Smooth semi-inner-product spaces

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With a semi-inner-product on a normed linear space is associated a
certain mapping from the space into its dual, called a support mapping.
It is shown that properties of the support mapping are closely allied to
the Gâteaux differentials of the norm, and that the various
differentiability properties for the norm can be characterised by
corresponding continuity properties for the support mappings. Continuity
properties of the support mappings used in conjunction with the
subreflexivity of Banach spaces provide a simple technique in establishing
many results which involve differentiability of the norm.

Using the fact that smoothness is characterised by a continuity
property of the semi-inner-product, inner product arguments are extended
to the study of the geometry of smooth semi-inner-product spaces.
Rotundity can be expressed readily in terms of the semi-inner-product. A
generalised orthogonality relation is defined in terms of the
semi-inner-product and is used to give a representation theorem for
continuous linear functionals similar to the Riesz representation theorem
for Hilbert space. The semi-inner-product representation of the dual
space is studied and questions of the duality of smoothness and rotundity
are discussed. The class of uniform semi-inner-product spaces is defined
and attention drawn to the duality properties of such spaces.

H. Rund's approach to the differential geometry of Minkowskian space
was a motivating influence in the research for the thesis. In
establishing the uniform semi-inner-product character of a smooth and
rotund finite dimensional normed linear space it is shown that many of
Rund's results are independent of the context of classical differential
geometry.

An analysis of L_p-spaces, where $1 < p < \infty$, as uniform

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semi-inner-product spaces illustrates the facility of the theory in handling duality questions.

An application of support mapping properties is given in an examination of part of the theory of the numerical range of an operator on a normed linear space as developed by G. Lumer and F.F. Bonsall.

Several areas for possible further research are outlined.

The following papers have arisen from the research for the thesis:

J.R. Giles, "Classes of semi-inner-product spaces", *Trans. Amer. Math. Soc.* 129 (1967), 436-446.

J.R. Giles, "On a characterisation of differentiability of the norm of a normed linear space", *J. Austral. Math. Soc.* (to appear).