THE SUBVARIETY LATTICE OF THE VARIETY
OF DISTRIBUTIVE DOUBLE p-ALGEBRAS

WIESŁAW DZIOBIAK

Let L denote the subvariety lattice of the variety of
distributive double p-algebras, that is, the lattice whose
universe consists of all varieties of distributive double
p-algebras and whose ordering is the inclusion relation. We
prove in this paper that each proper filter in L is
uncountable. Moreover, we prove that except for the trivial
variety (the zero in L) and the variety of Boolean algebras
(the unique atom in L) every other element of L , generated by
a finite algebra, has infinitely many covers in L , among which
at least one is not generated by any finite algebra. The former
result strengthens a result of Urquhart who showed that the
lattice L is uncountable. On the other hand, both of our
results indicate a high complexity of the lattice L at least in
comparison with the subvariety lattice of the variety of
distributive p-algebras, since a result of Lee shows that the
latter lattice forms a chain of type $\omega + 1$ and every cover in
it of the variety generated by a finite algebra is itself
generated by a finite algebra.

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1. Preliminaries

An algebra $(A, \wedge, \vee, *, +, 0, 1)$ is called a distributive double
p-algebra if $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $*$
and $+$ are unary operations on A such that $a \wedge x = 0$ if and only if
$x \leq a^*$, and $a \vee x = 1$ if and only if $a^+ \leq x$.

A distributive double p-algebra A is said to be of finite range if
for each $a \in A$ there exists $n < \omega$ such that $a^{n(+*)} = a^{(n+1)(+*)}$,
where $a^{0(+*)} := a$ and $a^{(k+1)(+*)} := a^{k(+*)+*}$ for every $k \geq 0$. When
$a^{n(+*)} = a^{(n+1)(+*)}$ for every $a \in A$, the algebra A is said to be of
range n . A is said to be regular if any two congruences on A with a
class in common are equal. By a result of Varlet [11], A is regular if
and only if it satisfies the quasi-identity: $x^* = y^*$ and $x^+ = y^+$ imply
$x = y$.

With each distributive double p-algebra A there are associated the
following sets: $D(A) = \{a \in A; a^* = 0\}$, $\overline{D}(A) = \{a \in A; a^+ = 1\}$ and
$\mathrm{Cen}(A) = \{a \in A; a = a^{+*}\}$.

A lattice filter F on a distributive double p-algebra A is called
normal if $a \in F$ implies $a^{+*} \in F$. If F is normal then
$C(F) = \{(a, b) \in A \times A; a \wedge c = b \wedge c$ for some $c \in F\}$ is a congruence
relation on A .

For a poset P , by $\mathrm{Min}\ P$ and $\mathrm{Max}\ P$ we denote the set of all
minimal and the set of all maximal elements in P , respectively. If
$X \subseteq P$ then by $[X)$ we denote the set of all $a \in P$ such that $x \leq_P a$
for some $x \in X$. $(X]$ is defined dually. Instead of $[\{x\})$ and $(\{x\}]$
we shall write $[x)$ and $(x]$, respectively. We call a map $f : P \to Q$
admissible if it is isotone and satisfies $f((x] \cap \mathrm{Min}\ P) = (f(\dot{x})] \cap \mathrm{Min}\ Q$
and $f([x) \cap \mathrm{Max}\ P) = [f(x)) \cap \mathrm{Max}\ Q$ for every $x \in P$. Recall that the
set of all increasing subsets of a poset P with respect to the set-
theoretical meet and join, and the operations $*, +$ defined by $X^* = P \backslash (X]$
and $X^+ = [P \backslash X)$ forms a distributive double p-algebra. This algebra will
be denoted by DP .

For a given distributive double p-algebra A let $P(A)$ denote the
set of all prime filters on A . We have a poset $(P(A), \subseteq)$. The map

$P : A \rightarrow DP(A)$ defined by $P(a) = \{X \in P(A); a \in X\}$ is an embedding of A
into $DP(A)$.   If A is finite then P is an isomorphism.

Throughout this paper by   **2**   and   **3**   we denote, respectively, two and
three element chains, considered as distributive double   *p*-algebras.   By
$V(K)$,   K   is a class of similar algebras, we denote the least variety
containing all algebras of   K .   We shall write $V(A)$ instead of $V(\{A\})$.

For a fuller account concerning distributive double   *p*-algebras we
refer to [2], [3], [7], [8].   For lattice theory and universal algebra we
refer to [4] and [5].

2. A sequence of finite simple algebras

In this section we define an infinite sequence of finite simple
distributive double   *p*-algebras whose every proper subalgebra is isomorphic
to   **2**   or to   **3** .   This sequence will be used in proofs of our results.

For each $5 \leq k$ let P_k denote a poset with the universe
$\{-i; 1 \leq i \leq k\} \cup \{i; 1 \leq i \leq k\}$, partially ordered by the relation $\leq_k$
defined as follows: for all x, y , $x \leq_k y$ holds if and only if $x = y$,
or $x = -y$ and $1 \leq y \leq k$, or $x = -(y-1)$ and $2 \leq y \leq k$, or
$-(k-3) \leq x \leq -2$ and $y = k - 1$.   The diagrams of P_5 and P_6 look as
follows:
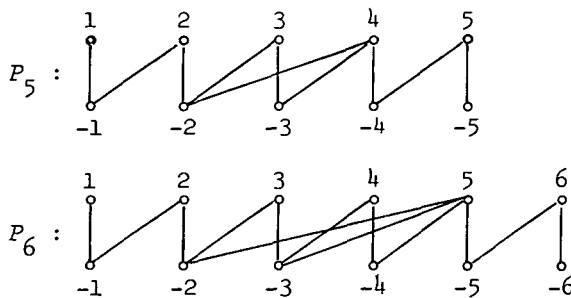
LEMMA 2.1.   *For each*   *k*   *and a poset*   *P* ,   *if* $f : P_k \rightarrow P$ *is*
admissible and not one-to-one then   Im *f*   *is a*   1-   *or*   2-*element chain*.

Proof.   Since P_k is connected and f is admissible, it suffices to
show that $|\text{Im } f| \leq 2$.

CLAIM 1. If $1 \le i \ne j \le k$ and $f(i) = f(-j)$ then $|\text{Im } f| = 1$.

Let $1 \le i \ne j \le k$ and $f(i) = f(-j)$. Then $f(i) \in \text{Min } P$ since $-j \in \text{Min } P_k$. But $f(i) \in \text{Max } P$ because $i \in \text{Max } P_k$. Hence $f(i)$ is both minimal and maximal in P . Therefore $f(i) = f(x)$ for all $x \in P_k$, proving the claim.

For $1 \le i \le k$ denote by $N(i)$ the set of all $1 \le j \le k$ such that $i \ne j$ and $f(i) = f(j)$ imply $|\text{Im } f| \le 2$. By induction one can verify the following

CLAIM 2. $N(i) = \{1, 2, \ldots, k\} \backslash \{i\}$ for every $1 \le i \le k$.

The lemma follows from Claims 1 and 2 and the observation that if $f(-i) = f(-j)$ for some $1 \le i \ne j \le k$ then there exist $1 \le m \ne n \le k$ with $f(m) = f(n)$.

LEMMA 2.2 (see [3]). *For finite posets P and Q , DP is embeddable into DQ if and only if there exists an admissible map from Q onto P .*

Proof. The "if" part is a particular case of Proposition 2.2 from Davey [3]. For the "only if" part let $f : DP \to DQ$ be an embedding. Then the map $g(X) = f^{-1}(X)$ for all X , is onto and admissible. Hence the observation that $Q \cong (P(DQ), \subseteq)$ and $P \cong (P(DP), \subseteq)$ concludes the proof.

LEMMA 2.3 (see [3]). *For a finite poset P ,*

(i) DP is simple if and only if P is connected and $P = \text{Min } P \cup \text{Max } P$,

(ii) DP is subdirectly irreducible but not simple if and only if $|P| > 1$, P is connected and there exists a unique x of P with $P = (\text{Min } P \cup \text{Max } P) \cup \{x\}$.

COROLLARY 2.4. *Each DP_k is simple and of range 4 . Moreover, the only, up to isomorphism, proper subalgebras of DP_k are 2 and 3 .*

Proof. The range of DP_k is verified easily. The rest follows from Lemmas 2.1, 2.2 and 2.3.

3. Cardinality of a proper filter in L

We prove here that each proper filter in L is of cardinality $2^{\aleph_0}$.
We begin with the following lemma.

LEMMA 3.1. *Let* K *be a variety of distributive double p-algebras, all being of the same finite range. Then both the class* K_{si} *of sub-directly irreducible algebras in* K *and the class* K_{sim} *of simple algebras in* K *form universal classes.*

Proof. That K_{si} is a universal class follows directly from a result of Katrinǎk [8] which says that a distributive double p-algebra A of finite range and with $|D(A)| \geq 2$ is subdirectly irreducible if and only if $|D(A) \cap \overline{D}(A)| \leq 2$ and $Cen(A) = \{0, 1\}$, and from the observation that A is a Boolean algebra if and only if $|D(A)| = 1$. That K_{sim} forms a universal class is obvious since, by a result of Beazer [1], a nontrivial distributive double p-algebra A of range k is simple if and only if A is regular and satisfies $x^{k(+*)} = 0$ for every $x \in A\backslash\{1\}$.

THEOREM 3.2. *Each proper filter in* L *is of cardinality* $2^{\aleph_0}$.

Proof. It suffices to show that the principal filter in L generated by a proper subvariety K of the variety of distributive double p-algebras is of cardinality $2^{\aleph_0}$. As the variety of distributive double p-algebras is generated by its finite members, there exists a finite poset P having at least two elements and whose distributive double p-algebra is subdirectly irreducible and does not belong to K . For each $n \geq 5$, let $P + P_n$ denote a poset with universe $P \cup P_n$ (we assume here that P and P_n are disjoint) and whose ordering is given by

$$\leq_P \cup \leq_n \cup \{(x, y);\ x \in \text{Min } P_n,\ y \in P,\ a \leq_P y \text{ and } a \neq y\}$$
$$\cup \{(x, y);\ x \in P,\ y \in \text{Max } P_n,\ x \leq_P b \text{ and } x \neq b\} ,$$

where a and b are fixed elements of P such that $a \in \text{Min } P$, $b \in \text{Max } P$ and $a \leq_P b$; of course, such elements exist in P since $|P| \geq 2$ and DP is subdirectly irreducible. We claim that DP is

embeddable into $D\left(P+P_n\right)$ for every n . Indeed, in view of Lemma 2.2, it suffices to consider a map $f : P \cup P_n \rightarrow P$ defined by $f(x) = a$ when $x \in \text{Min } P_n$, $f(x) = x$ when $x \in P$ and $f(x) = b$ otherwise.

Now we select inductively a sequence $Q_0, Q_1, \ldots$ of finite posets as follows: $Q_0 := P + P_5$ and $Q_{n+1} := P + P_{|Q_n|}$ for every $n = 0, 1, 2, \ldots$, and, for every $\emptyset \neq I \subseteq \omega$, we define $K(I) := K \vee \text{HSP}\left(\left\{DQ_n; \; n \in I\right\}\right)$. Obviously, $K \subseteq K(I)$ for every $I \subseteq \omega$. It remains then to show that $K(I)$ and $K(J)$ are different for different I and J . To this end we suppose $K(I) = K(J)$ and consider a fixed $m \in I$. We have $DQ_m \in K(J)$ which, in conjunction with Jónsson's Lemma (see [6]), Corollary 2.4 and the above claim, yields that DQ_m is a homomorphic image of a subalgebra A of an ultraproduct of some family of algebras from among DQ_n's , where $n \in J$. We claim that DQ_m is isomorphic to A which would yield that DQ_m is embeddable into some DQ_n , where $n \in J$. Since every DP_i is of range 4 and P is finite, then due to the definition of $P + P_i$ it follows that all algebras of the form DQ_n have the same finite range k . If DP is simple then, by Lemma 2.3, every algebra DQ_n is simple which, together with Lemma 3.1, implies that A is simple as well, and thus DQ_m is isomorphic to A . So assume that DP is not simple. Then, by Lemma 2.3, DQ_m is not simple either. Evidently, A is of range k and, by Lemma 3.1, it is sub-directly irreducible which, by Corollary 9 from [2], implies that the lattice of congruence relations on A is a 2- or 3-element chain. From this, since DQ_m is not simple, it follows that DQ_m is isomorphic to A , completing the proof of the claim. As DQ_m is finite, then by the claim we conclude that DQ_m is embeddable into some DQ_n , where $n \in J$. Hence, by Lemma 2.2, there exists an admissible map f from Q_n onto Q_m . We show $m = n$. To the contrary suppose $m \neq n$. Then $m < n$,

since f is onto, and hence $\left|Q_m\right| < \left|P_{|Q_{n-1}|}\right|$. Therefore, the map

$f{\uparrow}P_{|Q_{n-1}|}$ is not one-to-one which, together with Lemma 2.1, yields

$\left|\operatorname{Im} f{\uparrow}P_{|Q_{n-1}|}\right| \leq 2$. So, we have

$$\left|Q_m\right| = \left|\operatorname{Im} f{\uparrow}P \cup \operatorname{Im} f{\uparrow}P_{|Q_{n-1}|}\right| \leq |P| + 2 \ ,$$

a contradiction. Thus $m = n$, and so $m \in J$ which proves $I \subseteq J$.
Similarly, we show $J \subseteq I$. Thus $I = J$, concluding the proof.

4. On the bottom part of L

In this section we will be concerned with the bottom part of L . For
our efforts the following two lemmas are crucial.

LEMMA 4.1. *Let A be a subdirectly irreducible distributive double
p-algebra with $|A| \geq 3$. Then there exist a set I and a congruence
relation $\theta \neq \mathit{1}$ on A^I such that $D\big(A^I/\theta\big) \cap \overline{D}\big(A^I/\theta\big) \neq \emptyset$.*

Proof. Since A is subdirectly irreducible, then by Theorem 4 from
Katriňák [8], $\operatorname{Cen}(A) = \{0, 1\}$. This yields $x \vee x^* \in D(A)\backslash\{1\}$ for every
$x \in A\backslash\{0, 1\}$. Hence $D(A)\backslash\{1\} \neq \emptyset$. On the other hand, by Lemma 7 from
Katriňák [8], $\bigwedge\big(x^{n(+*)}; n < \omega\big) = 0$ for all $x \in D(A)\backslash\{1\}$. Therefore, in
the case when A is of finite range, we have $x^{n(+*)} = 0$ for some
$x \in D(A)$ and an integer n . But this, in view of the corollary
succeeding Theorem 7 from Katriňák [8], implies that $D(A) \cap \overline{D}(A)$ is non-
empty. So, in the case when A is of finite range, it suffices to take
$I = \{0\}$ and $\theta = \operatorname{id}$.

Let A be not of finite range. Then, for a certain element a of
A , the sequence $a^{n(+*)}$, $n < \omega$, is strictly descending. Put
$b_n := a \wedge a^{2(n+1)(+*)+}$ and $c_n := \vee\big(a^{j(+*)} \wedge a^{j(+*)+}; j \leq 2(n+1)\big)$ where
$n < \omega$.

CLAIM 1. $b_n \wedge c_n^* = 0$ for every $n < \omega$.

Proof. It suffices to prove that $P\big(b_n\big) \cap P\big(c_n^*\big) = \emptyset$. But each prime
filter on A is contained in some maximal prime filter, so in fact it is

enough to prove that for each maximal prime filter X on A the following condition holds: $X \in P(b_n)$ if and only if $X \in P(c_n)$. The "if" part directly follows from the observation that $c_n \leq b_n$. For the "only if" part, let $X \in P(b_n)$ be maximal. Then $X \in P(a)$ and

$X \not\in P\left(a^{(2(n+1)+1)(+*)}\right)$ because $a^{2(n+1)(+*)+} \wedge a^{(2(n+1)+1)(+*)} = 0$.

Hence, there exists a greatest $k < \omega$ such that $X \in P\left(a^{k(+*)}\right)$ and $X \not\in P\left(a^{(k+1)(+*)}\right)$. Since $X \not\in P\left(a^{(k+1)(+*)}\right)$, $X \not\in P(A) \setminus [P\left(a^{k(+*)+}\right)]$. Hence $X \in [P\left(a^{k(+*)+}\right)]$ and, therefore, $X \in P\left(a^{k(+*)+}\right)$ because X is maximal. Thus $X \in P\left(a^{k(+*)} \wedge a^{k(+*)+}\right)$ which, by $k \leq 2(n+1)$, implies $X \in P(c_n)$.

CLAIM 2. $b_n \wedge c_n^+ = b_n$ for every $n < \omega$.

Proof. $\overline{D}(A)$ is an ideal on A and $a^{j(+*)} \wedge a^{j(+*)+} \in \overline{D}(A)$. Hence $c_n \in \overline{D}(A)$, and so $b_n \wedge c_n^+ = b_n$.

CLAIM 3. $b_n^{n(+*)} \neq 0$ for every $n < \omega$.

Proof. For j, m we have

$$a^{(j+1)(+*)} \wedge a^{(m-1)(+*)+} \leq a^{(j+1)(+*)} \wedge a^{(m-1)(+*)+**}$$
$$= a^{(j+1)(+*)} \wedge a^{m(+*)*} \leq a^{(j+1)(+*)} \wedge a^{m(+*)++*} = \left(a^{j(+*)} \wedge a^{m(+*)+}\right)^{+*} .$$

Moreover, $a^{j(+*)} \wedge a^{j(+*)+} \neq 0$ for every $j < \omega$ because the sequence $a^{j(+*)}$, $j < \omega$, is strictly descending. Hence $b_n^{n(+*)} \neq 0$ for every $n < \omega$.

Let $b = (b_n; n < \omega)$ and $c = (c_n; n < \omega)$. By Claim 3, the normal filter F on A^{ω} generated by b is proper. So $C(F) \neq 1$. But, by Claims 1 and 2, $[c]C(F) \in D\left(A^{\omega}/C(F)\right) \cap \overline{D}\left(A^{\omega}/C(F)\right)$, proving the lemma.

LEMMA 4.2. *Let U be a non-principal ultrafilter on $\omega \setminus \{0, 1, 2, 3, 4\}$. Then 2 and 3 are, up to isomorphism, the only finite subdirectly irreducible algebras in $V\left(\Pi_U(DP_k; 5 \leq k < \omega)\right)$.*

Proof. That 2 and 3 are subdirectly irreducible and belong to
$V\bigl(\Pi_U\bigl(DP_k;\ 5 \le k < \omega\bigr)\bigr)$ is obvious. Let $A \in V\bigl(\Pi_U\bigl(DP_k;\ 5 \le k < \omega\bigr)\bigr)$ be
finite and subdirectly irreducible. Then, by Jónsson's Lemma,
$A \in \mathrm{HSP}_U\bigl(\bigl\{\Pi_U\bigl(DP_k;\ 5 \le k < \omega\bigr)\bigr\}\bigr)$. By Corollary 2.4 and Lemma 3.1, each
algebra belonging to $\mathrm{SP}_U\bigl(\bigl\{\Pi_U\bigl(DP_k;\ 5 \le k < \omega\bigr)\bigr\}\bigr)$ is simple. From this we
get that A is simple and, by the assumption that A is finite, we
conclude that A is embedded into some direct power of
$\Pi_U\bigl(DP_k;\ 5 \le k < \omega\bigr)$. This yields that A is embeddable into
$\Pi_U\bigl(DP_k;\ 5 \le k < \omega\bigr)$. Therefore there exists $I \in U$ such that A is
isomorphic to a subalgebra of $\Pi\bigl(DP_k;\ k \in I\bigr)$; here we used again the
assumption that A is finite. Since U is non-principal and A is
simple, then by Corollary 2.4 we get that A is isomorphic to 3 or to
2 .

THEOREM **4.3.** *(i)* $V(3)$ *is a unique cover of* $V(2)$ *in* L .

(ii) $V(3)$ *has infinitely many covers in* L *, among which at least
one is not generated by any finite algebra.*

Proof. *(i)* That $V(3)$ covers $V(2)$ is obvious. That $V(3)$ is
unique follows from Lemma 4.1.

(ii) By Corollary 2.4, every variety $V\bigl(DP_k\bigr)$, where $5 \le k < \omega$,
covers $V(3)$. To prove the rest, let U be a fixed non-principal ultra-
filter on $\omega\backslash\{0, 1, 2, 3, 4\}$. We have $V(3) \subsetneq V\bigl(\Pi_U\bigl(DP_k;\ 5 \le k < \omega\bigr)\bigr)$.

So, as $V(3)$ is finitely based, there exists a cover K of $V(3)$
contained in $V\bigl(\Pi_U\bigl(DP_k;\ 5 \le k < \omega\bigr)\bigr)$. By Lemma 4.2, K is not generated
by any finite algebra.

THEOREM **4.4.** *For a variety* K *generated by a finite distributive
double p-algebra the following conditions are equivalent:*

>   *(i)* *every variety that covers* K *in* L *is generated by a
>   finite algebra;*

>   *(ii)* K *has finitely many covers in* L *;*

>   *(iii)* K *is the trivial variety or the variety of Boolean
>   algebras.*

Proof. By Theorem 4.3, *(iii)* implies *(i)* and *(ii)*. It remains then
to show that both *(i)* and *(ii)* imply *(iii)*.

(i) $\Rightarrow$ *(iii)*. Suppose that K is a nontrivial variety and $K \neq V(2)$.
Then, by Theorem 4.3, $V(3) \subseteq K$. Let L be a fixed variety not generated
by any finite algebra that covers $V(3)$ in L; by Theorem 4.3, such a
cover really exists. We claim that $V(K \cup L)$ covers K in L, and that
$V(K \cup L)$ is not generated by any finite algebra. The latter claim is
immediate since otherwise, by Jónsson's Lemma, L would be generated by a
finite algebra. To prove the former one, let $K \subseteq M \subseteq V(K \cup L)$. Then, by
Jónsson's Lemma, $K_{si} \subseteq M_{si} \subseteq K_{si} \cup L_{si}$. When $M_{si} \cap L_{si} \subseteq I(\{2, 3\})$,
then $M_{si} = K_{si}$, and so $M = K$. Otherwise, $A \in M_{si} \cap L_{si}$ for some
$A \not\cong 2$ and $A \not\cong 3$. But L covers $V(3)$, so $L = V(\{3, A\})$. Hence
$M = V(K \cup L)$. Obviously, $K \neq V(K \cup L)$. Thus $V(K \cup L)$ covers K in
L.

(ii) $\Rightarrow$ *(iii)*. Suppose that K is nontrivial and $K \neq V(2)$. Then
$V(3) \subseteq K$ by Theorem 4.3. Let n be a least natural number such that
$|A| < n$ for every subdirectly irreducible algebra A from K. Clearly,
such a number exists because K is generated by a finite algebra and, by
Jónsson's Lemma, every subdirectly irreducible algebra in K is finite and
there are, up to isomorphism, only finitely many of them. Notice that
$K \neq V\left(K \cup \{DP_k\}\right)$ for every $n \leq k$. Moreover, by Corollary 2.4,
$V\left(K \cup \{DP_k\}\right)$ covers K in L, and $V\left(K \cup \{DP_k\}\right) \neq V\left(K \cup \{DP_m\}\right)$ for all
$n \leq k \neq m$, proving that *(ii)* implies *(iii)*.

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Section of Logic,
Polish Academy of Sciences,
Łódź,
Piotrkowska 179,
Poland.

Correspondence address:
Institute of Mathematics,
N. Copernicus University,
Toruń,
Poland.