c}{1-c^2}\sqrt{(n + d/q)(n + 1 + d/q)}X_{n+1} + \left(\frac{1+c^2}{1-c^2}(n + d/q) - z\right)X_n \\ + \frac{c}{1-c^2}\sqrt{(n + d/q)(n - 1 + d/q)}X_{n-1} = 0, \quad n \geq 0.\end{aligned} \tag{4.13}$$

By Theorem 6, $\mathrm{CF}_1(z) = \frac{c}{1-c^2}\sqrt{(d/q)(-1 + d/q)}\left(X^s_{-1}(z)/X^s_0(z)\right)$, where $X^s_n(z)$ is the minimal solution of (4.13). Applying [M-R, (3.2)] one obtains:

$$X^s_n = \left(\frac{-c}{1-c^2}\right)^{n+1} \frac{\sqrt{\Gamma(n + d/q)\Gamma(n + 1 + d/q)}}{\Gamma(n + 1 + d/q - \lambda)} \, {}_2F_1\left(\begin{matrix} n + d/q, n + 1 + d/q, -c^2/(1 - c^2) \\ n + 1 + d/q - \lambda \end{matrix}\right).$$

Thus

$$\mathrm{CF}_1(z) = (d/q - \lambda)\frac{{}_2F_1\left(\begin{matrix} -1+d/q, d/q, -c^2/(1-c^2) \\ d/q-\lambda \end{matrix}\right)}{{}_2F_1\left(\begin{matrix} d/q, 1+d/q, -c^2/(1-c^2) \\ 1+d/q-\lambda \end{matrix}\right)}. \tag{4.14}$$

From (4.11) and (4.12), it is clear that one obtains $\mathrm{CF}_2(z)$ from $\mathrm{CF}_1(z)$ by substituting $q - d$ for d. Hence

$$\mathrm{CF}_2(z) = (1 - d/q - \lambda)\frac{{}_2F_1\left(\begin{matrix} -d/q, 1-d/q, -c^2/(1-c^2) \\ 1-d/1-\lambda \end{matrix}\right)}{{}_2F_1\left(\begin{matrix} 1-d/q, 2-d/q, -c^2/(1-c^2) \\ 2-d/q-\lambda \end{matrix}\right)}. \tag{4.5}$$

Thus the theorem is obtained using Theorem 8 and (4.10).

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5. **Bounds on the eigenvalues.** Using Theorem 12, it is easy to show that the eigenvalues of B_d^c are bounded as functions of c:

THEOREM 2. *$\sigma(B_d^c) \cap \mathbb{N} = \emptyset$ for $0 < c < 1$.*

PROOF. Suppose $m \in \sigma(B_d^c) \cap \mathbb{N}$. Applying (4.9), (4.7), and (4.3) one obtains:

(5.1)
$$
\begin{aligned}
0 &= (d/q-m)(1-d/q-m) \\
&\qquad {}_2F_1\begin{pmatrix} -1+d/q, d/q, -c^2/(1-c^2) \\ d/q-m \end{pmatrix} {}_2F_1\begin{pmatrix} d/q, 1-d/q, -c^2/(1-c^2) \\ 1-d/q-m \end{pmatrix} \\
&\quad - d/q(1-d/q)\frac{c^2}{(1-c^2)^2} \\
&\quad {}_2F_1\begin{pmatrix} d/q, 1+d/q, -c^2/(1-c^2) \\ 1+d/q-m \end{pmatrix} {}_2F_1\begin{pmatrix} 1-d/q, 2-d/q, -c^2/(1-c^2) \\ 2-d/q-m \end{pmatrix} \\
&= (d/q-m)(1-d/q-m){}_2F_1\begin{pmatrix} 1-m, -m, -c^2/(1-c^2) \\ d/q-m \end{pmatrix} \\
&\qquad {}_2F_1\begin{pmatrix} 1-m, -m, -c^2/(1-c^2) \\ d/q-m \end{pmatrix} \\
&\quad - d/q(1-d/q)c^2{}_2F_1\begin{pmatrix} 1-m, -m, -c^2/(1-c^2) \\ 1+d/q-m \end{pmatrix} \\
&\qquad {}_2F_1\begin{pmatrix} 1-m, -m, c/(1-c^2) \\ 1+d/q-m \end{pmatrix} \\
&= \sum_{n=0}^{\infty}\Big(\frac{-c^2}{1-c^2}\Big)^n \sum_{j+k=n} \frac{(1-m)_j(-m)_j(1-m)_k(-m)_k(1-d/q-m)(d/q-m)}{j!\,k!(d/q-m)_j(1-d/q-m)_k} \\
&\quad \Big(1-\frac{d/q(1-d/q)c^2}{(d/q-m+j)(1-d/q-m+k)}\Big).
\end{aligned}
$$

Recall that $(1-m)_j = (1-m)\cdots(1-m+j-1)$. Hence one can assume $j \leq m-1$, and similarly $k \leq m-1$, in the sum above. It follows that

$$
1-\frac{d/q(1-d/q)c^2}{(d/q-m+j)(1-d/q-m+k)} > 0 \text{ for } c < 1, \tag{5.2}
$$

and

$$
\operatorname{sgn}\Big(\frac{(1-m)_j(-m)_j(1-m)_k(-m)_k}{j!\,k!(d/q-m)_j(1-d/q-m)_k}\Big) = (-1)^n. \tag{5.3}
$$

Here $\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$. From (5.2) and (5.3), it follows that the addends in (5.1) are all the same sign. Hence (5.1) is non-zero for $c \in (0,1)$, and so by contradiction, $m \notin \sigma(B_d^c)$.

THEOREM 3. *Let $q > 2$. Then $\{d/q, 1-d/q\} \cap \sigma(B_c^d) = \emptyset$ for $0 < c < 1$.*

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PROOF. Suppose $d/q \in \sigma(B_c^d)$. Then by Theorem 12,

$$\lim\Big\{(d/q-\lambda)(1-d/q-\lambda)\, {}_2F_1\begin{pmatrix}-d/q, 1-d/q, -c^2/(1-c^2)\\ 1-d/q-\lambda\end{pmatrix} {}_2F_1\begin{pmatrix}-1+d/q, d/q, -c^2/(1-c^2)\\ d/q-\lambda\end{pmatrix}$$
$$-d/q(1-d/q)\frac{c^2}{(1-c^2)}{}_2F_1\begin{pmatrix}d/q, 1+d/q, -c^2/(1-c^2)\\ 1+d/q-\lambda\end{pmatrix}$$
$$(5.4)\qquad {}_2F_1\begin{pmatrix}1-d/q, 2-d/q, -c^2/(1-c^2)\\ 2-d/q-\lambda\end{pmatrix}\Big\} = 0.$$

By (4.4),

$$\lim_{\lambda\to d/q}(d/q-\lambda){}_2F_1\begin{pmatrix}-1+d/q, d/q, -c^2/(1-c^2)\\ d/q-\lambda\end{pmatrix}$$
$$= (d/q)(1+d/q)\Big(\frac{-c^2}{1-c^2}\Big){}_2F_1\begin{pmatrix}d/q, 1+d/q, -c^2/(1-c^2)\\ 2\end{pmatrix}.$$

Thus (5.4) becomes the equation:

$$(1-2d/q){}_2F_1\begin{pmatrix}d/q, 1+d/q, -c^2/(1-c^2)\\ 2\end{pmatrix}{}_2F_1\begin{pmatrix}-d/q, 1-d/q, -c^2/(1-c^2)\\ 1-2d/2\end{pmatrix}$$
$$-{}_2F_1\begin{pmatrix}d/q, 1+d/q, -c^2/(1-c^2)\\ 1\end{pmatrix}{}_2F_1\begin{pmatrix}1-d/q, 2-d/q, -c^2/(1-c^2)\\ 2-2d/q\end{pmatrix} = 0.$$

Applying (4.6), one obtains

$$(5.5)\qquad (1-2d/q){}_2F_1\begin{pmatrix}d/q, 1-d/q, c^2\\ 2\end{pmatrix}{}_2F_1\begin{pmatrix}-d/q, -d/q, c^2\\ 1-2d/q\end{pmatrix}$$
$$= {}_2F_1\begin{pmatrix}d/q, -d/q, -c^2/(1-c^2)\\ 1\end{pmatrix}{}_2F_1\begin{pmatrix}1-d/q, -d/q, c^2\\ 2-2d/q\end{pmatrix}.$$

By (4.3), the left-hand side of (5.5) is an increasing function of c, while the right hand side is a decreasing function. By (4.5), the limit as $c \to 1$ of both sides of the equation is

$$\frac{\Gamma(2-2d/q)}{\Gamma(2-d/q)\Gamma(1-d/q)^2\Gamma(1+d/q)}.$$

Thus the (5.5) cannot hold for $c < 1$, and so by contradiction, $d/q \notin \sigma(B_d^c)$. Similarly, $1-d/q \notin \sigma(B_d^c)$.

THEOREM 4. *Let* $q = 2$, $d = 1$, *and* $c \in (0,1)$. *Then* $\{n-1/2\,;\, n \in \mathbb{N}\} \cap \sigma(B_1^c) = \emptyset$.

PROOF. Suppose $\lambda = n - 1/2 \in \sigma(B_1^d)$, with $n \in \mathbb{N}$. Then by Theorem 12,

$$\lim_{\lambda\to n-1/2}\Big[(1/2-\lambda){}_2F_1\begin{pmatrix}-1/2, 1/2, -c^2/(1-c^2)\\ 1/2-\lambda\end{pmatrix}$$
$$(5.6)\qquad \pm 1/2\frac{c}{1-c^2}{}_2F_1\begin{pmatrix}1/2, 3/2, -c^2/(1-c^2)\\ 3/2-\lambda\end{pmatrix}\Big] = 0.$$

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CASE 1. $n = 1$. By (4.4), (5.6) becomes

$$(1/2)c_2F_1\left(\begin{matrix}1/2, 3/2, -c^2/(1-c^2)\\ 2\end{matrix}\right) \pm {}_2F_1\left(\begin{matrix}1/2, 3/2, -c^2/(1-c^2)\\ 1\end{matrix}\right) = 0.$$

Applying (4.6), one gets

$$\underbrace{\frac{c}{2}{}_2F_1\left(\begin{matrix}1/2, 1/2, c^2\\ 2\end{matrix}\right)}_{A} \pm \underbrace{{}_2F_1\left(\begin{matrix}1/2, -1/2, c^2\\ 1\end{matrix}\right)}_{B} = 0.$$

By (4.3) and (4.5), $A - B < 0$ and $A + B > 0$ for $0 \leq c < 1$. Thus by contradiction, $1/2 \notin \sigma(B_1^c)$.

CASE 2. $n = 2, 3, \ldots$. Suppose (5.6) holds. Then by (4.4) and (4.6),

$$\begin{aligned}
0 &= \lim_{\lambda \to n-1/2} \frac{1}{\Gamma(1/2-\lambda)}\Bigg[(1/2-\lambda){}_2F_1\left(\begin{matrix}-1/2, 1/2, -c^2/(1-c^2)\\ 1/2-\lambda\end{matrix}\right)\\
&\qquad \pm \frac{1}{2}\frac{c}{1-c^2}{}_2F_1\left(\begin{matrix}1/2, 3/2, -c^2/(1-c^2)\\ 3/2-\lambda\end{matrix}\right)\Bigg]\\
&= \frac{-1}{2}\frac{n}{n!}\left(\frac{1}{1-c^2}\right)^{n+1}(-c)^{2n+1}\\
&\qquad \Bigg[c\frac{(-1/2)(1/2)}{n+1}{}_2F_1\left(\begin{matrix}n+1/2, n+3/2, -c^2/(1-c^2)\\ n+2\end{matrix}\right)\\
&\qquad \pm \frac{1}{2}{}_2F_1\left(\begin{matrix}n+1/2, n+3/2, -c^2/(1-c^2)\\ n+1\end{matrix}\right)\Bigg]\\
&= \frac{n}{n!}c^{2n+1}\left(\frac{1}{1-c^2}\right)^{1/2}\frac{1}{4}\Bigg[\frac{-c}{2(n+1)}{}_2F_1\left(\begin{matrix}n+1/2, 1/2, c^2\\ n+2\end{matrix}\right)\\
&\qquad \pm {}_2F_1\left(\begin{matrix}n+1/2, 1/2, c^2\\ n+1\end{matrix}\right)\Bigg].
\end{aligned}$$

Arguing as in Case 1, this equation cannot hold for $0 \leq c < 1$. Thus $n - 1/2 \notin \sigma(B_1^c)$.

6. **Asymptotics of the eigenvalues.** Assume $d \neq 0$ and $0 < c < 1$. By Theorem 12, $\lambda \in B_d^c$ if and only if

(6.1)

$$\begin{aligned}
0 &= d(d-q)c^2\frac{\Gamma(-1+d/q)\Gamma(d/q)}{\Gamma(d/q-\lambda)}{}_2F_1\left(\begin{matrix}-1+d/q, d/q, -c^2/(1-c^2)\\ d/q-\lambda\end{matrix}\right)\\
&\qquad \frac{\Gamma(-d/q)\Gamma(1-d/q)}{\Gamma(1-d/q-\lambda)}{}_2F_1\left(\begin{matrix}-d/q, 1-d/q, -c^2/(1-c^2)\\ 1-d/q-\lambda\end{matrix}\right)\\
&\quad + \left(\frac{qc^2}{1-c^2}\right)^2\frac{\Gamma(1-d/q)\Gamma(2-d/q)}{\Gamma(2-d/q-\lambda)}{}_2F_1\left(\begin{matrix}1-d/q, 2-d/q, -c^2/(1-c^2)\\ 2-d/q-\lambda\end{matrix}\right)\\
&\qquad \frac{\Gamma(d/q)\Gamma(1+d/q)}{\Gamma(1+d/q-\lambda)}{}_2F_1\left(\begin{matrix}d/q, 1+d/q, -c^2/(1-c^2)\\ 1+d/q-\lambda\end{matrix}\right).
\end{aligned}$$

Applying (4.8), one obtains

$$
\begin{aligned}
0 = d(d-q)c^2 \frac{\sin \pi(1-\lambda)}{\pi} &\Big[\Gamma(-1+d/q)\Gamma(\lambda)\Big(\frac{c^2}{1-c^2}\Big)^{1-d/q} \\
&+ \Big(\frac{c^2}{1-c^2}\Big)^{-d/q} \sum_{k=0}^{\infty} \Gamma(d/q+k)\Gamma(\lambda+1+k) g_1(k) \Big(\frac{1-c^2}{-c^2}\Big)^k\Big] \\
&\frac{\sin \pi(1-\lambda)}{\pi}\Big[\Gamma(-d/q)\Gamma(\lambda)\Big(\frac{c^2}{1-c^2}\Big)^{d/q} \\
&+ \Big(\frac{c^2}{1-c^2}\Big)^{d/q-1} \sum_{k=0}^{\infty} \Gamma(1-d/q+k)\Gamma(\lambda+1+k) g_2(k) \Big(\frac{1-c^2}{-c^2}\Big)^k\Big] \\
&+ \Big(\frac{qc^2}{1-c^2}\Big)^2 \frac{\sin \pi(1-\lambda)}{\pi}\Big[\Gamma(1-d/q)\Gamma(\lambda)\Big(\frac{c^2}{1-c^2}\Big)^{d/q-1} \\
&+ \Big(\frac{c^2}{1-c^2}\Big)^{d/q-2} \sum_{k=0}^{\infty} \Gamma(2-d/q+k)\Gamma(\lambda+1+k) g_3(k) \Big(\frac{1-c^2}{-c^2}\Big)^k\Big] \\
&\frac{\sin \pi(1-\lambda)}{\pi}\Big[\Gamma(d/q)\Gamma(\lambda)\Big(\frac{c^2}{1-c^2}\Big)^{-d/q} \\
&+ \Big(\frac{c^2}{1-c^2}\Big)^{-d/q-1} \sum_{k=0}^{\infty} \Gamma(d/q+1+k)\Gamma(\lambda+1+k) g_4(k) \Big(\frac{1-c^2}{-c^2}\Big)^k\Big].
\end{aligned} \tag{6.2}
$$

Here $g_\iota(k) = \ln\big(c^2/(1-c^2)\big) + \pi \cot \pi(1-\lambda) + \alpha_\iota$, with α_ι bounded functions of c. By Theorem 2, $\sin\big(\pi(1-\lambda)\big) \neq 0$, and so

$$
\begin{aligned}
0 = c^2 &\Big[1 + \frac{1-c^2}{c^2} \sum_{k=0}^{\infty} (k-1+d/q)(-1+d/q)_k (\lambda+k)(\lambda)_k g_1(k) \Big(\frac{1-c^2}{-c^2}\Big)^k\Big] \\
&\Big[1 + \frac{1-c^2}{c^2} \sum_{k=0}^{\infty} (k-d/q)(-d/q)_k (\lambda+k)(\lambda)_k g_2(k) \Big(\frac{1-c^2}{-c^2}\Big)^k\Big] \\
&- \Big[1 + \frac{1-c^2}{c^2} \sum_{k=0}^{\infty} (k+1-d/q)(1-d/q)_k (\lambda+k)(\lambda)_k g_3(k) \Big(\frac{1-c^2}{-c^2}\Big)^k\Big] \\
&\Big[1 + \frac{1-c^2}{c^2} \sum_{k=0}^{\infty} (k+d/q)(d/q)_k (\lambda+k)(\lambda)_k g_4(k) \Big(\frac{1-c^2}{-c^2}\Big)^k\Big].
\end{aligned} \tag{6.3}
$$

Eliminating higher order terms, (6.3) becomes

(6.4)

$$
\begin{aligned}
0 \sim 1 &+ \frac{\lambda}{c^2}(c^2+1)\ln\frac{c^2}{1-c^2} + \frac{\lambda}{c^2}(c^2+1)\pi\cot\pi(1-\lambda) - \Big(\pi\cot\big(\pi(1-\lambda)\big)\Big)^2 \\
&\times \frac{(1-c^2)^2}{c^4}\lambda^2\Big[-(1-d/q)(d/q) - (-1+d/q)(1-d/q)(-d/q)(\lambda+1) \\
&- (d/q)(-d/q)(-1+d/q)(\lambda+1) + \frac{1}{c^2}(2-d/q)(1-d/q)(d/q)(\lambda+1) \\
&+ \frac{1}{c^2}(1-d/q)(1+d/q)(1+d/q)(d/q)(\lambda+1)\Big].
\end{aligned}
$$

Solving for $\pi\cot\big(\pi(1-\lambda)\big)$ using the quadratic formula, and using the approximation $\sqrt{1+x}\sim 1+\frac{1}{2}x$, one obtains:

$$\pi\cot\big(\pi(1-\lambda)\big)\sim\frac{c^2(1+c^2)}{\lambda(1-d/q)(d/q)[-1+(\lambda+1)(3/c^2-1)]\big(\frac{1}{1-c^2}\big)^2} \tag{6.5}$$

or

$$\pi\cot\big(\pi(1-\lambda)\big)\sim-\frac{c^2}{c^2+1}\left(\frac{1+\lambda\frac{c^2+1}{c^2}\ln\frac{c^2}{1-c^2}}{\lambda}\right) \tag{6.6}$$

This shows that for $\lambda(c)\in\sigma(B_d^c)$, $\lim_{c\to 1}\lambda(c)=m\in\mathbb{Z}$. In fact, if (6.5) holds, then

$$\lambda(c)\sim m-(1-c^2)^2\cdot\frac{\lambda(1-d/q)(d/q)[-1+(\lambda+1)(\frac{3}{c^2}-1)]}{c^2(1+c^2)}$$

If (6.6) holds and $m\neq 0$, then $\lambda(c)\sim m+\frac{1}{\ln(c^2/1-c^2)}$. Finally, if (6.6) holds and $\lambda(c)\to 0$, then $\lambda(c)\sim\frac{1}{2\ln(c^2/1-c^2)}$. This completes the proof of Theorem 5.

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